DIRECT UNIVERSALITY OF LIE–SANTILLI ISOSYMMETRIES IN GRAVITATION

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We review the nonlinear, nonlocal and noncanonical, axiom–preserving isotopies of Lie's theory introduced by the physicist R. M. Santilli back in 1978 while at the Department of Mathematics of Harvard University, developed by numerous mathematicians and theoreticians and today called Lie–Santilli theory. We present seemingly novel advances in the structure theory of isotopic algebras and groups. We then show that the emerging isotopic covering of the Poincaré symmetry can provide the unified invariance of all possible Riemannian metrics for the exterior problem in vacuum as well as of all their possible nonlocal–integral generalizations for the interior gravitational problem (universality) directly in the frame of the experimenter (direct universality). We also show that the explicit form of the symmetry transformations can be readily computed from the knowledge of the conventional Poincaré symmetry of the tangent space and the generalized metric. We finally outline a number of other physical applications and recent experimental verifications of Lie–Santilli isosymmetries.

1. Statement of the Problem

1.A. Symmetries of exterior gravitation. One of the most visible differences in the transition from the special relativity ([9], [45]) with Minkowskian [34] line element

$$ds^2 = - dx^\mu \, \eta_{\mu\nu} \, dx^\nu \quad \eta = \text{diag.} \, (1, 1, 1, -1), \; x = \{ x, c_0 t \}, \qquad (1.1)$$

to the general relativity [10] for the exterior gravitational problem in vacuum with Riemannian [46] line element

$$ds^2 = - dx^\mu \, g_{\mu\nu}(x) \, dx^\nu \, , \qquad (1.2)$$

is that the former is reducible to the primitive Poincaré symmetry P(3.1), while the latter is not believed to be reducible to a single symmetry. Stated differently, a fundamental unresolved problem of the Riemannian geometry is the identification of their "universal symmetry", i.e., a

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symmetry holding for all infinitely possible metrics g.

This problem was solved by the theoretical physicist Ruggero Maria Santilli [47] in 1978 while at the Department of Mathematics of Harvard University under support from the U. S. Department of Energy, via the following main steps (see also the recent monographs [62]):

(1) the factorization of any given Riemannian metric g in (3+1)–dimension into the Minkowski metric η

$$g(x) = T(x) \, \eta \, , \qquad (1.3)$$

where T(x) is a positive–definite matrix (from the local Minkowskian character of g);

(2) The step-by-step generalization of Lie's theory (i.e., a generalization of the theory of numbers, vector and metric spaces, enveloping algebras, Lie algebras and groups, transformation and representation theory, functional analysis, etc.) into a covering theory characterized by the generalized unit

$$\hat{1}(x) = [T(x)]^{-1} > 0 \, , \qquad (1.4)$$

(3) The construction of the infinite family of generalizations $\hat{P}_{\hat{1}}(3.1)$ of the Poincaré symmetry P(3.1) constructed with respect to arbitrary generalized units (1.4),

(4) The proof that $\hat{P}_{\hat{1}}(3.1)$ leaves invariant all possible Riemannian line elements (1.2) and

(5) The proof (from the positive–definiteness of $\hat{1}$) that all generalized symmetries are locally isomorphic to the conventional symmetry

$$\hat{P}_{\hat{1}}(3.1) \sim P(3.1) \, . \qquad (1.5)$$

The above results permitted Santilli to reach a rather remarkable unification of the Minkowskian and Riemannian geometries and related relativities via one single primitive notion, the Poincaré symmetry $\hat{P}_{\hat{1}}(3.1)$ constructed with respect to arbitrary, positive–definite units $\hat{1}(x)$ with the Minkowskian form occurring for the simplest possible realization $\hat{1} = I = \text{diag.} \, (1, 1, 1, 1)$.

1.B. Symmetries of interior gravitation. Perhaps more remarkably, the above results were reached only as a preparatory ground for studies of the most general possible interior gravitational problems, those with a generally nonlinear, nonlocal–integral and nonlagrangian dependence not only in the coordinates x but also in the velocities $\dot{x}$, accelerations $\ddot{x}$, density μ, index of refraction n, temperature τ, frequency ω, and any needed interior quantity.

In fact, the latter interior gravitational theories in (3+1)–dimension are characterized by the line elements of the type

$$ds^2 = - dx^\mu \, \hat{g}_{\mu\nu}(x, \dot{x}, \ddot{x}, \mu, \tau, n, \omega, ...) \, dx^\nu \, , \qquad (1.6)$$

which also admit the factorization into the Minkowskian line element and related generalized unit

$$\hat{g}(x, \dot{x}, \ddot{x}, ...) = T(x, \dot{x}, \ddot{x}, ...)\, \eta, \qquad 1 = [\, T(x, \dot{x}, \ddot{x}, ...)\,]^{-1} > 0 . \qquad (1.7)$$

The applicability of the generalized symmetry $\hat{P}_1(3.1)$ to interior problems (1.6) then follows from the completely unrestricted functional dependence of the generalized unit 1 (provided that it is positive-definite, see below).

This permitted Santilli to achieve a generalization of the Poincaré symmetry, today called (for certain technical reasons review later on) *Santilli's isopoincaré symmetry,* which holds for all possible local-differential metrics $g(x)$ nonlinear in the coordinates x only, as well as for all their possible generalizations $\hat{g}(x, \dot{x}, \ddot{x}, ...)$ which are nonlinear *and* nonlocal also in the velocities and any needed additional quantities.

Stated differently, Santilli achieved a geometric unification into the single abstract isopoincaré symmetry $\hat{P}_1(3.1)$ of all infinitely possible (3+1)-dimensional geometries whose tangent space is Minkowskian, thus including not only the Riemannian geometry, but also all its possible integral generalizations.

These results have implications so deep to suggest a structural revision of our current gravitational views. As an example, irrespective of whether external or internal, gravitational fields must be covariant under the isopoincaré symmetry [60]. This implies the need for the gravitational field to be well defined for all infinitely possible isounits, *including the trivial unit* I = diag. (1, 1, 1, 1). In turn, this implies that a consistent gravitational theory must be first defined in the *flat* Minkowski space, and then admit higher formulations in the Riemannian space and its isotopies. As shown in the adjoining paper [60], this implies the need for a gravitational source which can be null nowhere in space-time, as requested by the electromagnetic origin of mass, but contrary to the known gravitational fields in vacuum $G_{\mu\nu} = 0$. The implications of the isopoincaré covariance are then evident, as discussed in [60].

In this paper we study the methods underlying the generalized symmetry $\hat{P}_1(3.1)$. The geometric and gravitational profiles are presented in the separate paper by Santilli [60] in this collection. .A summary of the content of this paper was presented by this author [25] at the *International Conference on Symmetry Methods in Physics,* held at the JINR, Dubna, in July 1993.

1.C. Bibliographical notes. The studies herein considered are based on the historical distinction introduced by Lagrange [29], Hamilton [14] and other founders of contemporary analytic dynamics between

(I) the *exterior dynamical problem,* i.e., particles which can be effectively approximated as being point-like while moving within the homogeneous and isotropic vacuum, resulting in local-differential-Lagrangian equations of motion, and

(II) the more general *interior dynamical problem,* i.e., extended, and therefore deformable particles moving within inhomogeneous and anisotropic physical media, thus resulting in the most general known nonlinear-nonlocal-nonlagrangian equations of motion historically represented with external terms in the analytic equations ([14], [29]).

The above distinction was kept in the early studies in gravitation (see, e.g., Schwartzchild's

two papers, the first celebrated paper [72] on the exterior problem and the second little known paper [73] on the interior problem) and kept in the early well written treatises in field (see, e.g., [4], [38]), but the distinction was progressively abandoned up to the virtually complete silence in the contemporary literature.

The latter occurrence is generally based on the belief that interior systems can be reduced to a collection of exterior systems via the reduction of macroscopic objects to their elementary, point-like constituents. This reduction has been disproved on numerous counts. We here recall the following main aspects.

First, the so-called *No-Reduction Theorems* [54] prohibit any consistent realization of the reduction. In fact, an macroscopic interior systems, such as satellite during re-entry *in an unstable-irreversible trajectory with a monotonically decreasing angular momentum,* simply cannot be decomposed into a (finite) collection of elementary particles each one with a *conserved angular momentum in a stable-reversible orbit.*

Second, interior gravitational systems must be first represented in a physically consistent way at the purely *classical* level. Only after achieving such a consistent classical description, operator-particle descriptions *may* have a physical value. At any rate, we do not possess today an operator version of gravitation which is accepted by the scientific community at large.

Third, interior gravitational systems, such as collapsing stars, are not composed of abstract points, but instead of a large number of extended wavepackets, wavelengths and charge distributions not only in conditions of total mutual penetration, but also of compression in large number into small regions of space. The emergence for these interior conditions of a nonlocal-integral structure is then beyond credible doubts. In fact, the belief that the Riemannian geometry is *exactly* valid in these interior conditions implies exiting the boundaries of science, and raising rather serious problems of scientific ethics [76].

The understanding is that the Riemannian geometry is *exactly* valid for *exterior* problems, while its *approximate* validity for *interior* problems is undeniable (see, e.g., Schwartzchild's insistence in the *exact* character of the *exterior* solution [72], and in the *approximate* nature of his *interior* solution [73]).

At any rate, nonlinear-nonlocal-noncanonical systems have forcefully emerged in a variety of problems of contemporary science outside gravitation, such as: trajectory control in atmosphere; quasar redshifts and blueshifts; Cooper pair in superconductivity, etc. Santilli's studies on nonlinear-nonlocal-noncanonical systems are therefore applicable to all systems of the class considered besides gravitation. The inapplicability of the conventional formulation of Lie's theory for the problems considered is so transparent from the different topological structures to needs no further comments.

Santilli originally submitted his methods under name of *isotopy* [47] suggested from the Greek meaning of "preserving configuration" interpreted as *axiom-preserving.* In fact, under the conditions on $\hat{g}$ being locally Minkowskian (i.e., on 1 being positive-definite) all novel algebraic and geometric structures coincide with their original form at the abstract, realization-free level. The emerging generalization of Lie's theory was therefore submitted under the name of *Lie-isotopic theory* and is today called *Lie-Santilli theory* (see papers [1], [2], [8], [11], [12], [16]-[23], [25],

[32], [33], [35]–[37], [40]–[43] and monographs [3], [24], [31], [74]). As we shall see, it consists of a structural generalization of all branches of Lie's theory, including number theory, vector and metric spaces, Lie algebras and groups, transformation and representation theory, symmetries and conservation laws, etc. Santilli's still more general methods called *genotopies*, which lead to the so-called *Lie-admissible theories* for open nonconservative systems, will be ignored for brevity.

Another novelty of Santilli's isotopic methods is the identification of a new antiautomorphic conjugation submitted the name of *isoduality* ([52], [53]), characterized by the map $1 \Rightarrow 1^d = -1$, which yields new *isodual fields, isodual spaces, isodual algebras, isodual symmetries*, etc. In particular, physical quantities change sign under isoduality, as a result of which time flows backward, the energy is negative-definite, etc.

The latter studies permitted the discovery of a hitherto unknown *isodual universe* particularly suited for the characterization of antiparticles (recall the negative-energy solutions of relativistic field equations), and which characterizes a new *universal invariance law under isoduality* [62]. In turn, these advances have permitted truly remarkable novel insights, such as the conception of the universe as having a null total energy, time and other physical quantities, the representation of the gravitational horizons and singularities as the zeros of the isotopic element T and isounit 1, respectively, the reduction of the interior gravitational problem to a problem on the origin of the gravitational field, the identification of the gravitational field with the electromagnetic field at the origin of mass (plus second order corrections due to weak and strong interactions), and others for which we refer to Santilli's paper [60] in this collection.

The literature on isotopies, including the isotopic lifting of the various branches of Lie's theory, classical, quantum and statistical mechanics, has now passed the mark of ten thousand pages of published research, yet it does not appear to have been sufficiently propagated in the mathematical literature. It is therefore recommendable that in this paper we first review the essential elements of the isotopic and isodual theories, and then enter into the presentation of advances in the structure of isotopic algebras, groups and their inter-relations. The understanding is that, particularly when compared to the truly vast studies conducted on Lie's theory for about a century, studies on the isotopies of Lie's theory are still at the beginning and so much remains to be done.

2. Isotopies and Isodualities of Numbers, Fields, Metric Spaces, Transformation Theory and Functional Analysis

One of the most insidious aspects for a researcher first approaching isotopic methods is the use of conventional mathematical thinking, because it leads to a number of inconsistencies which remain generally undetected.

In fact, the transition from conventional to the isotopic formulation of Lie's theory requires a generalization of *all* conventional mathematical tools, which is somewhat reminiscent of the mathematical generalization needed in the transition from Newtonian to quantum mechanics. In this section we shall indicate the basic isotopies and isodualities which are necessary for an understanding of the isotopic theory.

2.A. Isotopies and isodualities of the unit. The fundamental isotopies from which all others can be uniquely derived are the liftings $I \Rightarrow 1$ of the unit of the current formulation of Lie's theory. These liftings were classified by this author [22] into:

Class I (generalized units that are smooth, bounded, nondegenerate, Hermitean and positive-definite, characterizing the *isotopies* properly speaking);

Class II (the same as Class I although 1 is negative-definite, characterizing *isodualities*);

Class III (the union of Class I and II);

Class IV (singular isounits); and

Class V (generalized units with discrete structure, discontinuous functions, etc., characterizing discrete-time theories, deformable discrete structures, etc.).

All isotopic structures identified below also admit the same classification which will be omitted for brevity. In this note we shall generally study isotopies of Class III with Classes I and II as particular cases. Classes IV and V are vastly unexplored at this writing.

2.B. Isotopies and isodualities of fields. Let $F(a,+,\times)$ represent ordinary fields of characteristic zero (the fields of real $\mathfrak{R}$, complex C and quaternionic numbers Q) with generic elements a, addition $a_1 + a_2$, multiplication $a_1 a_2 := a_1 \times a_2$, additive unit 0, $a + 0 = 0 + a = a$, and multiplicative unit I, $a I = I a = a \; \forall \, a$, $a_1, a_2 \in F$.

The lifting $I \Rightarrow 1$ requires, for necessary compatibility, a generalization of the conventional associative multiplication ab into the so-called *isomultiplication* [46]

$$ab := a \times b \Rightarrow a * b := a T b, \quad T = \text{fixed}. \tag{2.1}$$

where the quantity T is called the *isotopic element*. Whenever $1 = T^{-1}$, 1 is the correct left and right unit of the theory, $1*a = a*1 = a \; \forall \, a$, in which case (only) 1 is called the *isounit*. In turn, the liftings $I \to 1$ and $\times \to *$, imply a generalization of the very notion of numbers and of fields into the structure of Class I

$$\hat{F}_I = \{ (\hat{a},+,*) \,|\, \hat{a} = a1, \; a = n, c, q \in F, \; \times \Rightarrow * = \times T \times, 1 = T^{-1} = \text{isounit} \}, \tag{2.2}$$

called *isofields*, with elements $\hat{a} \in \hat{F}$ called *isonumbers* ([56], [59]).

All conventional operations are evidently generalized in the transition from numbers to isonumbers. In fact, we have:

$$a + b \Rightarrow \hat{a} + \hat{b} = (a + b)1; \qquad a_1 \times a_2 \Rightarrow \hat{a}_1 * \hat{a}_2 = \hat{a}_1 T \hat{a}_2 = (a_1 a_2)1;$$

$$a^{-1} \Rightarrow \hat{a}^{\hat{1}} = a^{-1}1; \quad a/b = c \Rightarrow \hat{a}\hat{7}\hat{b} = \hat{c}, \quad \hat{c} = c1; \quad a^{\frac{1}{2}} \Rightarrow \hat{a}^{\frac{1}{2}} = a^{\frac{1}{2}}1^{\frac{1}{2}},$$

etc. Thus, conventional squares $a^2 = aa$ have no meaning under isotopy and must be lifted into the *isosquare* $\hat{a}^2 = \hat{a}*\hat{a}$. The *isonorm* is

$$\lceil \hat{a} \rceil = \overline{\hat{a}} * \hat{a} = (\overline{a}\,a)\hat{1} = |a|\hat{1} \in \hat{F},$$

where $\overline{a}$ denote the conventional conjugation and $|a|$ the conventional norm, and it is positive-definite (for isofields of Class I).

The isotopic character of the lifting $1 \Rightarrow \hat{1}$ is then confirmed by the fact that the isounit $\hat{1}$ verifies all axioms of 1. In fact,

$$\hat{1} * \hat{1} * ... * \hat{1} \equiv \hat{1}, \qquad \hat{1}^2 \hat{1} \equiv \hat{1}, \qquad \hat{1}^3 \equiv \hat{1}, \quad \text{etc.}$$

The *isodual isofields* are characterized by the antihomomorphic map $\hat{1} \Rightarrow \hat{1}^d = -\hat{1}$ and are given by the Class II structures

$$\hat{F}^d_{II} = \{ (\hat{a}^d, +, *^d) | \hat{a}^d = a\hat{1}^d, \ a = n, c, q \in F, * \Rightarrow *^d = \times T^d \times, T^d = -T, \hat{1}^d = -\hat{1} \}, \qquad (2.3)$$

in which the elements, called *isodual isonumbers*, are given by $\hat{a} = -\hat{a}$, the sum is again unchanged, and the multiplication is

$$\hat{a}^d *^d \hat{a}^d = \hat{a}^d T^d \hat{a}^d = -\hat{a}^d T \hat{a}^d = -\hat{a} T \hat{a}.$$

An important property of the isodual isofields is that their norm is *negative-definite* because characterized by

$$\lceil \hat{a}^{d} \rceil^d = |a|\hat{1}^d = -\lceil \hat{a} \rceil.$$

The latter property has the nontrivial implications that *physical quantities defined on an isodual isofield, such as time, energy, etc., are negative-definite.*

One can then begin to see the inconsistencies in the use of conventional mathematical thinking under isotopies. For instance, statements such as "two multiplied by two equals four" are correct for the conventional Lie theory, but they generally have no mathematical sense for the covering isotopic theory, evidently because they lack the identification of the basic unit as well as of the multiplication.

The *theory of isonumbers* is today sufficiently well known [59], and includes the lifting of all conventional numbers (real, complex and quaternionic numbers) into the following four classes used in this paper: (A) *ordinary numbers* with unit 1; (B) *isonumbers* with isounits of Class I, $\hat{1} > 0$; (C) *isodual numbers* with isodual unit $1^d = -1$; (D) *isodual isonumbers* with isounits of Class II, $\hat{1}^d < 0$. In addition, the theory of isonumbers includes isonumbers and isodual isonumbers of Class IV (this is a basically new notion of number with singular isounits) and Class V (another new type of number with distributions or discontinuous functions as isounits) which are not studied in this paper for brevity. The reader should be aware that the distinction between real, complex and quaternionic numbers is lost under isotopies because all possible numbers are

unified by the isoreals $\hat{\mathfrak{R}}_{III}$ owing to the freedom in $\hat{a}$ and $\hat{1}$ [26].

Note that the lifting $a \Rightarrow \hat{a} = a\hat{1}$ is *necessary* for $\hat{F}$ to preserve the axioms of F whenever the isounit $\hat{1}$ is not an element of the original field F, as generally occurring in physical applications (for details, see Section 3 of [59]). This implies that the "numbers" used in the Lie-Santilli theory generally have an *integral* structure.

2.C. Isospaces and their isoduals. The liftings of the unit $I \Rightarrow \hat{1}$, of the product $\times \Rightarrow *$ and of the fields $F \Rightarrow \hat{F}$ demand, for evident mathematical consistency, the corresponding lifting of conventional, N-dimensional, metric or pseudometric spaces $S(x,g,\mathfrak{R})$ with (real) local coordinates x and (nowhere singular, Hermitian) metric g over the reals $\mathfrak{R}$, into the *isospaces* first introduced in [51]

$$S_I(x,g,\mathfrak{R}): x^2 = x^t g x \in \mathfrak{R} \Rightarrow \hat{S}(x,\hat{g},\hat{\mathfrak{R}}): \hat{g} = Tg, \ \hat{\mathfrak{R}} \sim \mathfrak{R}, \hat{1} = T^{-1}, x^2 = (x^t \hat{g} x)\hat{1} \in \hat{\mathfrak{R}}, \quad (2.4)$$

where $\hat{g} = Tg$ is called the *isometric*. The *isodual isospaces* are then given by $\hat{S}^d_{II}(x,\hat{g}^d,\hat{\mathfrak{R}}^d)$, where $\hat{g}^d = T^d\eta$, $T^d = -T$, $\hat{1}^d = (T^d)^{-1} = -\hat{1}$, and the subindices I or II will be dropped for simplicity hereon.

The most salient properties of the isotopies of metric or pseudo-metric spaces are [56]-[60]: (1) *preservation of the original dimensionality*; (2) *preservation of the original basis (except for renormalization factors)*; and (3) *the isospaces $\hat{S}(x,\hat{g},\hat{\mathfrak{R}})$ (isodual isospaces $\hat{S}^d(x,\hat{g}^d,\hat{\mathfrak{R}}^d)$) are locally isomorphic to the original spaces $S(x,g,\mathfrak{R})$ ($S^d(x,g^d,\mathfrak{R}^d)$)*. Note the necessity for the last result of the *joint* liftings $g \Rightarrow \hat{g} = T\eta$ *and* $\mathfrak{R} \Rightarrow \hat{\mathfrak{R}}, \hat{1} = T^{-1}$.

The above properties begin to illustrate Santilli's geometric unification of the special and general relativities [57] (see also [60]). In fact, all their geometric distinctions cease to exist when the conventional Riemannian spaces $R(x,g,\mathfrak{R})$ are reinterpreted as isominkowski spaces $\hat{M}(x,\hat{\eta},\hat{\mathfrak{R}})$ with chain of isotopic equivalences

$$R(x,g,\mathfrak{R}) \sim \hat{R}(x,g,\hat{\mathfrak{R}}) \sim \hat{M}(x,\hat{\eta},\hat{\mathfrak{R}}) \sim M(x,\eta,\mathfrak{R}), \ g = T(x)\eta \equiv \hat{\eta}, \ \hat{1} = [T(x)]^{-1}, \qquad (2.5)$$

where $\hat{1}$ is called the *gravitational isounit*. One can then anticipate that the isotopy $\hat{P}(3.1)$ of the Poincaré symmetry P(3.1) of $M(x,\eta,\mathfrak{R})$ will produce the symmetry of $R(x,g,\mathfrak{R})$, as we shall see. Intriguingly, the above unification was conceived by Santilli only as a basis for the isotopic generalizations of conventional relativities for nonlocal-nonhamiltonian interior dynamical problems, under the condition of admitting the latter as particular cases whenever motion returns to be in vacuum [62].

The most salient physical application of isospaces is the geometrization of physical media, that is, the geometrization of the departures from empty space caused by matter. A well known example is a plane electromagnetic wave propagating in the homogeneous and isotropic empty space, in which case the Minkowski space is exactly applicable, and then propagating within our atmosphere which, being inhomogeneous and anisotropic, implies the loss of exact applicability of the Minkowski space in favor of its isotopic covering.

Because of the assumed properties, the isounits of Class I or II can always be diagonalized, resulting in expressions of the type in four dimensions [51]

$$\hat{1} = \text{diag.} \, (\pm b_1^{-2}, \pm b_2^{-2}, \pm b_3^{-2}, b_4^2) > 0, \quad b_\mu = b_\mu(x, \dot{x}, \ddot{x}, \ldots) > 0, \quad \mu = 1, 2, 3, 4. \quad (2.6)$$

where the b's are called the *characteristic quantities of the medium*, they generally vary from medium to medium, and can be averaged into constants $b°_\mu$ when total properties are needed. It should be stressed, however, that the above diagonalization is not necessary, and numerous applications exist for nondiagonal isounits.

Note also that, because of the functional dependence of the isotopic element T (Section 1), isospaces are *bona-fide* nonlocal-integral generalizations of the original local-differential spaces. In fact, isospaces characterize new geometries called: *isoeuclidean geometry* for $\hat{S} = \hat{E}(r,\hat{\delta},\hat{\Re})$, $\hat{\delta} = T\delta$; *isominkowskian geometry* for $\hat{S} = \hat{M}(x,\hat{\eta},\hat{\Re})$, $\hat{\eta} = T\eta$; *isoriemannian geometry* for $\hat{S} = \hat{R}(x,\hat{g},\hat{\Re})$, $\hat{g} = Tg$; etc.; each of them admitting an *isodual geometry* for $\hat{S}^d$ $(x,\hat{g}^d,\hat{\Re}^d)$, $\hat{g}^d = -\hat{g}$, $\hat{\Re}^d \sim \hat{\Re}\,\hat{1}^d$, $\hat{1}^d = -\hat{1}$. A knowledge of the isogeometries from the companion paper [60] is hereon assumed.

2.D. Isotopies and isodualities of the transformation theory. Let $S(x,F)$ be a conventional vector space with local coordinates x over a field F, and let $x' = A(w)x$ be a linear, local and canonical transformation on $S(x,F)$, $w \in F$. The lifting $S(x,F) \Rightarrow \hat{S}(x,\hat{F})$ requires a corresponding necessary isotopy [47]

$$x' = \hat{O}(\hat{w}) * x = \hat{A}(\hat{w}) \, T \, x, \quad T \text{ fixed}, \quad x \in \hat{S}(x,\hat{F}), \quad \hat{\omega} \quad \hat{1} = T^{-1}, \quad (2.7)$$

called *isotransformations*, with isodual form $x' = \hat{A}(\hat{w}) *^d x = \hat{A}(\hat{w}) T^d x = -\hat{A}(\hat{w}) * x$.

Isotransformations verify the condition of linearity in isospaces, called *isolinearity*

$$A * (\hat{a} * x + \hat{b} * y) = \hat{a} * (A * x) + \hat{b} * (A * y), \quad \forall x, y \in \hat{S}(x,\hat{F}), \quad \hat{a}, \hat{b} \in \hat{F}, \quad (2.8)$$

although their projection in the original space $S(x,F)$ is nonlinear, because $x' = AT(x, \dot{x},...)x$. Isotransformations (2.7) are also *isolocal* because the theory formally deals with the local variables x while all nonlocal terms are embedded in the isounit. Nevertheless, they too are nonlocal when projected in the original space. Similarly, isotransformations (2.7) are *isocanonical* because they are formally derivable from a variational principle on the *isosymplectic geometry* [60], although they are noncanonical when projected in $\hat{S}(x,\hat{F})$. Note that nonlinear, nonlocal and noncanonical transforms can always be identically rewritten an in isotopic form.

2.E. Isotopies and isodualities of functional analysis. As indicated earlier, the isotopies imply nontrivial generalizations of *all* mathematical structures used in physics, beginning from elementary notions such as numbers and then passing to angles and leading inevitably to a generalization of functional analysis called by this author *functional isoanalysis* [22].

The generalized discipline begins with the isotopy of continuity (whose knowledge is assumed when dealing with technical aspects of Section 3), and includes the isotopies of conventional square-integrable, Banach and Hilbert spaces, the isotopies of all operations on them.

The functional isoanalysis includes a generalization of all conventional special functions, distributions and transforms. For instance, the isotopies of Dirac's δ-function permit a direct treatment of nonlocality in a fully causal way (because the isoexpectation value of the nonlocal isounit $\hat{1}$ is I); the isotopies of the Fourier transform imply a necessary generalization of Heisenberg uncertainties, evidently applicable only for particles in interior conditions (see Section 3.A); the isotopies of the Legendre polynomials imply a generalization of potential scattering theory (initiated by Mignani ([35], [36], [37]) and completed by Santilli [61]) with expected revision of the data elaboration of inelastic collisions with nonlinear-nonlocal-noncanonical internal effects; etc.

Note that functional isoanalysis is fundamentally different than the known q-analysis [75], evidently because the latter is based on the use of conventional fields, spaces, etc., while the former is based on covering structures. The reformulation of the latter in terms of the former is intriguing and deserving specialized studies.

3. Isotopies and Isodualities of Enveloping Agebras, Lie Algebras, Lie Groups, Symmetries, Representation Theory and Their Applications

As well known, Lie's theory (see, e.g., [13], [15] and [75]) is centrally dependent on the basic n-dimensional unit I = diag. (1, 1, ..., 1) in all its major branches, such as enveloping algebras, Lie algebras, Lie groups, representation theory, etc. The main idea of the Lie-Santilli theory [47], [49], [61], [62] is the reformulation of the entire conventional theory with respect to the most general possible, integro-differential isounit $\hat{1}(x, \dot{x}, \ddot{x}, \ldots)$.

One can therefore see from the very outset the richness and novelty of the isotopic theory. In fact, it can be classified into five main classes as occurring for isofields, isospaces, etc., and admits a variety of fundamentally novel concepts indicated in Section 1.

But the most intriguing results occur for the construction of the symmetry $\hat{G}$ of a generalized metric $\hat{g} = Tg$ via an isotopy of the original symmetry G of g. In this case: (A) the Lie-Santilli theory is *directly universal*, that is, capable of providing the nonlinear-nonlocal-noncanonical symmetries $\hat{G}$ for all infinitely possible deformations $\hat{g} = Tg$ ("universality"), directly in the given frame x of the observer ("direct universality"); (B) the isotopic symmetry $\hat{G}$ is connected to G via *nonunitary* transformations, the two symmetries thus resulting to be physically inequivalent in their local realizations; while (C) the two symmetries $\hat{G}$ and G coincide at the abstract level by construction (for T > 0).

The central motivation is that, in the transition from the Minkowskian line element to its most general possible nonlinear-nonlocal-noncanonical generalization, the Lorentz and Poincaré *transformations* are evidently inapplicable (and not "violated"), yet the geometric axioms of the Lorentz and Poincaré *symmetries* persist. We merely have their realization in the most general

possible nonlinear–nonlocal–noncanonical form. In turn, this sets the foundations for the generalization of conventional relativities for interior dynamical problems for which the isotopic methods were conceived in the first place.

3.A. Isotopies and isodualities of universal enveloping associative algebras. Let ξ be a universal enveloping associative algebra [15] over a field F (of characteristic zero) with generic elements A, B, C,... , trivial associative product AB and unit I. Their isotopes $\hat\xi$ were first introduced in [47] under the name of *isoassociative envelopes*. They coincide with ξ as vector spaces but are equipped with the isoproduct so as to admit $\hat 1$ as the correct (right and left) unit

$$\hat\xi : \quad A * B = ATB, \quad T \text{ fixed}, \quad \hat 1 * A = A * \hat 1 \equiv A \quad \forall A \; \hat\xi, \; \hat 1 = T^{-1}. \qquad (3.1)$$

Let $\xi = \xi(L)$ be the universal enveloping algebra of an N-dimensional Lie algebra L with ordered basis $\{X_k\}$, $k = 1, 2, ..., N$, $[\xi(L)]^- \sim L$ over F, and let the infinite-dimensional basis of $\xi(L)$ be given by the Poincaré–Birkhoff–Witt theorem [15]. A fundamental result achieved by Santilli in the original proposal [47] (see also [59, Vol. II, p. 154–163]) is the following

Theorem 3.1. *The cosets of $\hat 1$ and the standard, isotopically mapped monomials*

$$\hat 1, \quad X_k \qquad X_i * X_j \quad (i \leq j), \qquad X_i * X_j * X_k \quad (i \leq j \leq k), \; . . . \qquad (3.2)$$

form a basis of the universal enveloping isoassociative algebra $\hat\xi(L)$ of a Lie algebra L.

A first important consequence is that the isotopies of conventional exponentiation are given by the expression, called *isoexponentiation*, for $\hat w \in \hat F$,

$$e_{\hat\xi}^{i\hat w * X} = \hat 1 + (i\hat w * X)/1! + (i\hat w * X) * (i\hat w * X)/2! + ... = \hat 1 \{ e^{iwTX} \} = \{ e^{iXTw} \} \hat 1. \qquad (3.3)$$

As anticipated in Section 1, the nontriviality is illustrated by the emergence of the nonlinear–nonlocal isotopic element T directly in the exponent of the transformations, thus ensuring the desired generalization.

The implications of Theorem 3.1 also emerge at the level of functional analysis because all structures defined via the conventional exponentiation must be suitably lifted into a form compatible with Theorem 3.1. As an example, Fourier transforms are structurally dependent on the conventional exponentiation. As a result, they must be lifted under isotopies into the expressions [23]

$$f(x) = (1/2\pi) \int_{-\infty}^{+\infty} g(k) * e_{\hat\xi}^{ikx} \, dk, \quad g(k) = (1/2\pi) \int_{-\infty}^{+\infty} f(x) * e_{\hat\xi}^{-ikx} \, dx, \qquad (3.4)$$

with similar liftings for Laplace transforms, Dirac–delta distribution, etc., not reviewed here for brevity.

On physical grounds, Theorem 3.1 implies that the isotransform of a gaussian in isofunctional analysis is given by [23]

$$f(x) = N * e_{\hat\xi}^{-x/2 a^2} = N e^{-x^2 T/2 a^2} \Rightarrow g(k) = N * e_{\hat\xi}^{-k^2 a^2/2} = N'e^{-k^2 Ta^2/2} \qquad (3.5)$$

As a result, the widths are of the type $\Delta x \sim aT^{-\frac12}$, $\Delta k \sim a^{-1}T^{-\frac12}$. It then follows that the isotopies imply the loss of the conventional uncertainties $\Delta x \; \Delta k \sim 1$ in favor of the covering *isouncertainties* [61]

$$\Delta x \; \Delta k \sim \hat 1, \qquad (3.6)$$

which are solely applicable for interior conditions of particles, e.g., for an electron in the core of a star. In fact, whenever the particle returns to move in vacuum, the isounit recovers Planck constant identically, $\hat 1 \equiv \hbar$. Intriguingly, isouncertainties (3.6) recover classical determinism for a particle in the interior of gravitational collapse into a singularity (because the isoexpectation values depend on $T\hat 1 T = T \to 0$ for $\hat 1 \to \infty$). This illustrates that the isotopies permit a "completion" of quantum mechanics much along the historical E–P–R argument (see [61] for details).

The *isodual isoenvelopes* $\hat\xi^d$ are characterized by the isodual basis $X_k^d = - X_k$ defined with respect to the isodual isounits $\hat 1^d = -\hat 1$ and isodual isotopic element $T^d = -T$ over the isodual isofields $\hat F^d$. The *isodual isoex-ponentiation* is then given by

$$e_{\hat\xi^d}^{i^d w^d \times d_X^d} = \hat 1^d \{ e^{iwTX} \} = - e_{\hat\xi}^{iwX} \qquad (3.7)$$

and plays an important role for the characterization of antiparticles as possessing negative-definite energy and moving backward in time (as necessary when using isodual isofields).

It is easy to see that Theorem 3.1 holds, as originally formulated [47], for envelopes now called of Class III, thus unifying isoenvelopes $\hat\xi$ and their isoduals $\hat\xi^d$. In fact, the theorem was conceived to unify with one single Lie algebra basis X_k nonisomorphic compact and noncompact algebras of the same dimension N (see the example of Section 3.E).

The isotopy $\xi \Rightarrow \hat\xi$ is not a conventional map because the local coordinates x, the infinitesimal generators X_k and the parameters w_k are not changed by assumption. Also, ξ and $\hat\xi$ can be linked by nonunitary transformations $UU^\dagger = \hat 1 \neq I$, for which

$$A \Rightarrow A' = UAU^\dagger, \; B \Rightarrow B' = UBU^\dagger, \; AB \Rightarrow UABU\dagger = A'*B' = A' T B', \; T = (UU^\dagger)^{-1}. \qquad (3.8)$$

The lack of equivalence of the two theories is then further illustrated by the inequivalence between conventional eigenvalue equations, e.g., the conventional form for Lie's theory

$$H | b > = E | b >,$$

and their isotopic form

$$H*|\hat{b}> = \hat{E}*|\hat{b}> = E'|\hat{b}>,$$

with consequential *different eigenvalues for the same operator* H, $E' \neq E$ (see Section 3.E for an example).

3.B. Isotopies and isodualities of Lie algebras. A (finite-dimensional) isospace $\hat{L}$ over the isofield $\hat{F}$ of isoreal $\hat{\Re}(\hat{n},+,*)$, isocomplex numbers $\hat{C}(\hat{c},+,*)$ or isoquaternions $\hat{Q}(\hat{q},+,*)$ with isotopic element T and isounit $\hat{1} = T^{-1}$ is called a *Lie-Santilli algebra* [3], [25], [31], [74] over $\hat{F}$ (at times also referred as *isoalgebra* when no confusion may arise) when there is a composition [A,̂B] in $\hat{L}$, called *isocom-mutator*, which is isolinear and such that for all $\hat{a}, \hat{b} \in \hat{F}$ and A, B, C $\in \hat{L}$

$$[A\hat{,}B] = - [B\hat{,}A], \quad [A\hat{,}[B\hat{,}C]] + [B\hat{,}[C\hat{,}A]] + [C\hat{,}[A\hat{,}B]] = 0, \quad (3.9a)$$

$$[A*B\hat{,}C] = A*[B\hat{,}C] + [A\hat{,}C]*B. \quad (3.9b)$$

The isoalgebras are said to be: *isoreal (isocomplex)* when $\hat{F} = \hat{\Re}$ ($\hat{F} = \hat{C}$), and *isoabelian* when [A,̂B] ≡ 0 ∀ A, B ∈ $\hat{L}$. A subset $\hat{L}_0$ of $\hat{L}$ is said to be an *isosubalgebra* of $\hat{L}$ when [$\hat{L}_0$,̂$\hat{L}_0$] ⊂ $\hat{L}_0$ and an *isoideal* when [$\hat{L}$,̂$\hat{L}_0$] ⊂ $\hat{L}_0$. A maximal isoideal which verifies the property [$\hat{L}$,̂$\hat{L}_0$] = 0 is called the *isocenter* of $\hat{L}$. For the isotopies of conventional notions, theorems and properties of Lie algebras, one may seet monograph [74].

We can here only recall the existence, proved in the original memoir [47], of consistent *isotopic generalizations of the celebrated Lie's First, Second and Third Theorems*. For instance, the isotopic second theorem reads

$$[X_i\hat{,}X_j] = X_i * X_j - X_j * X_i = X_i T(x, ...) X_j - X_j T(x, ...) X_i = \hat{C}_{ij}^k(x, \dot{x}, \ddot{x}, ...) * X_k, \quad (3.10)$$

where the $\hat{C}$'s, called the *structure functions*, generally have an explicit dependence on the underlying isospace (see the example of Section 3.E), and verify certain restrictions from the Isotopic Third Theorem.

Let L be an N-dimensional Lie algebra with conventional commutation rules and structure constants C_{ij}^k on a space S(x,F) with local coordinates x over a field F, and let L be (homomorphic to) the antisymmetric algebra [ξ(L)]⁻ attached to the associative envelope ξ(L). Then $\hat{L}$ can be equivalently defined as (homomorphic to) the antisymmetric algebra [ξ̂(L)]⁻ attached to the isoassociative envelope ξ̂(L) ([47], [49], [74]). In this way, an infinite number of isoalgebras $\hat{L}$, depending on all possible isounits $\hat{1}$, can be constructed via the isotopies of one single Lie algebra L. It is easy to prove the following result:

Theorem 3.2. *The isotopies* L ⇒ $\hat{L}$ *of an N-dimensional Lie algebra L preserve the original dimensionality.*

In fact, the basis e_k, k = 1, 2, ..., N of a Lie algebra L is not changed under isotopy, except for renormalization factors denoted $\hat{e}_k$. Let then the commutation rules of L be given by

$$[e_i, e_j] = C_{ij}^k e_k.$$

The isocommutation rules of the isotopes $\hat{L}$ are

$$[\hat{e}_i\hat{,}\hat{e}_j] = \hat{e}_i T \hat{e}_j - \hat{e}_j T \hat{e}_i = \hat{C}_{ij}^k(x, \dot{x}, \ddot{x}, ...) \hat{e}_k, C = \hat{C} T. \quad (3.11)$$

One can then see in this way the necessity of lifting the structure <constants> into structure <functions>, as correctly predicted by the Isotopic Second Theorem.

The structure theory of isoalgebras is still unexplored to a considerable extent. In the following we shall show that the main lines of the conventional structure of Lie theory do indeed admit a consistent isotopic lifting. To begin, we here introduce the *general isolinear and isocomplex Lie-Santilli algebras* denoted G$\hat{L}$(n,$\hat{C}$) as the vector isospaces of all n×n complex matrices over $\hat{C}$. It is easy to see that they are closed under isocommutators as in the conventional case. The *isocenter* of G$\hat{L}$(n,$\hat{C}$) is then given by $\hat{a} * \hat{1}$, ∀ $\hat{a} \in \hat{\Re}$. The subset of all complex n×n matrices with null trace is also closed under isocommutators. We shall call it the *special, complex, isolinear isoalgebra* and denote it with S$\hat{L}$(n,$\hat{C}$). The subset of all antisymmetric n×n real matrices X, $X^t = - X$, is also closed under isocommutators, it is called the *isoorthogonal algebra*, and it is denoted with $\hat{O}$(n).

By proceeding along similar lines, we classify all classical, non-exceptional, Lie-Santilli algebras over an isofield of characteristic zero into the isotopes of the conventional forms, denoted with $\hat{A}_n$, $\hat{B}_n$, $\hat{C}_n$ and $\hat{D}_n$ each one admitting realizations of Classes I, II, III, IV and V (of which only Classes I, II and III are studied herein). In fact, $\hat{A}_{n-1} = $ S$\hat{L}$(n,$\hat{C}$); $\hat{B}_n = \hat{O}$(2n+1, $\hat{C}$); $\hat{C}_n = $ S$\hat{P}$(n,$\hat{C}$); and $\hat{D}_n = \hat{O}$(2n,$\hat{C}$). One can begin to see in this way the richness of the isotopic theory as compared to the conventional theory.

The notions of *homomorphism, automorphism* and *isomorphism* of two isoalgebras $\hat{L}$ and $\hat{L}'$, as well as of *simplicity* and *semisimplicity* are the conventional ones. Similarly, all properties of Lie algebras based on the addition, such as the *direct and semidirect sums*, carry over to the isotopic context unchanged (because of the preservation of the conventional additive unit 0).

An *isoderivation* $\hat{D}$ of an isoalgebra $\hat{L}$ is an isolinear mapping of $\hat{L}$ into itself satisfying the property

$$\hat{D}([A, B]) = [\hat{D}(A)\hat{,}B] + [A\hat{,}\hat{D}(B)] \quad \forall A, B \in \hat{L}. \quad (3.12)$$

If two maps $\hat{D}_1$ and $\hat{D}_2$ are isoderivations, then $\hat{a}*\hat{D}_1 + \hat{b}*\hat{D}_2$ is also an isoderivation, and the isocommutators of $\hat{D}_1$ and $\hat{D}_2$ is also an isoderivation. Thus, the set of all isoderivations forms a Lie-Santilli algebra as in the conventional case.

The isolinear map a$\hat{d}$($\hat{L}$) of $\hat{L}$ into itself defined by

$$\text{ad } A(B) = [A \overset{\frown}{,} B], \qquad \forall \ A, B \in \mathcal{L}, \tag{3.13}$$

is called the *isoadjoint map*. It is an isoderivation, as one can prove via the iso–Jacobi identity. The set of all ad(A) is therefore an isolinear isoalgebra, called *isoadjoint algebra* and denoted $\mathcal{L}_a$. It also results to be an isoideal of the algebra of all isoderivations as in the conventional case.

Let $\mathcal{L}^{(0)} = \mathcal{L}$. Then $\mathcal{L}^{(1)} = [\mathcal{L}^{(0)} \overset{\frown}{,} \mathcal{L}^{(0)}]$, $\mathcal{L}^{(2)} = [\mathcal{L}^{(1)} \overset{\frown}{,} \mathcal{L}^{(1)}]$, etc., are also isoideals of $\mathcal{L}$. $\mathcal{L}$ is then called *isosolvable* if, for some positive integer n, $\mathcal{L}^{(n)} \equiv 0$. Consider also the sequence

$$\mathcal{L}_{(0)} = \mathcal{L}, \qquad \mathcal{L}_{(1)} = [\mathcal{L}_{(0)} \overset{\frown}{,} \mathcal{L}], \qquad \mathcal{L}_{(2)} = [\mathcal{L}_{(1)} \overset{\frown}{,} \mathcal{L}], \quad \text{etc.},$$

Then $\mathcal{L}$ is said to be *isonilpotent* if, for some positive integer n, $\mathcal{L}_{(n)} \equiv 0$. One can then see that, as in the conventional case, an isonilpotent algebra is also isosolvable, but the converse is not necessarily true.

Let the *isotrace* of a matrix be given by the element of the isofield [61]

$$\hat{\text{Tr}} \ A = (\text{Tr } A) \hat{1} \in \hat{F}, \tag{3.14}$$

where Tr A is the conventional trace. Then

$$\hat{\text{Tr}} \ (A * B) = (\hat{\text{Tr}} \ A) * (\hat{\text{Tr}} \ B), \qquad \hat{\text{Tr}} \ (B A B^{-1}) = \hat{\text{Tr}} \ A.$$

Thus, the $\hat{\text{Tr}} \ A$ preserves the axioms of Tr A, by therefore being a correct isotopy. Then the isoscalar product

$$(A \overset{\frown}{,} B) = \hat{\text{Tr}} \ [(\text{Ad}X) * (\text{Ad } B)], \tag{3.15}$$

is here called the *isokilling form*. It is easy to see that $(A \overset{\frown}{,} B)$ is symmetric, bilinear, and verifies the property $(\text{Ad}X(Y) \overset{\frown}{,} Z) + (Y \overset{\frown}{,} \text{Ad } X(Z)) = 0$, thus being a correct, axiom–preserving isotopy of the conventional Killing form.

Let e_k, $k = 1, 2, ..., N$, be the basis of L with one–to–one invertible map $e_k \to \hat{e}_k$ to the basis of $\mathcal{L}$. Generic elements in $\mathcal{L}$ can then be written in terms of local coordinates x, y, z, $A = x^i \hat{e}_i$ and $B = y^j \hat{e}_j$, and

$$C = z^k \hat{e}_k = [A \overset{\frown}{,} B] = x^i y^j [\hat{e}_i \overset{\frown}{,} \hat{e}_j] = x^i x^j \tilde{C}_{ij}{}^k \hat{e}_k.$$

Thus,

$$[\text{Ad } A(B)]^k = [A \overset{\frown}{,} B]^k = \tilde{C}_{ij}{}^k x^i x^j. \tag{3.16}$$

We now introduce the *isocartan tensor* $\tilde{g}_{ij}$ of an isoalgebra $\mathcal{L}$ via the definition

$$(A \overset{\frown}{,} B) = \tilde{g}_{ij} x^i y^j \text{ yielding}$$

$$\tilde{g}_{ij}(x, \dot{x}, \ddot{x}, ...) = \tilde{C}_{ip}{}^k \tilde{C}_{jk}{}^p. \tag{3.17}$$

Note that the isocartan tensor has the general dependence of the isometric tensor of Section 2.C, thus confirming the inner consistency among the various branches of the isotopic theory. In particular, the isocartan tensor is generally *nonlinear, nonlocal and noncanonical* in all variables x, $\dot{x}$, $\ddot{x}$, ... This clarifies that isotopic generalization of the Riemannian spaces studied in the companion paper [60] $R(x, g, \Re) \Rightarrow \hat{R}(x, \hat{g}, \hat{\Re})$, $\hat{g} = \tilde{g}(s, x, \dot{x}, \ddot{x}, ...)$, has its origin in the very structure of the Lie–isotopic theory.

The isocartan tensor also clarifies another fundamental point of Section 1, that the isotopies naturally lead to an arbitrary dependence in the velocities and accelerations, exactly as needed for realistic treatment of the problems identified in Section 1, and that their restriction to the nonlinear dependence on the coordinates x only, as generally needed for the exterior (e.g., gravitational) problem, would be manifestly un–necessary.

The isotopies of the remaining aspects of the structure theory of Lie algebras can be completed by the interested reader. Here we limit ourselves to recall that when the isocartan form is positive– (or negative–) definite, $\mathcal{L}$ is compact, otherwise it is noncompact. Then it is easy to prove the following

Theorem 3.3. *The Class* III *liftings* $\mathcal{L}$ *of a compact (noncompact) Lie algebra L are not necessarily compact (noncompact).*

The identification of the remaining properties which are not preserved under liftings of Class III is an instructive task for the interested reader. For instance, if the original structure is irreducible, its isotopic image is not necessarily so even for Class I, trivially, because the isounit itself can be reducible, thus yielding a reducible isotopic structure.

Let $\mathcal{L}$ be an isoalgebra with generators X_k and isounit $\hat{1} = T^{-1} > 0$. From Equations (3.7) we then see that the *isodual Lie–Santilli algebras* $\mathcal{L}^d$ of $\mathcal{L}$ is characterized by the isocommutators

$$[X_i \overset{\frown}{,} X_j]^d = -[X_i \overset{\frown}{,} X_j] = \tilde{C}_{ij}{}^{k \ (d)} X_k, \qquad \tilde{C}_{ij}{}^{k(d)} = -\tilde{C}_{ij}{}^k. \tag{3.18}$$

$\mathcal{L}$ and $\mathcal{L}^d$ are then (anti) isomorphic. Note that the isoalgebras of Class III contain all Class I isoalgebras $\mathcal{L}$ and all their isoduals $\mathcal{L}^d$. The above remarks therefore show that the Lie–Santilli theory can be naturally formulated for Class III, as implicitly done in the original proposal [47]. The formulation of the same theory for Class IV or V is however considerably involved on technical grounds thus requiring specific studies.

The notion of isoduality applies also to conventional Lie algebras L, by permitting the identification of their *isodual Lie algebras* L^d via the rule [52], [53]

$$[X_i, X_j]^d = X^d_i \hat{1}^d X^d_j - X^d_j \hat{1}^d X^d_i = -[X_i, X_j] = C_{ij}{}^{k \ (d)} X^d_k, \qquad C_{ij}{}^{k(d)} = -C_{ij}{}^k. \tag{3.19}$$

Note the necessity of the isotopies for the very construction of the isodual of conventional Lie algebras. In fact, they require the nontrivial lift of the unit $I \Rightarrow I^d = (-I)$, with consequential necessary generalization of the Lie product $AB - BA$ into the isotopic form $ATB - BTA$.

The following property is mathematically trivial, yet carries important physical applications.

Theorem 3.4. *All infinitely possible, Class I isotopes $\hat{L}$ of a (finite-dimensional) Lie algebra L are locally isomorphic to L, and all infinitely possible, Class II isodual isotopes $\hat{L}^d$ of L are (anti) isomorphic to L.*

The realization of isoalgebras in particle physics is essentially given by the formulation via operators X_k defined on a suitable isohilbert space to ensure the preservation of Hermiticity/observability [61]. In classical mechanics, the most direct realization of the isoalgebra is that given in terms of vector fields X_k on the isocotangent bundle $T^*\hat{E}(x,\hat{\delta},\hat{\Re})$, $\hat{\delta} = T\delta$, $\hat{\Re} = \Re \hat{1}$, $\hat{1} = T^{-1}$, resulting in the isotopy of the Poisson brackets first submitted in [56]

$$[A,B] = \frac{\partial A}{\partial x^\mu} \hat{1}^\mu_\nu(x, p, ...) \frac{\partial B}{\partial p_\nu} - \frac{\partial B}{\partial x^\mu} \hat{1}^\mu_\nu(x\, p, ...) \frac{\partial A}{\partial p_\nu}, \qquad (3.20)$$

where the isounit is constructed via a special procedure (the isotopic Poincaré Lemma) ensuring the verification of the Lie algebra axioms (see [56] and [62].

3.C. Isotopies and isodualities of Lie groups. The isotopies of topological spaces and groups are still lacking to this day. The only isotopies of Lie groups currently available are those of Lie's transformation groups [47], [49], [74].

A *right Lie-Santilli group* $\hat{G}$ on an isospace $\hat{S}(x,\hat{F})$ over an isofield $\hat{F}$, $\hat{1} = T^{-1}$ (of isoreal $\hat{\Re}$ or isocomplex numbers $\hat{C}$), also called *isotransformation group* or *isogroup*, is a group which maps each element $x \in \hat{S}(x,\hat{F})$ into a new element $x' \in \hat{S}(x,\hat{F})$ via the isotransformations $x' = \hat{0}*x = \hat{0}Tx$, T fixed, such that: (1) The map $(\hat{0}, x) \to \hat{0}*x$ of $\hat{G}\times\hat{S}(x,\hat{F})$ onto $\hat{S}(x,\hat{F})$ is isodifferentiable; (2) $\hat{1} * \hat{0} = \hat{0} * \hat{1} = \hat{0} \; \forall \hat{0} \in \hat{G}$; and (3) $\hat{0}_1 * (\hat{0}_2 * x) = (\hat{0}_1 * \hat{0}_2) * x$, $\forall x \in \hat{S}(x,\hat{F})$ and $\hat{0}_1, \hat{0}_2 \in \hat{G}$. A *left isotransformation group* is defined accordingly.

The notions of *connected or simply connected transformation groups* carry over to the isogroups in their entirety. We consider hereon the connected isotransformation groups. Right or left isogroups are characterized by the following laws [47]

$$\hat{0}(0) = \hat{1}, \quad \hat{0}(\hat{w}) * \hat{0}(\hat{w}') = \hat{0}(\hat{w}') * \hat{0}(\hat{w}) = \hat{0}(\hat{w} + \hat{w}'), \; \hat{0}(\hat{w}) * \hat{0}(-\hat{w}) = \hat{1}, \; \hat{w} \in \hat{F}. \quad (3.21)$$

Their most direct realization is that via isoexponentiation (3.3),

$$\hat{0}(w) = \prod_k e_{\hat{\xi}}^{i\hat{w}_k * X_k} = \hat{1}\{\prod_k e^{iw_k TX_k}\} = \{\prod_k e^{iX_k Tw_k}\}\hat{1}, \qquad (3.22)$$

where the X's and w's are the infinitesimal generator and parameters of the original algebra L. Equations (3.22) hold for some open neighborhood of N of the isoorigin of $\hat{L}$ and, in this way, characterize some open neighborhood of the isounit of $\hat{G}$. Then the isotransformations can be reduced to an ordinary transform for computational convenience,

$$x' = \hat{0}*x = \{\prod_k e_{\hat{\xi}}^{iX_k * w_k}\} * x = \{\prod_k e^{iX_k Tw_k}\} x, \qquad (3.23)$$

with the understanding that, on rigorous mathematical grounds, only the isotransform is correct.

Still another important result obtained in [47] is the proof that conventional group composition laws admit a consistent isotopic lifting, resulting in the following isotopy of the Baker-Campbell-Hausdorff Theorem

$$(e_{\hat{\xi}}^X)*(e_{\hat{\xi}}^X) = e_{\hat{\xi}}^{X_3}, \; X_3 = X_1 + X_2 + [X_1 \hat{,} X_2]/2 + \{(X_1 - X_2)\hat{,}[X_1 \hat{,} X_2]\}/12 + \ldots \quad (3.24)$$

Note the crucial appearance of the isotopic element $T(x, \dot{x}, \ddot{x}, ...)$ in the exponent of the isogroup. This ensures a structural generalization of Lie's theory of the desired nonlinear, nonlocal and noncanonical form. For details see [49] and [74].

The structure theory of isogroups is also vastly unexplored at this writing. In the following we shall point out that the conventional structure theory of Lie groups does indeed admit a consistent isotopic lifting. The isotopies of the notions of weak and strong continuity of [22] are a necessary pre-requisite. Let $\hat{L}$ be a (finite-dimensional) Lie-Santilli algebra with (ordered) basis $\{X_k\}$, $k = 1, 2, ..., N$. For a sufficiently small neighborhood N of the isoorigin of $\hat{L}$, a generic element of $\hat{G}$ can be written

$$\hat{0}(w) = \prod_{k=1,2,...,N} e_{\hat{\xi}}^{iX_k w_k}, \qquad (3.25)$$

which characterizes some open neighborhood M of the isounit $\hat{1}$ of $\hat{G}$. The map

$$\Phi_{\hat{0}_1}(\hat{0}_2) = \hat{0}_1 * \hat{0}_2 * \hat{0}_1^{-1}, \qquad (3.26)$$

for a fixed $\hat{0}_1 \in \hat{G}$, characterizes an *inner isoautomorphism* of $\hat{G}$ onto $\hat{G}$. The corresponding isoautomorphism of the algebra $\hat{L}$ can be readily computed by considering the above expression in the neighborhood of the isounit $\hat{1}$. In fact, we have

$$\hat{0}'_2 = \hat{0}_1 * \hat{0}_2 * \hat{0}_1^{-1} \cong \hat{0}_2 + w_1 w_2 [X_2 \hat{,} X_1] + \hat{0}^{(2)}. \qquad (3.27)$$

By recalling the differentiability property of $\hat{G}$, we also have the following isotopy of the conventional expression in one dimension:

$$X = (1/i) \frac{d}{d\hat{w}} \hat{0} \Big|_{w=0} = (1/i) \frac{d}{dw} e_{\hat{\xi}}^{iwX} \Big|_{w=0} = X * e_{\hat{\xi}}^{iwX} \Big|_{w=0} = X. \qquad (3.28)$$

Thus, to every inner isoautomorphism of $\hat{G}$, there corresponds an inner isoautomorphism of $\hat{L}$ which can be expressed in the form:

$$(\hat{L})_i{}^j = \check{C}_{ki}{}^j w^k . \tag{3.29}$$

The isogroup $\hat{G}_a$ of all inner isoautomorphism of $\hat{G}$ is called the *isoadjoint group*. It is possible to prove that the Lie-Santilli algebra of $\hat{G}_a$ is the isoadjoint algebra $\hat{L}_a$ of $\hat{L}$. This establishes that the connections between algebras and groups carry over in their entirety under isotopies.

We mentioned before that the direct sum of isoalgebras is the conventional operation because the addition is not lifted under isotopies (otherwise there will be the loss of distributivity, see [59]). The corresponding operation for groups is the semidirect product which, as such, demands care in its formulation.

Let $\hat{G}$ be an isogroup and $\hat{G}_a$ the group of all its inner isoautomorphisms. Let $\hat{G}^o{}_a$ be a subgroup of $\hat{G}_a$, and let $\hat{\Lambda}(\hat{g})$ be the image of $\hat{g} \in \hat{G}$ under $\hat{G}^o{}_a$. The *semidirect isoproduct* $\hat{G} \hat{\times} \hat{G}^o{}_a$ of $\hat{G}$ and $\hat{G}^o{}_a$ is the isogroup of all ordered pairs

$$(\hat{g}, \; \hat{\Lambda}) * (\hat{g}', \; \hat{\Lambda}') = (\hat{g} * \hat{\Lambda}(\hat{g}'), \; \hat{\Lambda} * \hat{\Lambda}') , \tag{3.30}$$

with total isounit given by $(1, 1_{\hat{\Lambda}})$ and inverse $\hat{g}, \hat{\Lambda}s)^{-1} = (\hat{\Lambda}^{-1}(\hat{g}^{-1}) , \hat{\Lambda}^{-1})$. The above notion plays an important role in the isotopies of the inhomogeneous space-time symmetries outlined later on.

Let $\hat{G}_1$ and $\hat{G}_2$ be two isogroups with respective isounits 1_1 and 1_2. The *direct isoproduct* $\hat{G}_1 \hat{\circ} \hat{G}_2$ of $\hat{G}_1$ and $\hat{G}_2$ is the isogroup of all ordered pairs $(\hat{g}_1, \hat{g}_2)$, $\hat{g}_1 \in \hat{G}_1$, $\hat{g}_2 \in \hat{G}_2$, with isomultiplication

$$(\hat{g}_1, \hat{g}_2) * (\hat{g}'_1, \hat{g}'_2) = (\hat{g}_1 * \hat{g}'_1, \hat{g}_2 * \hat{g}'_2) , \tag{3.31}$$

total isounit $(1_1, 1_2)$ and inverse $(\hat{g}_1{}^{-1}, \hat{g}_2{}^{-1})$. The isotopies of the remaining aspects of the structure theory of Lie groups can then be investigated by the interested reader.

Let $\hat{G}$ be an N-dimensional isotransformation group of Class I with infinitesimal generators X_k, $k = 1, 2, ..., N$. The *isodual Lie-Santilli group* $\hat{G}^d$ of $\hat{G}$ ([52], [53]) is the N-dimensional isogroup with generators $X_k{}^d = -X_k$ constructed with respect to the isodual isounit $1^d = -1$ over the isodual isofield $\hat{F}^d$. By recalling that $w \in F \Rightarrow w^d \in F^d$, $w^d = -w$, a generic element of $\hat{G}^d$ in a suitable neighborhood of 1^d is therefore given by

$$\hat{U}^d(\hat{w}^d) = e_{\hat{\xi}^d}{}^{i^d \hat{w}^d \hat{\times}^d X^d} = - e_{\hat{\xi}}{}^{i \hat{w} * X} = - \hat{U}(\hat{w}) . \tag{3.32}$$

The above antiautomorphic conjugation can also be defined for conventional Lie group, yielding the *isodual Lie group* G^d of G with generic elements $U^d(w^d) = e_{\hat{\xi}^d}{}^{iw^d X} = - e_{\hat{\xi}}{}^{iwX}$.

Thus, any symmetry can be formulated in four different ways: the conventional form G, its isodual G^d, the isotopic form $\hat{G}$ and the isodual isotopic form $\hat{G}^d$. These different forms are useful for the respective characterization of particles and antiparticles in vacuum (exterior problem) or within physical media (interior problem).

It is hoped that the reader can see from the above elements that the entire conventional Lie's theory does indeed admit a consistent and nontrivial lifting into the covering Lie-Santilli formulation. Particularly important are the isotopies of the conventional representation theory, known as the *isorepresentation theory*, which naturally yields the most general known, nonlinear, nonlocal and noncanonical representations of Lie groups. Studies along these latter lines were initiated by Santilli with the isorepresentations of SO(2) and of SO(3) [61], by Klimyk and Santilli Klimyk [27], and others.

As is well known, the operator realization of Lie's theory is given by quantum mechanics. The operator realization of the covering Lie-Santilli theory is given by a new mechanics known under the namer of *hadronic mechanics* ([58], [61]). Its fundamental time evolution is given by a one-parameter isogroup of inner isoautomorphisms, which can be written for an operator Q on an isohilbert space $\mathcal{H}$ in the finite form:

$$Q(t) = \{ e_{\hat{\xi}}{}^{-itH} \} * Q(0) * \{ e_{\hat{\xi}}{}^{itH} \} = \{ e^{-itTH} \} \; Q(0) \; \{ e^{itTH} \}, \; H = H\dagger , \tag{3.33}$$

with infinitesimal counterpart derivable from the former via the rules studied in this section.

$$i \frac{dQ}{dt} = [Q, \hat{,} H] = Q * H - H * Q = Q T H - H T Q . \tag{3.34}$$

This confirms that the isotopies of Lie's theory are indeed at the structural foundations of the isotopic completion of quantum mechanics.

A classical realization is given by the following relativistic *Hamilton-isotopic equations* on isominkowski space $\hat{M}(x, \hat{\eta}, \hat{R})$ first submitted in [56]

$$\dot{x}^\mu = 1^\mu{}_\nu \frac{\partial H}{\partial p_\nu} , \qquad \dot{p}_\mu = - 1_\mu{}^\nu \frac{\partial H}{\partial x^\nu} \tag{3.35a}$$

$$\Phi_k(s, x, p, ...) = 0 , \tag{3.35b}$$

where the Φ's represent all needed subsidiary *isoconstraints*, such as $\dot{x}^\mu \hat{\eta}_{\mu\nu} \dot{x}^\nu = -1$, where $\hat{\eta}$ is the isometric, and can be written in the unified form:

$$\dot{a} = \omega^{\mu\alpha} 1_{2\alpha}{}^\nu \frac{\partial H}{\partial a^\nu} , \qquad 1_2 = \text{diag.} (1, 1), \tag{3.36}$$

where ω is the familiar canonical Lie tensor. The exponentiated version of the above time evolution then reads

$$a(t) = \{ e^{t\omega^{\mu\alpha} 1_{2\alpha}{}^\nu (\partial H / \partial a^\mu) \partial / \partial a^\nu} \} \; a(0), \tag{3.37}$$