FOUNDATIONS OF PHYSICS

An International Journal Devoted to the Conceptual Bases and Fundamental Theories of Modern Physics, Biophysics, and Cosmology

Editor:
Alwyn van der Merwe

Volume 27, Number 8, August 1997

Plenum Press • New York–London

Special Issue of Invited Papers in Memory of and Honoring

Lawrence Biedenharn

1922–1996

Invariant Lie-Admissible Formulation of Quantum Deformations[1]

Ruggero Maria Santilli[2]

Received May 28, 1997

In this note we outline the history of q-deformations, indicate their physical shortcomings, suggest their apparent resolution via an invariant Lie-admissible formulation based on a new mathematics of genotopic type, and point out their expected physical significance.

1. INTRODUCTION

In 1948 Albert[1] introduced the notions of *Lie-admissible and Jordan-admissible algebras* as generally nonassociative algebras U with elements a, b, c, and abstract product ab which are such that the attached algebras U^- and U^+, which are the same vector spaces as U equipped with the products $[a, b]_U = ab - ba$ and $\{a, b\}_U = ab + ba$, are Lie and Jordan algebras, respectively. Albert then studied the algebra with product

$$(A, B) = p \times A \times B + (1 - p) \times B \times A \tag{1}$$

where p is a parameter, A, B are matrices or operators (hereon assumed to be Hermitian), and $A \times B$ is the conventional associative product.

It is easy to see that the above product is indeed jointly Lie- and Jordan-admissible because $[A, B]_U = (1 - 2p) \times (A \times B - B \times A)$ and $\{A, B\}_U = A \times B + B \times A$. However, there exist no (finite) value of p under which product (1) recovers the Lie product. As a result, product (1) cannot be used for possible coverings of current physical theories.

[1] This paper is dedicated to the memory of Larry Biedenharn, my teacher of the rotational symmetry.

[2] Institute for Basic Research, P.O. Box 1577, Palm Harbor, Florida 34682.

In view of the above occurrence, as part of my Ph.D. studies, I introduced in 1967[2] a new notion of Lie-admissibility which is Albert's definition (loc. cit.), plus the condition that the algebras U admit Lie algebras in their classification or, equivalently, that the generalized Lie product admits the conventional one as a particular case.

As an illustration, I introduced, apparently for the first time back in 1967, the algebra with product (Ref. 2, Eq. (8), p. 573)

$$(A, B) = p \times A \times B - q \times B \times A \tag{2}$$

and related time evolution in the infinitesimal and finite forms ($\hbar = = 1$)[2,3,4]

$$\begin{aligned} i \times dA/dt &= p \times A \times H - q \times H \times A \\ A(t) &= \{e^{i \times q \times H \times t}\} \times A(0) \times \{e^{-i \times p \times t \times H}\} \end{aligned} \tag{3}$$

where p and q are non-null parameters with non-null values $p \pm q$. It is easy to see that product (2) is Lie- and Jordan-admissible and admits the Lie and Jordan products as particular (nondegenerate) cases.

Structures (2) and (3) turned out to be insufficient for physical applications because, as we shall see in Sec. 3, the parameters p and q become operators P and Q under the time evolution of the theory. I therefore introduced in 1978[5] (see also monograph[6] of 1983) the notion of *general Lie-admissibility* which is the notion of Ref. 1 plus the conditions that algebras U admit *Lie-isotopic*[5,6] (rather than Lie) algebras in their attached antisymmetric form and admit ordinary Lie algebras in their classification.

The latter notion was realized via the *general Lie-admissible product* (first introduced in Ref. 5b, p. 719; see Ref. 6 for a more detailed treatment)

$$(A, B) = A \times P \times B - B \times Q \times A \tag{4}$$

and time evolution in infinitesimal and finite forms (Ref. 5b, pp. 741, 742, and Ref. 6)

$$\begin{aligned} i \times dA/dt &= A \times P \times H - H \times Q \times A \\ A(t) &= \{e^{i \times H \times Q \times t}\} \times A(0) \times \{e^{-i \times t \times P \times H}\} \end{aligned} \tag{5}$$

where H is Hermitian but P and Q are nonsingular, generally nonhermitian matrices or operators with nonsingular values $P \pm Q$ admitting of the parametric values p and q as particular cases. The conventional Heisenberg's equations are evidently recovered for $P = Q = 1$.

Note that the attached products $[A, B]_U = (A, B) - (B, A) = A \times T \times B - B \times T \times A$, $T = P + Q$, and $\{A, B\}_U = (A, B) + (B, A) = A \times T \times B + B \times T \times A$, $T = P - Q$, are still Lie and (commutative) Jordan, respectively, although of a more general type called *isotopic*.[5,6]

Note also that the P and Q operators must be sandwiched in between the elements A and B to characterize an algebra as commonly understood in mathematics.[5,6] It should be finally indicated that, when properly written, Hamilton's equations with external terms possess precisely a Lie-admissible structure.

Following these studies I had the opportunity of discussing Albert's Lie-admissibility with Larry Biedenharn in a number of occasions, including a visit to his department at Duke University in North Carolina in spring 1981. Subsequently, our scientific contacts were interrupted for several years.

In 1989 Biedenharn[7] and, independently, Macfarlane[8] introduced the so-called *q-deformations*, with a structure of the type

$$A \times B - B \times A \to A \times B - q \times A \times B \tag{6}$$

which are an evident particular case of structures (2), and which were followed by a number of papers so large as to discourage an outline (see, e.g., representative papers[9]). More recently, other types of deformations of relativistic quantum formulations appeared in the literature under the name of k-deformations (see, e.g., Refs. 10, quantum groups (see, e.g., Refs. 11), and other generalizations.

I saw Larry Biedenharn for the last time at the Third Wigner Symposium held at Oxford University, England, in September 1993. During that occasion, I communicated to him the existence of a number of physical shortcomings of the Lie-admissible models in general, and of the q-deformations in particular, which our group had identified following our last meeting of 1981, on which shortcomings he agreed immediately.

I then indicated to Larry Biedenharn, also at the Third Wigner Symposium, new lines of inquiries which apparently permit the resolution of the problematic aspects of Lie-admissible and q-deformations via their invariant formulation on generalized spaces and fields. He expressed interest and requested copies of our forthcoming papers in the field. I explained that this would take some time because the resolution of the physical shortcoming requires a new mathematics, called *genomathematics*, with new numbers, new Hilbert spaces, new geometries, etc., which had to be studied in mathematical journals prior to any possible physical application.

Memoir[12] on the new genomathematics was published only recently and I regret to have been unable to send a copy to Larry Biedenharn because of his, for me, unexpected departure.

The shortcomings of Lie-admissible theories or q-deformations, which were immediately understood and accepted by Larry Biebernarn, are the following. As a necessary condition to exit the class of equivalence of quantum mechanics, Lie-admissible theories, q-deformations, k-deformations, quantum groups, and all that must have a nonunitary time evolution, $U \times U^{\dagger} \neq I$. When these theories are formulated on conventional spaces over conventional fields, the following physical shortcomings are simply unavoidable:

(1) *Lack of invariance of the fundamental unit* (that of the enveloping operator algebra), because under nonunitary transforms we have $I \to I' = U \times I \times U^{\dagger} = U \times U^{\dagger} \neq I$. This implies lack of invariance of the basic units of space and time, with consequential lack of unambiguous applications of the theories to experiments, because it is not possible to conduct a meaningful measurement, say, of a length, with a stationary meter changing in time.

(2) *Lack of conservation of the Hermiticity in time*, with consequential lack of physically acceptable observables (see Sec. 3 for more details).

(3) *Lack of invariance of physical laws*, e.g., because of the lack of invariance of the deformed brackets under the time time evolution of the theory.

(3) *Lack of uniqueness and invariance of numerical predictions*, because of the lack of uniqueness (e.g., in the exponentiation) and invariance (e.g., of special functions and transforms) needed for data elaboration (for instance, the "q-parameter" becomes a "Q-operator" under a nonunitary transform, $Q = q \times (U \times U^{\dagger})^{-1}$, with consequential evident loss of all original special functions and transforms constructed for the q-parameter).

(4) Evident problematic aspects with causality and probability laws.

(5) *Loss of the axioms of the special relativity*, an occurrence of all generalizations under consideration, evidently because deformed spaces and symmetries are no longer isomorphic to the original ones. This creates the sizable problems of: first, identifying new axioms capable of replacing Einstein's axioms; second, proving their axiomatic consistency; and, third, establishing them experimentally.

In this note I shall present, apparently for the first time, a conceivable resolution of the above physical shortcomings. To render the note self-sufficient, I shall first present in Sec. 2 the rudiments of the genomathematics and then indicate in Sec. 3 the invariant formulations.

The reader should keep in mind that the most serious shortcoming of the generalized theories under consideration is the loss of Einsteins axioms. Our primary objective is therefore to attempt the formulation of generalized theories in such a way as to preserve the axioms of the special relativity, although in generalized spaces and fields. If achieved, this result will be sufficient, alone, to resolve all possible physical shortcomings.

2. ELEMENTS OF GENOMATHEMATICS

The main idea of the Lie-admissible theory[6] is that its structure (5) is inherent in the *conventional* Lie theory. In fact, a one-parameter connected Lie group realized via Hermitian operators $X = X^{\dagger}$ on a Hilbert space $\mathscr{H}$ has in reality the structure of *a bimodule* (also called in nonassociative algebras *spit-null extension*; see, e.g., Ref. 13).

In nontechnical terms, the structure of a Lie group as a bimodule is essentially characterized by an action from the left $U^{>}$ and an action from the right ${}^{<}U$ with explicit realization and interconnecting conjugation

$$\begin{aligned} A(w) &= U^{>} \times Q(0) \times {}^{<}U = \{e^{i \times X^{>} \times w}\} > A(0) < \{e^{-i \times w \times {}^{<}X}\} \\ &= (I^{>} + i \times X^{>} \times w + \cdots) > A(0) < ({}^{<}I - i \times w \times {}^{<}X + \cdots) \qquad (7) \\ U^{>} &= ({}^{<}U)^{\dagger} = U, \qquad X^{>} = ({}^{<}X)^{\dagger} = X, \qquad \hat{I}^{>} = {}^{<}I = I \end{aligned}$$

(where w is a Lie parameter and the multiplications $>$ and $<$ represent conventional associative products ordered to the right and to the left, respectively). The infinitesimal version in the neighborhood of the unit then acquires the familiar form

$$i[A(dw) - A(0)]/dw = A < X - X > A = A \times X - X \times A \qquad (8)$$

which clarifies that in the product $AxX = A < X(X \times A = X > A)$, X in actuality acts from the right (from the left).

The bimodular structure is generally ignored in the conventional formulation of Lie's theory because it is unnecessary. In fact, in a Lie bimodule $\{{}^{<}\mathscr{H}, \mathscr{H}^{>}\}$, where ${}^{<}\mathscr{H} = \mathscr{H}^{>} = \mathscr{H}$ is a conventional Hilbert space, the modular action to the right and to the left are interconnected with the simple bimodular rules[14] $X^{>} > \psi^{>} = X \times \psi = -{}^{<}\psi < {}^{<}X = -\psi \times X$, where $\psi^{>} \in \mathscr{H}^{>}$, ${}^{<}\psi \in {}^{<}\mathscr{H}$, $X^{>}$ is an element of the universal enveloping associative algebra $\xi^{>}(L)$ of the considered Lie algebra $L \approx [\xi^{>}(L)]^{-}$ for the action to the right, and ${}^{<}X \in {}^{<}\xi(L)$.[14] Since $\mathscr{H}^{>} = {}^{<}\mathscr{H} = \mathscr{H}$, $\xi^{>}(L) = {}^{<}\xi(L) = \xi(L)$. The *birepresentations* of the bimodular structure $\{{}^{<}\xi(L), \xi^{>}(L)\}$ over $\{{}^{<}\mathscr{H}, \mathscr{H}^{>}\}$ can then be effectively reduced to the *one-sided representations*, or just *representations* for short, of $\xi(L)$ over $\mathscr{H}$, as well known. However, as we shall see shortly, the original bimodular structure of Lie's theory is no longer trivial for broader realizations of axioms (7).

Lie-admissible structure (5) was proposed[5a] on the basis of the mere observation that the abstract axioms of the bimodular structure (7) do not necessarily require that the multiplications $>$ and $<$ must be conventional, because they can also be generalized, provided that they remain *associative*. In other words, the abstract axiomatic structure of the action from the right, $U^{>} > A(0)$, is that of a *right modular associative action*, with no restriction on the realization of the associative product, and the same occurs for the action from the left $A(0) < {}^{<}U$.

The simplest possible broadening of the Lie version is given by the *isotopies of Lie's theory*, first proposed in Refs. 5, then studied in various works (see Ref. 6 for a comprehensive presentation as of 1983), and it is called the *Lie–Santilli isotheory* (see, e.g., Refs. 15–18). It is essentially characterized by the lifting of the conventional right modular associative product $U^{>} > A(0) = U^{>} \times T \times A(a)$ with conjugate from the left $A(0) < {}^{<}U$, where $T = T^{\dagger}$ is a fixed, well-behaved, nowhere singular and Hermitian matrix or operator of the same dimension of the considered representation. Its inverse $\hat{I} = T^{-1}$ is then a fully acceptable, generalized, left and right unit, $I \times A = A > \hat{I} = \hat{I} < A = A < \hat{I} = \hat{I}$ for all possible elements A.

The isotopies then require, for mathematical and physical consistency, the reconstruction of the *entire* Lie theory with respect to the new unit $\hat{I}$ and isoproduct $> \, = \, < \, = \hat{\times}$, including: numbers and fields; vector, metric, and Hilbert spaces; Lie algebras, groups, and symmetries; transformation and representation theories; etc.[15–18] This intermediate level of study also possesses a trivial bimodular structure, in the sense that its two-sided representations can be effectively reduced to the one-sided form.

Following the prior achievement of sufficient mathematical maturity in Ref. 12, the physical profiles of the isotopic realization of axioms (7) have been studied in details in the recent memoir,[19] including most importantly the resolution of problematic aspects (1)–(5) of the preceding section. A knowledge of Ref. 19 is useful for a technical understanding of this note.

Our objective is the realization of the abstract axioms of bimodular structure (7) via the generalized associative laws originally submitted in Ref. 5b of 1978, under the name of *genoassociative multiplication and unit* (*or genomultiplication and genounit* for short), then studies in Refs. 6, 20, and more recently studied in detail in Ref. 12,

$$\begin{gathered} A > B = A \times P \times B, \qquad A < B = A \times Q \times B, \\ \hat{I}^{>} = P^{-1}, \qquad I^{>} > A = A > \hat{I}^{>} = A, \qquad {}^{<}I = = Q^{-1}, \\ {}^{<}I < A = A < {}^{<}I = A, \qquad \hat{I}^{>} = P^{-1} = ({}^{<}\hat{I})^{\dagger} = Q^{\dagger} \end{gathered} \tag{9}$$

where $P \neq Q$ are well-behaved, everywhere invertible, nonhermitian matrices or operators generally realized via real-valued nonsymmetric matrices of the same dimension of the considered Lie representation.

Moreover, it is suggested that $P + Q$ and $P - Q$ be nonsingular to preserve a well-defined Lie and Jordan content, respectively. To differentiate forms (9) from the isotopic ones, I called them *genotopic* in Ref. 5, to denote their character of inducing a more general realization. $I^{>}$ and ${}^{<}I$ are then called *genotopic units* and P and Q the *genotopic elements*.

Broader products and units (9) characterize the following more general realization of the abstract axioms (7) I tentatively called *Lie-admissible transformation group*[5, 6, 12, 20]

$$\begin{aligned} A(w) &= U^{>} > A(0) < {}^{<}U = \{e_{>}^{i \times X \times w}\} > A(0) < \{{}_{<}e^{-i \times w \times X}\} \\ &= \{e^{i \times X \times P \times w}\} \times A(0) \times \{e^{-i \times w \times Q \times X}\} \\ &= (I + i \times X \times P \times w + \cdots) \times A(0) \times ({}^{<}I - i \times w \times {}^{<}Q \times {}^{<}X + \cdots) \\ &= (I^{>} + i \times X \times w + \cdots) > A(0) < ({}^{<}I - i \times w \times {}^{<}X + \cdots) \\ U^{>} &= ({}^{<}U)^{\dagger}, \qquad X^{>} = ({}^{<}X)^{\dagger} = {}^{<}X = X, \qquad P^{>} = P = ({}^{<}Q)^{\dagger} = Q^{\dagger} \\ & \qquad \hat{I}^{>} = P^{-1} = ({}^{<}I)^{\dagger} = (Q^{\dagger})^{-1} \end{aligned} \tag{10}$$

with infinitesimal version in the neighborhood of the genounits characterized by the *general Lie-admissible algebra* (loc. cit.)

$$i \times [A(dw) - A(0)]/dw = A < X - X > A = A \times P \times X - X \times Q \times A \tag{11}$$

where we have used the *genoexponentiation* to the *right* and *to the left*[12, 19]

$$\begin{aligned} e_{>}^{i \times X \times w} &= I> + i \times X \times w/1! + (i \times X \times w) > (i \times X \times w)/2! + \cdots \\ &= \{e^{i \times X \times P \times w}\} \times I^{>} \\ {}_{<}e^{i \times X \times w} &= {}^{<}I + i \times X \times w/1! + (i \times X \times w) < (i \times X \times w)/2! + \cdots \\ &= {}^{<}I \times \{e^{i \times w \times Q \times X}\} \end{aligned} \tag{12}$$

It is at this point where the essential bimodular character of axioms (7) acquire their full light because they are no longer effectively reducible to a one-sided form. It is evident that realization (10) and (11) of the conventional Lie axioms (7) coincides with the Lie-admissible equations (4) and (5). For this reason, realizations (9)–(12) are assumed as the foundation of the Lie-admissible theory under study in this section.

The central assumption we are studying herein is the bimodular lifting of the unit of Lie's theory $I \rightarrow \{{}^{<}\hat{I}, \hat{I}^{>}\}$, ${}^{<}\hat{I} = (\hat{I}^{>})^{\dagger}$. To achieve consistency, the *entirety* of the Lie theory must be lifted into a dual genotopic form, with no known exception. A rudimentary review of the emerging *genotopic mathematics* or *genomathematics* for short of Ref. 12 plus unpublished aspects is the following.

Definition 1[21,12]. Let $F = F(a, +, \times)$ be a conventional field of (real R, complex C, or quaternionic Q) numbers a with additive unit 0, multiplicative unit $I = 1$, sum $a + b$, and product $a \times b$. The *genofields* to the right $F^> = F^>(a^>, +^>, \times^>)$ are rings with elements $a^> = a \times I^>$ called *genonumbers*, where a is an element of F, $\times$ is the multiplication in F, and $I^> = P^{-1}$ is a well-behaved, everywhere invertible and non-Hermitian quantity generally outside F, equipped with all operations *ordered to the right, i.e., the ordered genosum to the right, ordered genoproduct to the right*, etc.,

$$(a^>) +^> (b^>) = (a + b) \times I^>,$$
$$(a^>) > (b^>) = (a^>) \times P \times (b^>) = (a \times b) \times I^> \qquad (13)$$

genoadditive unit to the right $0^> = 0$ and *multiplicative genounit to the right* $I^>$. The *genofields to the left* ${}^<F = {}^<F({}^<a, {}^<+, {}^<\times)$ are rings with genonumbers ${}^<a = {}^<I \times a$, all operations ordered to the left, such as genosum $({}^<a) {}^<+ ({}^<b) = {}^<I \times (a + b)$, genoproduct $({}^<a) < ({}^<b) = ({}^<a) \times Q \times ({}^<b) = {}^<I \times (a \times b)$, etc., with *additive genounit to the left* ${}^<0 = 0$ and *multiplicative genounit to the left* ${}^<I = Q^{-1}$ which is generally different from the genounit $I^>$ to the right. A *bigenofield* is the structure $\{{}^<F, F^>\}$ with corresponding bielements, biunits, bioperations, etc. holding jointly to the left and right under the condition $\hat{I}^> = ({}^<I)^\dagger$.

Lemma 1[21]. Each individual genofield to the right $F^>$ or to the left ${}^<F$ is a field isomorphic to the original field F. Thus, the liftings $F \to F^>$, $F \to {}^<F$, and $\{F, F\} \to \{{}^<F, F^>\}$ are axiom-preserving.

Remarks. In the definition of fields (and isofields[21]) there is no ordering of the multiplication in the sense that in the products $a \times b$ and $a \hat{\times} b = a \times T \times b$, $T = T^\dagger$, one can either select a multiplying b from the left, $a > b$, or b multiplying a from the right, $a < b$, because $a > b = a < b$ (even for noncommutative isofields such as the isoquaternions). A genofield requires that all multiplications and related operations (division, moduli, etc.) be ordered *either* to the right *or* to the left because now, for a commutative field $F = R$ or C, we have the properties $a > b = b > a$ and $a < b = b < a$, but in general $a > b = a \times P \times b \neq a < b = a \times Q \times b$. Note that in each case the *genounit* is the *left and* right unit, Eqs. (9). The important advances of Ref. 21 are therefore the identification, first, that the axioms of a field remain valid when the multiplication is ordered to the right or to the left, and, second, each ordered multiplication can be generalized, provided that it remains associative. The above mathematical occurrences permit the axiomatization of irreversibility beginning with the most fundamental quantities, units and numbers. In fact, the unit and product to the right, $I^>$ and $>$, characterize *motion forward in time* while the conjugate quantities ${}^<I$ and $<$ characterize *motion backward in time*. Irreversibility is then ensured under the condition $I^> \neq {}^<I$ because all subsequent mathematical structures, being always built on numbers, must preserve the same axiomatization of irreversibility, as a necessary condition for consistency.

Definition 2[12]. Let $S = S(r, g, R)$ be a conventional n-dimensional metric or pseudo-metric space with local chart $r = \{r^k\}$, $k = 1, 2, \ldots, n$, nowhere singular, real-valued and symmetric metric $g = g(r, \ldots)$ and invariant $r^2 = r^t \times g \times r$ (where t denotes transpose) over a conventional real field $R = R(a, +, \times)$. The n-dimensional *genospaces to the right* $S^> = S^>(r^>, G^>, R^>)$ are vector spaces with local *genocoordinates to the right* $r^> = r \times I^>$, *genometric* $G^> = P \times g \times I^> = (g^>) \times I^>$, $g^> = P \times g$, and *genoinvariant to the right*

$$(r^>)^{2>} = (r^>)^t > (G^>) > r^> = [r^t \times (g^>) \times r] \times I^> \in R^> \qquad (14)$$

which, for consistency, must be a genoscalar to the right with structure $n \times I^>$ and be an element of the genofield $R^>$ with common genounit to the right $I^> = P^{-1}$, where P is given by an everywhere invertible, real-valued, nonsymmetric $n \times n$ matrix. The n-dimensional *genospaces to the left* ${}^<S = {}^<S({}^<r, {}^<+, {}^<F)$ are genospaces over genofields with all operations ordered to the left and a common $n \times n$-dimensional genounit to the left ${}^<I = Q^{-1}$ which is generally different from that to the right but verifying the interconnecting condition $P = Q^\dagger$. The *bigenospaces* are the structures $\{{}^<S, S^>\}$ with bigenocoordinates, etc., defined over the bigenofield $\{{}^<R, R^>\}$ under the condition $I^> = ({}^<I)^\dagger$.

Lemma 2[12]. Genospaces to the right $S^>$ and, independently, those to the left ${}^<S$ (thus bigenospaces $\{{}^<S, S^>\}$) are locally isomorphic to the original spaces $S(\{S, S\})$.

Proof. The original metric g is lifted in the form $g \to P \times g$, but the unit is lifted by the *inverse* amount $I \to I^> = P^{-1}$ thus preserving the original axioms [because the invariant is $(\text{length})^2 \times (\text{unit})^2$], and the same occurs for the other cases.

Remarks. The best way to see the local isomorphism between conventional and genospaces is by nothing that the latter are the results of the following novel degree of freedom of the former (here expressed for the case of a scalar complex function P)

$$
\begin{aligned}
r^t \times g \times r \times I &\equiv r^t \times g \times r \times Q \times Q^{-1} \equiv (r^t \times g^{>} \times r) \times I^{>} \\
&\equiv P^{-1} \times P \times (r^t \times g \times r \times I) \equiv {}^{<}I \times (r \times {}^{<}g \times r^t)
\end{aligned} \tag{15}
$$

which is another illustration of the structure of the basic invariant of metric spaces $(\text{length})^2 \times (\text{unit})^2$.

Definition 3[12]. The *genodifferential calculus to the right* on a genospace $S^{>}(r^{>}, R^{>})$ over $R^{>}$ is the image of the conventional differential calculus characterized by the expressions (where we have ignored for notational simplicity the multiplication to the right by $I^{>}$)

$$
\begin{aligned}
&dr^k \to d^{>} r^k = (I^{>})^k_i \times dr^i, \; dr_k \to d^{>} r_k = P^i_k \times dr_i \\
&\partial/\partial r^k \to \partial^{>}/\partial^{>} r^k = P^i_k \times \partial/\partial r^i, \; \partial/\partial r_k \to \partial^{>}/\partial^{>} r_k = I^{>k}_i \times \partial/\partial r_i
\end{aligned} \tag{16}
$$

with all operations ordered to the right and main properties

$$
\partial^{>} r^i/\partial^{>} r^j = \delta^i_j, \qquad \partial^{>} r_i/\partial^{>} r_j = \delta^j_i, \text{ etc.} \tag{17}
$$

The *genodifferential calculus to the left* is the conjugate of the preceding one for the genounit to the left ${}^{<}I \neq I^{>}$. The *bigenodifferential calculus* is that acting on $\{{}^{<}S, S^{>}\}$ over $\{{}^{<}R, R^{>}\}$ for $I^{>} = ({}^{<}I)^{\dagger}$.

Lemma 3[12]. The genocalculus to the right and, independently, that to the left on genospaces over genofields, preserve all original properties, such as commutativity of the second-order derivative, etc.

Remarks. A important advance of Ref. 12 is the identification of an insidious lack of invariance where one would expect it the least, in the conventional differential calculus, because it is traditionally formulated without indicating its dependence on the selected unit. As a result, all *generalized* equations of motion expressed in terms of *conventional* derivatives, e.g., *dA/dt, are not* invariant.

Definition 4[12]. The *genogeometries to the right*, or *to the left* or the *bigenogeometries* are the geometries of the corresponding genospaces when entirely expressed via the applicable genomathematics, including the genodifferential calculus.

Lemma 4 (loc. cit.). The *genoeuclidean*, *genominkowskian*, *genoriemannian*, and *genosymplectic geometries to the right* and, independently, to the left and their combined bimodular form, are locally isomorphic to the original geometries (i.e., they verify their abstract axioms).

Remarks. Another intriguing property identified in memoir[12] is that the *Riemannian axioms do not necessarily need symmetric metrics* because the metrics can also be *nonsymmetric* with structure $g^{>} = P \times g$, $P \neq P^t$ real-valued but nonsymmetric, provided that the geometry is formulated on a genofield with genounit given by the *inverse* of the nonsymmetric part, $I^{>} = P^{-1}$, and the same occurs for the case to the left. This property has permitted the first quantitative studies on the *irreversibility* of interior gravitational problems via the conventional *Riemannian axioms*,[20] e.g., the geometrization of the irreversible black hole model by Ellis, Nonopoulos, and Mavromatos,[22] which has precisely a Lie-admissible structure, and other models. These remarks are important to begin to see the physical relevance of quantum deformations when written in an axiomatically correct form.

Definition 5[12]. Let $\mathscr{H}$ be a conventional Hilbert space with states $|\psi\rangle, |\varphi\rangle, \ldots$, inner product $\langle\varphi| \times |\psi\rangle$ over the field $C = C(c, +, \times)$ of complex numbers and normalization $\langle\psi| \times |\psi\rangle = 1$. A *genohilbert space to the right* $\mathscr{H}^{>}$ is a right genolinear space with genostates $|\psi^{>}\rangle$, $|\varphi^{>}\rangle, \ldots$, *genoinner product and genonormalization to the right*

$$
\begin{aligned}
\langle\varphi^{>}|{>}|\psi^{>}\rangle &= \langle\varphi^{>}| \times P \times |\psi^{>}\rangle \times I^{>} \in C^{>}(c^{>} +^{>}, \times^{>}), \\
\langle\varphi^{>}|{>}|\psi^{>}\rangle &= I^{>}
\end{aligned} \tag{18}
$$

defined over a genocomplex field to the right $C^{>}(c^{>}, +^{>}, \times^{>})$ with a common genounit $I^{>} = P^{-1}$. A *genohilbert space to the left* ${}^{<}\mathscr{H}$ is the left conjugate of $\mathscr{H}^{>}$ with left genounit ${}^{<}L = Q^{-1}$ generally different from $I^{>}$. A *bigenohilbert space* is the bistructure $\{{}^{<}\mathscr{H}, \mathscr{H}^{>}\}$ over the bigenofield $\{{}^{<}C, C^{>}\}$ under the conjugation $I^{>} = ({}^{<}I)^{\dagger}$.

Lemma 5. The right- and left-spaces are locally isomorphic to the original space $\mathscr{H}$.

Proof. The original inner product is lifted by the amount $\langle | \times | \rangle \to \langle | \times P \times | \rangle$, but the underlying unit is lifted by the *inverse* amount, $1 \to P^{-1}$, thus leaving the original axiomatic structure unchanged.

Remark. The understanding of genooperator theory requires the knowledge that it is a consequence of the following, hitherto unknown degree of freedom of conventional Hilbert spaces (where P is independent from the integration variable for simplicity):

$$< \varphi| \times |\psi> \equiv <\psi| \times |\psi> \times P \times P^{-1}$$
$$\equiv <\varphi| \times P \times |\psi> \times P^{-1} = <\varphi| < |\psi> \times {}^{<}I$$
$$\equiv <\varphi| \times |\psi> \times Q \times Q^{-1} \equiv <\varphi| > |\psi> \times I^{>} \qquad (19)$$

which is evidently the Hilbert space counterpart of the novel invariance (15). It should be noted that new invariances (15) and (19) have remained undetected since Riemannian's and Hilbert's times, respectively, because they required the prior discovery of *new numbers*, those with an arbitrary, generally nonhermitian unit.

Definition 6. *Genolinear operators to the right* are operators $A, B,...,$ of a genoenveloping algebra to the right verifying the condition of genolinearity (i.e., linearity on $\mathscr{H}^{>}$ over $C^{>}$), and a similar occurrence holds for the left case. In particular, we have the *genounitary operators to the right and to the left*

$$U^{>} > U^{>\dagger} = U^{>\dagger} > U = I^{>}, \qquad {}^{<}U < {}^{<}U^{\dagger} = {}^{<}U^{\dagger} < {}^{<}U = {}^{<}I \qquad (20)$$

When applied on the bistructure $\{{}^{<}\mathscr{H}, \mathscr{H}^{>}\}$ over $\{{}^{<}C, C^{>}\}$, the theory is *bigenolinear.*

Lemma 6. Operators X which are originally Hermitian on $\mathscr{H}$ over C remains Hermitian on $\mathscr{H}^{>}$ over $C^{>}$, or on ${}^{<}\mathscr{H}$ over ${}^{<}C$ (i.e., genotopies preserve the original observables).

Proof. The condition of genohermiticity on $\mathscr{H}^{>}$ reads $X^{\dagger>} = Q \times Q^{-1} \times X^{\dagger} \times Q \times Q^{-1} = X^{\dagger}$.

Lemma 7. Under sufficient topological conditions, any conventionally nonunitary operator on $\mathscr{H}$ can be identically written in a genounitary form to the right or to the left.

Proof. Any operator U of the considered class such that $U \times U^{\dagger} \neq I$ can always be written

$$U = (U^{>}) \times Q^{1/2} \quad \text{or} \quad P^{1/2} \times ({}^{<}U) \qquad (21)$$

and properties (20) follow.

Remarks. The reader should be aware that the entire theory of linear operators on a Hilbert spaces must be lifted into a genotopic form for consistency. For instance, conventional operations, such as Tr X, Det X, etc. can be easily proved to be inapplicable for genomathematics, and must be replaced with the corresponding genoforms. The same happens for *all* conventional and special functions and transforms. A systematic study of the theory of genolinear operators will be conducted elsewhere.

We are now equipped to present, apparently for the first time, the central notion of this note which consists of the old notion of Lie-admissibility(5) upgraded with the systematic use of genomathematics.

Definition 7. Consider the conventional Lie theory with ordered N-dimensional basis of Hermitian operators $X = \{X_k\}$, parameters $w = \{w_k\}$, universal enveloping associative algebra $\xi = \xi(L)$, Lie algebra $L \approx [\xi(L)]^{-}$, and corresponding (connected) Lie transformation group G on a space $S(r, F)$ with local coordinates $r = \{r^k\}$ over a field F.

The *Lie-admissible theory* (also called *Lie–Santilli genotheory*(15–18)) is here defined as a step-by-step bimodular lifting of the conventional Lie theory defined on bigenospaces over bigenofields, and includes:

(7.A) The *universal genoenveloping associative algebra to the right* $\xi^{>}(L)$ of an N-dimensional Lie algebra L with ordered basis $X^{>} \equiv X = \{X_k\}$, $k = 1, 2,..., N$, genounit $I^{>} = Q^{-1}$, genoassociative product $X_i > X_j = X_i \times Q \times X_j$ and infinite-dimensional genobasis characterized by the *genotopic Poincaré–Birkhoff–Witt theorem to the right*

$$I^{>} = Q^{-1}, X_k, X_i > X_j (i \leqslant j), X_i > X_j > X_k (i \leqslant j \leqslant k), \ldots \qquad (22)$$

and genoexponentiation (12); the *universal genoassociative algebra to the left* ${}^{<}\xi(L)$ with genounit ${}^{<}I = P^{-1}$ and genoproduct $X_i < X_j = X_i \times P \times X_j$, with infinite-dimensional genobasis characterized by the *genotopic Poincaré–Birkhoff–Witt theorem to the left*

$${}^{<}I = P^{-1}, X_k, X_i < X_j (i \leqslant j), X_i < X_j < X_k (i \leqslant j \leqslant k), \ldots \qquad (23)$$

and genoexponentiation to the left (12); the *bigenoenvelope* is the bistructure $\{{}^{<}\xi, \xi^{>}\}$ defined on corresponding bigenospaces and bigenofields under the condition $I^{>} = ({}^{<}I)^{\dagger}$.

(7.B) A *Lie–Santilli genoalgebra* is a bigenolinear bigenoalgebra defined on $\{{}^{<}\xi, \xi^{>}\}$ over $\{{}^{<}F, F^{>}\}$ with Lie-admissible product

$$(X_i, X_j) = X_i < X_j - X_j > X_i = X_i \times P \times X_j - X_j \times Q \times X_i \qquad (24)$$

(7.C) A (connected) *Lie–Santilli genotransformation group* is the biset $\{{}^<G, G^>\}$ of bigenotransforms on $\{{}^<S, S^>\}$ over $\{{}^<F, F^>\}$ with genounits ${}^<I = (I^>)^\dagger$

$$r^{>\prime} = (U^>) > r^> = (U^>) \times Q \times r \times I^> = V \times r \times I^>, \quad U^> = V \times I^>,$$
$${}^<r' = {}^<r < ({}^<U) = {}^<I \times r \times P \times ({}^<U) = {}^<I \times r \times W, {}^<U = {}^<I \times W \tag{25}$$

verifying the following conditions: genodifferentiability of the maps $G^> > S^> \to S^>$ and ${}^<S \leftarrow {}^<S < {}^<G$, invariance of the genounits and genolinearity, with realizations $U^> = \exp_>(i \times w \times X)$ and ${}^<U = \exp_<(-i \times w \times X)$, genolaws

$$U^>(w^>) > U^>(w^{>\prime}) = U^>(w^> + w^{>\prime}),$$
$$U^>(w^>) > U^>(-w^>) = U^>(0^>) = I^> \tag{26}$$

and Lie-admissible algebra in the neighborhood of the genounits $\{{}^<I, I^>\}$ according to rule (10).

Lemma 8. Lie-admissible product (24) verifies the *Lie* axioms when defined on $\{{}^<\xi, \xi^>\}$ over $\{{}^<F, F^>\}$.

Proof. The genoenvelopes to the left ${}^<\xi$ and to the right $\xi^>$ are isomorphic to the original envelope ξ, thus implying ${}_{<I}(A < B) = (A > B)_{I^>}$, i.e., the value of the genoproduct $A < B = A \times P \times B$, when measured with respect to the genounit ${}^<I = P^{-1}$, is equal to that of the genoproduct $A > B = A \times Q \times B$ measured with respect to the genounit $I^> = Q^{-1}$.

The most important property of this section, which is an evident consequence of the preceding analysis, can be expressed as follows:

Theorem 1. Lie-admissible groups as per Definition 7 coincide at the abstract level with the original Lie-transformation groups.

Remarks. Note that the generators of the original Lie algebra are not lifted under genotopies, evidently because they represent conventional physical quantities, such as energy, linear momentum, angular momentum, etc. Only the *operations* defined on them are lifted. Note also that, when the conjugation $P = Q^\dagger$ is violated, the Lie axioms are lost. Note also that the genotheory is highly nonlinear, because the elements P and Q in genotransforms (25) have an unrestricted functional dependence, thus including that in the local coordinates. Nevertheless, genomathematics reconstructs linearity in genospaces over genofields. The same happens for nonlocality, noncanonicity, nonunitarity, and irreversibility.[20] In fact, on genospaces over genofields, genotheories are fully linear, local, canonical unitary and reversible. Departures from these axiomatic properties occur only in their *projection* over conventional spaces and fields. These are evident fundamental conditions to lift nonlinear, nonlocal, noncanonical, nonunitary, and irreversible theories into a form compatible with the notoriously linear, local, canonical, unitary, and reversible axioms of the special relativity.

Needless to say, we have been able to present in this note only the rudiments of the needed genomathematics, with the understanding that its detailed study is rather vast indeed. Also, by no means should genomathematics be considered as the most general possible form admitted by the Lie axioms. Mathematics and physics are disciplines which will never admit "final theories." In fact, a still broader multivalued hyper-realization of Lie's theory has already been identified in Ref. 12 and cannot be treated here for brevity.

3. INVARIANT FORMULATION OF QUANTUM DEFORMATIONS

We are now equipped to submit the suggested invariant formulation of the (p, q)-[2] or q-deformations.[2,7,8] First, we have to identify the following insufficiencies:

(I) No invariant formulation is possible for (p, q)-parameters because, under the nonunitary time evolution of the theory, brackets (2) or (8) assume the general Lie-admissible form (4) (for which reason the latter was submitted in the first place,[5b,6]

$$\begin{aligned} U \times (A, B) \times U^\dagger &= p \times U \times A \times B \times U^\dagger - q \times U \times B \times A \times U^\dagger \\ &= A' \times P \times B' - B' \times Q \times A' \\ P &= p \times (U \times U^\dagger)^{-1}, \\ Q &= q \times (U \times u^\dagger)^{-1}, \\ A' &= U \times A \times U^\dagger, \\ B' &= U \times B \times U^\dagger \end{aligned} \tag{27}$$

(II) Despite such a generality, the formulation are still not physically acceptable because they generally violate the crucial conjugation