NONLOCAL FORMULATION OF THE BOSE-EINSTEIN CORRELATION WITHIN THE CONTEXT OF HADRONIC MECHANICS,

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Abstract

In this paper we submit the hypothesis that the Bose-Einstein correlation originates from a nonlocal component of the strong interactions in the interior of the fireball caused by the deep overlapping of the charge distributions-wavepackets of the $p-\bar{p}$ collision at high energy, along the historical legacy of Bogoliubov, Fermi and others. Owing to their contact nature, the notion of potential energy has no meaning for the nonlocal internal interactions here considered which, as such, are structurally outside the representational capabilities of quantum mechanics on analytic, topological, operator and other grounds. We therefore study the Bose-Einstein correlation via the covering hadronic mechanics, which consists of axiom-preserving isotopies of quantum mechanics derived from the fundamental isotopy, the generalization of Planck's constant $\hbar = 1$ into an integrodifferential unit $\hat{1} = \hat{1}(t, r, p, \dot{p}, \psi, \psi^\dagger, \partial\psi, \partial\psi^\dagger, ...)$, det. $\hat{1} \neq 0, \hat{1} = \hat{1}^\dagger, \hat{1} > 0$. Because of to the disparate character of the existing literature, we first review for the reader's convenience the elements of: the classical foundations of the isotopies; the isotopic quantization, the elements of nonrelativistic and relativistic hadronic mechanics; the underlying generalized notions of isoparticle and isocomposite system; the isotopic

symmetries and relativities that are applicable to the fireball with internal nonlocal and nonhamiltonian interactions; and other aspects. We then study the interior problem of the Bose–Einstein correlation via isorelativistic hadronic mechanics, and shows that it is indeed directly derivable from first principles via the joint use of conventional Hamiltonians $H = H(r, p)$ and generalized units $\hat{1} = \hat{1}(t, r, p, \dot{p}, \psi, \psi^{\dagger}, \partial\psi, \partial\psi^{\dagger}, ...)$. More specifically, we show that the Bose-Einstein correlation is outside the structural axioms of quantum mechanics, e.g., that of the conventional expectation values $< \psi \mid \psi >$, while it is directly derivable from the corresponding generalizedaxioms, e.g., that of the isoexpectation values $< \psi \hat{|} \psi > = < \psi \mid T \mid \psi > \hat{1}, \hat{1} = T^{-1}$. We then study the exterior problem of the Bose–Einstein correlation via a direct representation of the prolate spheroidal ellipsoidical shape of the fireball, achieved by means of the factorization of the isounit $\hat{1} = \{ \text{diag.} (b_1^2, b_2^2, b_3^2, - b_3^2, b_4^2) \} \hat{1}(0)$ where the b's are certain "characteristics quantities of the fireball" and $\hat{1}(0)$ is a certain nonlocal and nondiagonal operator. In the final section we identify our properly normalized, two-particle isocorrelation function in a form ready for plots with experimental data, as well as preliminary estimates of the value of its four characteristics parameters b_μ.

In summary, our analysis shows that, on rigorous grounds, the Bose-Einstein correlation is outside the structure of quantum mechanics, while its representation via the covering hadronic mechanics can:

1) identify the origin of the Bose–Einstein correlation in the nonlocal internal effects due to deep wave-overlapping of the p-$\bar{p}$ collision at high energy;

2) provide a quantitative representation of thecorrelation from basic principles under the sole conventional approximation that the q_0 and q_4 components of the momentum transfer are very small;

3) directly represent the actual shape of the fireball, a prolate spheroidal ellipsoid oriented along the original p-$\bar{p}$ direction, as well as its rapid expansion in time, , all this prior to any second quantization;

4) identify the maximal value 1.67 for the two-points correlation function on exact grounds from basic axioms without approximations; and

5) provide a satisfactory representation of available experimental data on the Bose–Einstein correlation.

But, in the opinion of the author, the most important result of this analysis is that the current experimental data on Bose–Einstein correlation can result to be a direct experimental evidence on the validity of the historical legacy of Bogoliubov, Fermi and others on the ultimate nonlocal structure of strong interactions, with consequential far reaching implications, such as the need for a new generation of covering relativities.

dedicated to the memory of

N. N. BOGOLIUBOV,

because of his teaching on truly fundamental physical issues,
the only one which passes the test of time.

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1. INTRODUCTION

The physical event generally known as the *Bose-Einstein correlation* (see, e.g., ref. [1] and quoted papers) constitutes an intriguing but vastly unexplained phenomenon, whereby two or more uncorrelated particles (such as the proton-antiproton annihilation in UA1 experiments at CERN) first interact, then coalesce into what is generally called a *fireball* [2], and finally decay in a variety of unstable particles whose final detectable product is a set of correlated bosons.

In this paper we submit the hypothesis that the correlation originates in nonlocal interactions expected in the $p-\bar{p}$ annihilation, as well as in the interior of strong interactions at large, according to the historical legacy of Bogoliubov, Fermi and others. Stated differently, our hypothesis implies that the correlation is a phenomenon strictly due to the extended character of the charge distribution of hadrons and of their wavepackets, and that, whenever the interacting particles are at sufficiently large distances to permit their effective point-like approximation, no correlation is possible.

The above hypothesis also explains the inability to reach a final understanding of the phenomenon with ordinary quantum mechanics. In fact, quantum mechanics is strictly local-differential in topological structure. This implies the quantum mechanical inability to provide a quantitative treatment of nonlocal interactions via first principles, as well known.

Moreover, quantum mechanics is structurally of potential-Hamiltonian type, namely, it can only represent action-at-distance interactions described by a potential. On the contrary, nonlocal effects due to deep mutual penetration of wavepackets are well known at the classical level to be of *contact type*, namely, interactions for which the notion of potential energy has no physical meaning [3]. As such, contact, nonlocal interactions are conceptually, topologically and analytically outside the representational capabilities of quantum mechanics.

In an attempt to overcome these limitations, we submitted back in 1978[1] the proposal to construct the so-called axiom-preserving *isotopies* of the conventional Lie's theory, under the name of *Lie-isotopic theory* [3] and of quantum mechanics under the name of *hadronic mechanics* [4], while providing certain technical means to study nonhamiltonian forces, called *conditions of variational nonselfadjointness* [5].

[1] When at Harvard university under support from the U.S.Department of Energy, contract numbers ER-78-S-02-4742, AS02-78ER-4742, and DE-Ac02-8-ER10651.

These proposals were subsequently studied by numerous authors (see, e.g., the recent Proceedings [6] and quoted literature). In particular, the proposal to build hadronic mechanics reached sufficient maturity of formulation in papers [7,8] and in the more recent memoirs [9].

The classical foundations of the theory were first identified in the *Birkhoffian generalization of conventional Hamiltonian mechanics* [10], as a first step in the study of the so-called *interior dynamical problem*, i.e., extended and deformable particles moving within inhomogeneous and anisotropy physical media. In particular, Birkhoffian mechanics resulted to be "directly universal"[2] for all nonlinear and nonhamiltonian vector-fields in local-differential approximation verifying certain topological conditions.

The extension of these results to the most general possible (classical) nonlinear, nonlocal-integral and nonhamiltonian systems was first done in memoirs [11]. These latter studies reached mathematical maturity in memoirs [12,13] via the identification of axioms-preserving isotopies of the conventional symplectic, affine and Riemannian geometries, under the corresponding names of *isosymplectic, isoaffine* and *isoriemannian geometries.*

Sufficient physical maturity in the nonrelativistic, relativistic and gravitational treatment of the most general known, nonlinear, nonlocal and nonhamiltonian system of the interior problem was then reached in monographs [14,15] via the submission of certain isotopies of conventional relativities under the names of *isogalilean, isospecial and isogeneral relativities* . Independent reviews can be found in ref.s [16–18].

Additional studies have shown the existence of a unique and unambiguous *isotopic quantization* mapping the classical isotopic formulations [14,15] into the corresponding operator forms [7–9], today knows under the name of *hadronization* [9,19,20]. This indicates the existence of a unity of thought between the classical and operator formulations, which is mathematically expressed in both cases by the notion of isotopy, and physically represented, also in both cases, by the treatment of nonlocal and nonhamiltonian interactions.

The operator formulation of the isotopic theories follows very closely the classical ones. A central conceptual idea is the admission that there exist physical conditions in the universe simply outside the representational capabilities of a Lagrangian or Hamiltonian operator. This is the case for: the deformation of shape of extended charge

[2] Throughout this analysis, by "direct universality" we shall mean the capability of representing all systems of the class admitted ("universality"), directly in the frame of the experimenter ("direct universality").

distributions; the inhomogenuity and anisotropy of physical media in the interior of stars, nuclei and hadrons; the nonlinear, nonlocal and nonpotential interactions originating from mutual overlappings of the wavepackets of particles, as expected in the structure of strong interactions.

For any given mathematical or physical structure, its *isotopic images* can be essentially conceived as the most general possible, nonlinear, nonlocal and noncanonical realizations of the original axioms.

The fundamental isotopy characterizing hadronic mechanics, from which all other aspects of the theory can be derived, is the generalization of Plack's unit $\hbar = 1$ of quantum mechanics into the most general possible unit $\hat{1}$, called *isounit* , which verifies the original axioms of $\hbar$ (nonsingularity, Hermiticity and positive-definiteness), but possesses an unrestricted, integro-differential dependence on all local variables, wavefunctions and their derivatives, as well as any needed additional quantity,

$$\hbar = I \qquad \Rightarrow \qquad \hat{1} = \hat{1}(t, r, p, \dot{p}, \psi, \psi^{\dagger}, \partial\psi, \partial\psi^{\dagger}, \ldots), \qquad (1.1a)$$

$$\det. I \neq 0, \quad I = I^{\dagger}, \quad I > 0 \quad \Rightarrow \quad \det \hat{1} \neq 0, \qquad \hat{1} = \hat{1}^{\dagger}, \qquad \hat{1} > 0. \qquad (1.1b)$$

In a way much similar to the classical case [14,15], all action-at-a-distance interactions are represented with the Hamiltonian operator $H = T + V$, while all nonlocal and nonpotential interactions are embedded in the isounit $\hat{1}$. Conventional operator formulations must then be generalized in such a way to admit $\hat{1}$ as the correct, right and left operator unit.

The latter aspect essentially requires a suitable generalization of the entire structure of quantum mechanics, such as: the enveloping operator algebra, the attached Lie algebra, the underlying Hilbert space, the Heisenberg's and Schrödinger's representations, the unitary transformation theory, etc.

However, since the new mechanics is an isotopy of quantum mechanics, the generalization is axiom-preserving. In fact, the emerging covering mechanics coincides, by construction, with the conventional quantum mechanics at the abstract, realization-free level, thus establishing its mathematical consistency.

As a result of these efforts we can say today that hadronic mechanics has reached sufficient mathematical maturity as a covering of quantum mechanics for the quantitative representation of extended-deformable particles with linear and nonlinear, local and nonlocal, deformable particles with linear and nonlinear, local and nonlocal, Hamiltonian as well as nonhamiltonian interactions in a form suitable for experimental verification.

The Bose–Einstein correlation appears to be an ideal setting to test the physical effectiveness of the new mechanics, particularly when studied under our basic hypothesis of its nonlocal and nonhamiltonian origin.

This paper has been written to be as self-sufficient as possible, owing to the disparate nature of the existing literature, and to provide the interested phenomenologist in Bose–Einstein correlation with the necessary theoretical methods for experimental plots. In particular, a primary objective is to submit, apparently for the first time, an isotopic model of Bose–Einstein–correlation from basic principles in a form suitable for direct plotting with experimental data. The agreement of the model with available measures is rather impressive, with the understanding that it needs additional, independent experimental plots.

The paper is organized as follows. In Sect. 2 we identify the most important limitations of quantum mechanics, in general, and for the Bose–Einstein correlation, in particular. In Sect. 3 we outline the classical isotopic formulations, while in Sect. 4 we review the elements of their mapping into operators forms. In Sect. 5 we provide the elements of hadronic mechanics, with an outline of the apparent resolution of the limitations identified in Sect. 2. Sect. 6 is devoted to the identification of the generalized notion of composite hadronic systems characterizing the fireball. In Sect. 7 we outline the isotopic space–time symmetries and relativities applicable to a fireball with internal nonlocal and nonhamiltonian effects.

We then pass in Sect. 8 to the specific study of the Bose–Einstein correlation via *the isotopic description of the interior problem of the fireball with a nonlocal origin of correlation* ; then in Sect. 9 we introduce the *isotopic formulation of the exterior experimental detection of correlated bosons* ; and in Sect. 10 we present *the comparison of our isotopic model with experimental data* . The primary results are summarized in Sect. 11.

As we shall see, the results of this paper illustrate rather clearly the physical effectiveness, as well as the horizon of truly novel possibilities offered by a quantitative treatment of the historical legacy on the ultimate nonlocal structure of matter.

2. BASIC LIMITATIONS OF QUANTUM MECHANICS

As well known, *quantum mechanics* (see, e.g., ref. [21] and quoted literature) was conceived for the atomic structure, as well as, more generally, for the electromagnetic interactions at large, for which it subsequently resulted to be exact according to an overwhelming amount of experimental evidence.

The central notion of quantum mechanics is Planck's quantum of energy $\hbar$, which is the basic algebraic unit of the theory. The primary mathematical structures of the theory are given by:

A) The *universal, enveloping, associative, operator algebra* ξ with elements A, B, ... (say, matrices or local-differential operators) and product given by the trivial associative product AB, under which Planck's constant assumes the meaning of the *left and right unit* of the theory

$$\xi : \begin{cases} \hbar = 1, & (2.1a) \\ AB = \text{assoc.}, \qquad 1A \equiv A1 \equiv A \quad \forall\ A \in \xi; & (2.1b) \end{cases}$$

B) The *field* F of real numbers $\Re$ or of complex numbers C.

C) The *Hilbert space* $\mathcal{H}$ with states $|\psi>$, $|\phi>$,..., and inner product

$$\mathcal{H} : \qquad <\psi|\phi> \ = \ \int dr' \, \psi(t, r') \, \phi(t, r') \ \in C; \tag{2.2}$$

All familiar formulations of quantum mechanics can be derived from the above primitive mathematical structures either in a direct or an an indirect way. As an example, Schrödinger's equation for a given (Hermitean) Hamiltonian H

$$i \frac{\partial}{\partial t} |\psi> \ = \ H|\psi> = E|\psi>, \tag{2.3}$$

is a consequence of the original associativity of the envelope ξ which results in the action of the operator H on the state $|\psi>$ being right, modular and associative, i.e., such that

$$ABC|\psi> \ = \ A(BC|\psi>) \ = \ (AB)C|\psi> \ = (ABC)|\psi>. \tag{2.4}$$

Similarly, Heisenberg's celebrated equations for the time evolution of a physical quantity Q,

$$i\dot{Q} = [Q\,,H]_{\xi} = QH - HQ, \qquad (2.5)$$

are characterized by a Lie algebra L with brackets [A,B] which is homomorphic to the antisymmetric algebra ξ^{-} attached to the envelope ξ, $L \approx \xi^{-}$.

Finally, we recall that the exponentiation of Eq.s (2.5) into a finite Lie group is a power series expansion in the envelope ξ, namely, it is technically permitted by the infinite-dimensional basis in ξ originating from the Poincaré-Birkhoff-Witt theorem, with familiar expansion

$$e_{|\xi}^{i\alpha X} = I + i\alpha X / 1! + (i\alpha X)(i\alpha X) / 2! + \ldots.., \quad \alpha \in F, \quad X = X^{\dagger} \in \xi, \qquad (2.6)$$

under which the infinitesimal form (2.5) can be exponentiated to the finite group form

$$Q(t) = e_{|\xi}^{itH}\, Q(0)\, e_{|\xi}^{-Hti}. \qquad (2.7)$$

All other aspects of quantum mechanics, such as linear operations on $\mathcal{H}$, Heisenberg's uncertainty principle, Pauli exclusion principle, representation theory, etc., can be derived via a judicious use of formulations derivable from or compatible with the above fundamental structures ξ, $\mathfrak{R}$ (or C) and $\mathcal{H}$.

It is important for our analysis to identify the primary limitations of quantum mechanics for the Bose-Einstein correlation.

LIMITATION 1: LACK OF REPRESENTATION OF EXTENDED CHARGE DISTRIBUTIONS. As well known, the topological, geometrical and algebraic structures underlying quantum mechanics are strictly *local-differential* . As a result, *quantum mechanics can only represent extended charge distributions in a first point-like approximation* . Such a limitation is evidently not a problem for the electrons of an atomic structure, owing to their point-like charge structure, but it is a clear limitation for all hadrons in general (because they have a finite charge radius of about 1 Fermi), and, in particular, for the p-$\bar{p}$ inelastic scatterings of the Bose-Einstein correlation.

Stated differently, a primary limitation of the quantum mechanics, which is particularly relevant for the problem of boson correlation, is that of its original conception, namely, for sufficiently large mutual distances of particles under which their actual size is ignorable. Despite the large volume of experimental verifications recalled earlier, the limitation considered persists to this day because inherent in the very structure of the theory.

In order to reach a description of the *extended* character of the hadrons, one has to pass to the so-called second quantization. Even at that level, the theory cannot represent the *actual shape* of the charge distribution considered.

As an example, there are indications that the shape of the charge distribution of a nucleon is not perfectly spherical, but is instead an oblate spheroidal ellipsoid along, say, the z-axis with values for the semiaxes for the proton [22]

$$b^2_x = b^2_y = 1, \quad b^2_z = 0.60, \tag{2.8}$$

which provide one (not necessarily unique) explanation of the anomalous magnetic moments of the nucleons.

It is evident that quantum mechanics *cannot* represent non-spherical shape (2.8), whether directly or indirectly. In fact, even passing to second quantization, the form factors only provide a remnant of the actual shape, and not a representation of the actual shape itself. At any rate, the rotational invariance of the theory would eliminate all nonspherical shapes of type (2.8).

In regard to the boson correlation, the above limitation implies that *quantum mechanics can only represent perfectly spherical fireballs* , with evident limitations in the quantitative description of the phenomenon considered. In fact, there is clear theoretical and experimental evidence indicating that the fireball of boson correlation is not perfectly spherical, but a prolate spheroidal ellipsoid oriented along the direction of the original p-$\bar{p}$ collision (see Fig. 1).

LIMITATION 2: LACK OF REPRESENTATION OF THE DEFORMATIONS OF EXTENDED CHARGE DISTRIBUTIONS . Once the lack of representation of the actual shape of a charge distribution is understood, one can see that *quantum mechanics is intrinsically unable to represent the possible variations and/or deformations of given charge distributions* , e.g., because prohibited by the underlying rotational symmetry.

But the fireball is expanding immediately upon its formation. The above limitation therefore implies *the inability of quantum mechanics to represent the evolution of the fireball of Bose-Einstein correlation* (see Fig. 1).

In addition to the above *evolution* , we also have a possible *deformation* of the fireball caused by external, sufficiently intense external fields. It is important in this respect to recall that *rigid bodies do not exist in the physical reality.* Thus, shape (2.8) of a proton or of a neutron cannot be assumed to be perfectly rigid. Evidently the *amount* of the deformation for given external forces and/or collisions is unknown at this writing, and only tentatively indicated by neutron interferometric experiments (see ref. [23] and quoted papers for the experimental profile, and ref. [15], Chap. VII for a preliminary treatment). However, the basic concept which is relevant for this paper is that the *existence* of the deformations of shape (2.8) under sufficiently intense external forces or collisions is simply beyond scientific doubts.

When considered in the context of boson correlations, this second limitation implies that *quantum mechanics can only represent fireballs which, besides being perfectly spherical , are also perfectly rigid.* The ensuing limitations of the theory are also evident. In fact, the basic rotational symmetry is known to characterize a theory of *rigid bodies*.

LIMITATION 3: LACK OF REPRESENTATION OF NONLOCAL NONPOTENTIAL INTERACTIONS. Above all, the most important limitation for Bose-Einstein correlation is *the inherent inability of quantum mechanics to represent the nonlocal interactions expected in the hyperdense medium in the interior of the fireball*.

As well known, in the atomic structure we have constituents at large mutual distances with no appreciable overlapping of their wavepackets. Under these conditions, we have the lack of appreciable nonlocal effect (as well as the lack of effect nonlinear in the wavefunction [24]), with consequential *exact* validity of quantum mechanics.

In the interior of a fireball we have instead physical conditions fundamentally different then those of the original conception of quantum mechanics, inasmuch as we have *constituents in conditions of total mutual penetration, overlapping and compression of their wavepackets one inside the others* . It is evident that the latter conditions result in the most general physical conditions and interactions that are conceivable by contemporary mathematical knowledge.

These interactions are composed by conventional local-potential-Hamiltonian (e.g., external electromagnetic) interactions, plus the interactions caused by the mutual penetration, overlapping and compression of the wavepackets. The latter ones generally are:

a) *nonlinear* in all variables, i.e., nonlinear in the local coordinates r and wavefunctions ψ and $\psi^\dagger$ as well as in their *derivatives* $\dot{r}$ (p), $\ddot{r}$ ($\dot{p}$), $\partial\psi$, $\partial\psi^\dagger$, ...;

b) *nonlocal* in all variables, in the sense of having an integral dependence on the coordinates and wavefunctions, as well as in their *derivatives*; and, last but not least;

c) *nonpotential-nonhamiltonian,* namely, such to violate the integrability conditions for the existence of a Hamiltonian representation, the so-called *conditions of variational selfadjointness* [3,5] as typical for all contact interactions of our physical reality, whether in classical or particle physics[3].

When seen within the context of boson correlation, the above features imply *the inherent impossibility of quantum mechanics to provide an effective description of the expected physical origin of the correlation*. In fact, the particles originating the fireball begin their interaction at large mutual distances and are generally uncorrelated. The correlation evidently originates in the expected nonlocal interactions inside the fireball. At any rate, what is significant here is the inability of conventional, action-at-a-distance interactions to provide a satisfactory interpretation of the correlation (see also Fig. 1).

LIMITATION 4: LACK OF REPRESENTATION OF THE BOSE-EINSTEIN CORRELATION FROM BASIC AXIOMS. Expectedly, *the very notion of correlation is outside the representational capabilities of the basic axioms of quantum mechanics*. Consider a system of n particles represented with the symbol k = 1, 2, ..., n, each one possessing correlated and uncorrelated components represented with

[3] The reader should be warned against not uncommon models based on the simplistic addition of an "integral potential" to the Hamiltonian for the evident attempt of preserving old knowledge, because they can be proved to be inconsistent on numerous counts. In fact, mathematically, these additions are not allowed by the underlying local-differential topology, and, physically, they imply granting a potential energy to contact interactions which cannot have any, thus resulting in unphysical trajectories (this latter point is clearly illustrated at the Newtonianlevel by the familiar drag forces which, if treated via a potential energy, imply majorphysical inconsistencies). As we shall see later, this is the ultimate physical motivation for our representation of contact nonlocal effects in the interior of the fireball via the generalized unit of the theory, that is, *with a quantity positively other than the Hamiltonian*.

the symbol a and b, respectively. Let the states be given by $| k,a > \times | k,b>$, k = 1, 2, ..., n. According to quantum mechanics, the correlation probability is evidently given by

$$C_n = \overbrace{\underbrace{<1,a|<1,b|\ldots\ldots <n,a|<n,b|}} \begin{pmatrix} |1,a> \\ |1,b> \\ \ldots \\ |n,a> \\ |n,b> \end{pmatrix} =$$

$$= \sum_k (<k,a|k,a> + <k,b|k,b>), \qquad (2.9)$$

namely, the above expression lacks exactly the cross terms $< k, a | k, b >$ representing the correlation.

In fact, the Bose–Einstein correlation is currently represented via empirical phenomenological models [1]. A theoretical inspection of these models reveals that, strictly speaking, they are outside structure (2.9) and, therefore, outside the axiomatic structure of quantum mechanics.

LIMITATION 5: LOSS OF THE BASIC SPACE–TIME SYMMETRIES AND RELATIVITIES UNDER NONLOCAL NONHAMILTONIAN FORCES. The historical open legacy of Bogoliubov, Fermi and others on the ultimate nonlocality of the interior of the strong interactions has profound epistemological, theoretical and mathematical implications, because it implies the inapplicability of all conventional space–time symmetries and relativities for a number of independent reasons studied in details in volumes [10,14,15].

We can here quote only the inapplicability due to:

5–a) the homogeneous and isotropic character of the basic medium of conventional relativities, empty space, as compared to the generally inhomogeneous and anisotropic character of all physical media, whether of classical or operator type;

5–b) the Lie–Hamiltonian character of the conventional relativities, as compared to the nonhamiltonian structure of the interactions considered;

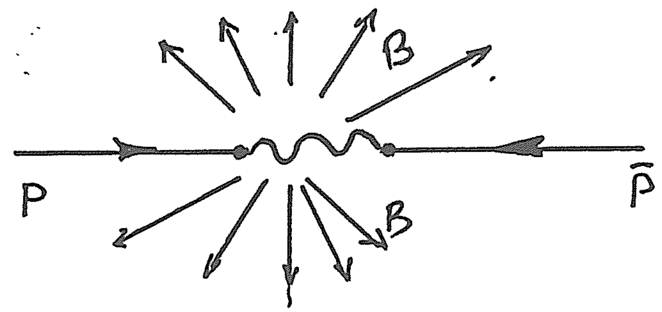
5–c) the local–differential character of the underlying topology (e.g., the Zeeman topology of the special relativity), as compared to the nonlocal–integral nature of the events considered; etc.

In conclusion, the viewpoint submitted in this paper is that

I) Quantum mechanics does indeed provide an exact description of the Bose–Einstein correlation during the approaching phase of the

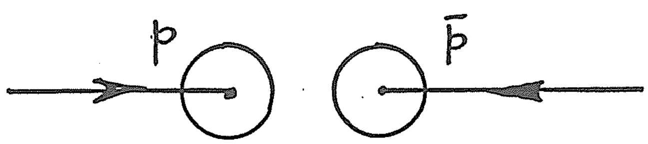
QUANTUM MECHANICAL DESCRIPTION OF CORRELATION

(1.1)



A MORE REALISTIC DESCRIPTION OF CORRELATION

(1.2)



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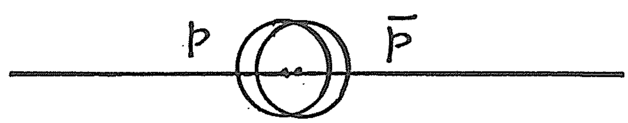

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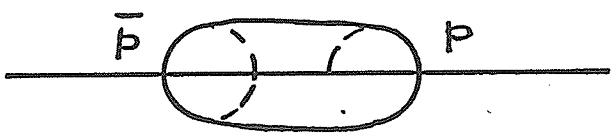

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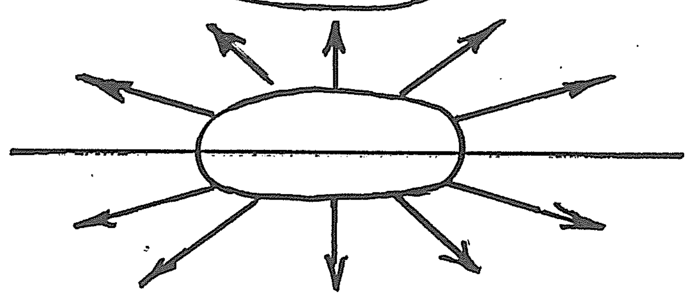

FIGURE 1: a schematic view of the quantum mechanical representation of the Bose-Einstein correlation (Diagram 1.1), and a

more realistic one suggested by available experimental information (Diagrams 1.2–1.5). In the quantum mechanical case the original proton and antiproton are represented as points. The correlation and production of the emitted bosons B is then reducible to virtual, action-at-a-distance exchanges, resulting in Limitations 1-5 pointed out in the text. In the physical reality, the proton and antiproton are extended charge distributions of radius 1Fermi (Diagram 1.2). Under very high energy, they annihilate in conditions of total mutual penetration and compression of their wavepackets (Diagram 1.3). This creates the fireball which is a prolate spheroidal ellipsoid oriented toward the original p–$\bar{p}$ direction (Diagram 1.4). This fireball rapidly expands and decomposes itself into the final correlated bosons B (Diagram 1.5). A satisfactory representation of the Bose-Einstein correlation must therefore be in a position to represent phases 1.2-1.5, as well as resolve Limitations 1-5 of the text from basic principles.

original p–$\bar{p}$ particles;

II) Quantum mechanics provides a description of boson correlations after the creation of the p–$\bar{p}$ fireball, which can only be valid in first approximation[4], and

III) A more accurate description of the phenomenon requires a suitable generalization of quantum mechanics itself.

3: CLASSICAL ISOTOPIC FORMULATIONS

No serious resolution of Limitations 1-5 of quantum mechanics of Sect. 2 can occur without first identifying the foundations at the purely classical level. This is the reason for the comprehensive classical studies of ref.s [11-15]. The identification of the operator formulation and of their compatibility with the classical profile is only consequential.

The starting point of the studies is the historical teaching by Lagrange and Hamilton according to which *physical reality cannot be*

[4] Under no circumstance the reader should think of quantum mechanics as being "wrong" under the broader conditions considered in this paper, because it remains "fully valid in first approximation". In fact, as we shall see better later on, quantum mechanics does indeed provide a first (although excessively crude) approximation of the Bose-Einstein correlation. Similarly, quantum mechanics should not be considered as being "violated" by nonlocal and nonhamiltonian interactions, but merely "inapplicable", in exactly the same way according to which Galilei's relativity is not "violated" but merely "inapplicable" under relativistic conditions. In the final analysis, quantum mechanics was conceived for physical conditions fundamentally different than those in the interior of the p–$\bar{p}$ fireball.

entirely represented via only one function, today called a *Lagrangian* or a *Hamiltonian*, because of the existence of forces (such as the contact forces of extended particles within physical media) and effects (such as the deformation of shape), which are physically and mathematically outside the notion of potential.

In fact, Lagrange and Hamilton formulated their historical equations with external terms, which were subsequently removed for reasons not yet analyzed by historians in the necessary details. For instance, the equations originally proposed by Hamilton are not those of the contemporary literature, but instead the expressions

$$\dot{r}_{ia} = \frac{\partial H(r, p)}{\partial p_{ia}}, \quad \dot{p} = - \frac{\partial H(r,p)}{\partial r_{ia}} + F_{ia}, i = x, y, z, \quad a = 1, 2, ..., n. \quad (3.1)$$

The functions $H = T + V$ then represent all action-at-a-distance, potential forces, while the contact forces due to motion within physical media are represented with the external terms F_{ka} (see ref. [14], Chap. I for historical details).

The need for isotopic formulations originates from the property that *the brackets of Hamilton's equations with external terms, not only violate the Lie algebra axioms, but actually violate the necessary conditions to characterize any algebra* (the scalar and distributive laws)[5]. The isotopies of conventional Hamiltonian mechanics (that without external terms) have been constructed for the objectives of preserving the Lie algebra structure, while possessing additional degrees of freedom equivalent to Hamilton's external terms.

These objectives were achieved via *axiom-preserving isotopies of contemporary mathematical structures*, such as fields, vector spaces, transformation theory, algebras, geometries, etc. [11, 12, 13].

The central idea of the classical formulations is the generalization of the trivial unit of conventional Lie's theory, $I = \text{diag.}\ (1, 1, ..., 1)$, into a matrix $\hat{1}$ of the same dimension, called (classical) *isounit*, with the most general possible nonlinear and nonlocal dependence on all quantities, such as time t, coordinates r, momenta p, accelerations[6] $\dot{p}/m$, as well as local density μ of the medium, its local temperature τ, its possible index of refraction n, etc, subject to the condition of preserving the original axioms of I, i.e., $\hat{1}$ is nowhere degenerate, Hermitean and positive-definite

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[5] See for details Appendix II.A. of Vol. I, ref. [14].

[6] Contact nonlocal and nonhamiltonian forces imply the appearance of acceleration dependent forces which, for this reason, are at times called *non-Newtonian* [15].

$$I = \mathrm{diag.}(1, 1, \ldots, 1) \quad \Rightarrow \quad \hat{I}(t, r, p, \dot{p}, \mu, \tau, n, .), \tag{3.2a}$$

$$\det I \neq 0, \quad I = \hat{I}, \quad I > 0 \quad \Rightarrow \quad \det \hat{I} \neq 0, \quad \hat{I} = \hat{I}^{\dagger}, \quad \hat{I} > 0. \tag{3.2b}$$

As shown in details in ref.s [14,15], the above isotopy then implies corresponding compatible isotopies of all mathematical structures used in mechanics, such as:

1) Isotopy of fields $F \Rightarrow \hat{F} = \{ \hat{n} \mid \hat{n} = n\,\hat{I},\ n \in F,\ \hat{I} = T^{-1} \}$, called *isofields*, with conventional sums and *isoproducts*

$$\hat{n}_1 * \hat{n}_2 = \hat{n}_1 T\, \hat{n}_2 = (n_1 n_2)\, \hat{I} \tag{3.3}$$

where T is called the *isotopic element*, under which $\hat{I}$ is the correct right and left unit of $\hat{F}$, $\hat{I} * \hat{n} = \hat{n} * \hat{I} \equiv \hat{n}, \forall\, \hat{n} \in \hat{F}$;

2) the isotopy of metric and pseudometric spaces $M(r,g,F)$ with local coordinates r and metric g over a field F

$$M(r,g,F) \Rightarrow \hat{M}(r,\hat{g},\hat{F}), \qquad \hat{g} = Tg, \qquad \hat{F} = F\,\hat{I}, \qquad \hat{I} = T^{-1}; \tag{3.4}$$

3) the isotopy of Lie's theory consisting of the isotopies of: universal enveloping associative algebras, Lie algebra, Lie groups, transformation theory, representation theory, etc.

4) the isotopy of the conventional symplectic, affine and Riemannian geometries into corresponding *isosymplectic, isoaffine and isoriemannian geometries*;

5) the isotopy of classical (conventional) Hamiltonian mechanics into a generalized mechanics tentatively called *Hamilton-isotopic mechanics*.

A rudimentary outline of the above results is important for the Bose-Einstein correlation because it shows that the capability of isotopic theories to represent extended and deformable particles under nonlocal and nonhamiltonian interactions originates at the purely classical level, and then simply persists under mapping to the corresponding operator formulations.

The basic carrier spaces of the classical nonrelativistic isotopies are the so-called *isoeuclidean spaces* $\hat{E}(r,\hat{\delta},\hat{\Re})$ which are characterized by the following isotopies of the conventional Euclidean space $E(r,\delta,\Re)$ over the reals $\Re$

$$E(r,\delta,\Re) \Rightarrow \hat{E}(r,\hat{\delta},\hat{\Re}), \tag{3.5a}$$

$$\delta = \text{diag.}\,(1,1,1) \Rightarrow \hat{\delta} = T\,\delta = \text{diag.}\,(b_1{}^2, b_2{}^2, b_3{}^2), \tag{3.5b}$$

$$r^2 = r_{ia}\,\delta_{ij}\,r_{ja} \quad \Rightarrow \quad r^{\hat{2}} = r_{ia}\,\hat{\delta}_{ij}\,r_{ja}, \tag{3.5c}$$

$$\mathfrak{R} \Rightarrow \hat{\mathfrak{R}} = \mathfrak{R}\,\hat{1}, \quad \hat{1} = T^{-1} = \hat{\delta}^{-1}, \tag{3.5d}$$

$$b_k = b_k(t, r, \dot{r}, \ddot{r}, \ldots) > 0, k = 1, 2, 3. \tag{3.5e}$$

where the diagonalization of the isotopic element is always possible owing to its positive-definiteness.

Despite the lifting of the separation into the most general possible nonlinear and nonlocal form (3.5c), *the isoeuclidean spaces $\hat{E}(r,\hat{\delta},\hat{\mathfrak{R}})$ are locally isomorphic to the conventional Euclidean space $E(r,\delta,\mathfrak{R})$ under the conditions $\hat{\delta} = T\,\delta$, $\hat{\mathfrak{R}} = \mathfrak{R}\hat{1}$, $\hat{1} = T^{-1} > 0$* .

Note also that, while the Euclidean space $E(r,\delta,\mathfrak{R})$ is unique, there exist an infinite number of geometrically equivalent, but physically inequivalent isotopes $\hat{E}(r,\hat{\delta},\hat{\mathfrak{R}})$, evidently due to the infinitely possible isometrics $\hat{\delta}$. This is requested to represent the infinitely possible interior physical conditions.

The "phase space" of the theory induced by $\hat{E}(r,\hat{\delta},\hat{\mathfrak{R}})$ is the so-called *iso-phase-space* (isocotangent bundle) in 6n coordinates which can be expressed via the unified notation [7]

$$T^*\hat{E}(r,\hat{\delta},\hat{\mathfrak{R}}): \quad a = (a^\mu) = (r, p) = (r_{ia}, p_{ia}), \mu = 1, 2, \ldots, 6n, \tag{3.6a}$$

$$\hat{\mathfrak{R}} = \mathfrak{R}\hat{1}, \; \hat{1} = \text{diag.}\,(\hat{\delta}^{-1}, \hat{\delta}^{-1}), \tag{3.6b}$$

A central objective of the formulations is to achieve a geometric, analytic and algebraic characterization on $T^*\hat{E}(r,\hat{\delta},\hat{\mathfrak{R}})$ of the most general known nonlinear, nonlocal and nonhamiltonian vector-fields verifying certain topological conditions (analyticity and regularity in all variables)

$$\dot{a} = \Gamma(t, a, \dot{a}) = \left\{ \begin{array}{l} p_{ia} / m_a \\ F^{SA}{}_{ia}(r) + F^{NSA}{}_{ia}(t, r, p) + \int_\sigma d\sigma\, \mathcal{F}^{NSA}(\sigma, t, r, p, ..) \end{array} \right\} \tag{3.7}$$

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[7] For simplicity of notation, we shall use lower indices only in isoeuclidean spaces (3.5), and preserve the distinction between upper and lower indices in iso-phase-space (3.6).

where: SA (NSA) stands for the verification (violation) of the conditions of variational selfadjointness for the existence of a potential [5], σ is the surface of the body considered, $F^{SA}(r)$ represents conventional local-potential forces, F^{NSA} represents forces still local-differential, but not derivable from a potential or a Hamiltonian[8], and $\mathcal{F}^{NSA}$ represents the correction of the trajectory of the center-of-mass caused by the actual shape σ of the body considered.

The primary aspects of the Hamilton-isotopic mechanics can be outlined as follows:

ISOSYMPLECTIC STRUCTURE (ref. [14], Sect. II.9). The conventional, linear and local transformations $r' = A\ r$ are no longer applicable to isospaces $\hat{E}(r,\hat{\delta},\hat{\mathfrak{R}})$ (e.g., because they would violate transitivity and linearity), and must be generalized into the *isotransformations*

$$r' = A * r = A\ T\ r = A\ T(t, r, \dot{r}, \ddot{r}, ...)\ r \tag{3.8}$$

which are linear and local at the abstract level (in which all distinctions between $A\ r$ and $A * r$ cease to exist), but are nonlinear and nonlocal when projected in the original space $E(r,\delta,\mathfrak{R})$. For these reasons transformations (3.8) are called *isolinear and isolocal*.

A consequence is that the conventional differential calculus with familiar expression $dr' = A\ dr$ is inapplicable to the isocotangent bundle $T^*\hat{E}(r,\hat{\delta},\hat{\mathfrak{R}})$, and must be lifted into the so-called *isodifferential calculus* for which

$$\hat{d}r' = A * \hat{d}r, = A\ T\ \hat{d}r = A\ T(t, r, \dot{r}, \ddot{r}, ...)\ \hat{d}r, \tag{3.9}$$

where the $\hat{d}$'s are called *isodifferentials*[9].

Isospaces (3.6) can be first equipped with the *canonical one-*

[8] One may think at the drag forces experienced nowadays by missiles in atmosphere which are proportional to the tenth power of the velocity and more. These systems are manifestly nonhamiltonian, but they always admit a Birkhoffian representation (see below) [10].

[9] All conventional operations on a continuous manifold are then generalized in a compatible way. We shall encounter in this paper only the *isoexponentiation* and a few other generalized operations. For the remaining ones (particularly for the crucial isotopy of the Dirac's delta function) we have to refer the reader to the quoted literature, e.g., ref.s [9].

isoforms

$$\hat{\Phi}_1 = \hat{\Re}^\circ \hat{*} \hat{d}a = R^\circ_\mu \hat{T}_1{}^\mu{}_\nu(t, a, \dot{a}, ...) \hat{d}a^\mu = R^\circ_\mu b_\mu{}^2 \hat{d}a^\mu =$$

$$= p * \hat{d}r = p_{ka} b_k{}^2(t, r, p, \dot{p}, ...) \hat{d}r_{ka}, \quad \Re^\circ = (p, 0), \tag{3.10}$$

in which case they are denoted with the symbol $T^*\hat{E}_1(r,\hat{\delta},\hat{\Re})$.

Note that isoforms (3.10) coincide with the conventional one-form $\Phi_1 = \Re^\circ da = p\, dr$ at the abstract, realization-free level, by construction. This implies that all nonlinear, nonlocal and nonhamiltonian terms are embedded in the isotopic element $\hat{T}_1$, as now familiar.

The *isoexterior derivative* of iso-one form (3.10) is given by

$$\hat{\Phi}^\circ_2 = \hat{d}\hat{\Phi}^\circ_1 = \tfrac{1}{2}[\omega_{\mu_1\mu_2} b_{\mu_1}{}^2 b_{\mu_2}{}^2 +$$

$$+ (R^\circ_{\mu_2} \frac{\partial b_{\mu_2}{}^2}{\partial a^{\mu_1}} b_{\mu_1}{}^2 - R^\circ_{\mu_1} \frac{\partial b_{\mu_1}{}^2}{\partial a^{\mu_2}} b_{\mu_2}{}^2)]\, \hat{d}a^{\mu_1} \hat{\wedge} \hat{d}a^{\mu_2} \tag{3.11}$$

and it always admits the factorization

$$\hat{\Phi}_2 = \hat{d}\hat{\Phi}_1 = \tfrac{1}{2} \omega_{\mu_1\nu} \times \hat{T}_2{}^\nu{}_{\mu_2}(t, x, \dot{x}, ...) \hat{d}a^{\mu_1} \hat{\wedge} \hat{d}a^{\mu_2} \tag{3.12}$$

where $\omega_{\mu\alpha}$ is the familiar canonical symplectic tensor

$$(\omega_{\mu\alpha}) = (\frac{\partial R^\circ_\nu}{\partial a^\mu} - \frac{\partial R^\circ_\mu}{\partial a^\nu}) = \begin{pmatrix} 0_{3n\times3n} & -I_{3n\times3n} \\ I_{3n\times3n} & 0_{3n\times3n} \end{pmatrix} \tag{3.13}$$

and

$$\hat{T}_2{}^{\mu_1\mu_2} = b^{\mu_1 2} b^{\mu_2 2} + \omega^{\mu_1\alpha} \omega^{\mu_2\beta} (R^\circ_\beta \frac{\partial b_\beta{}^2}{\partial a^\alpha} b_\alpha{}^2 - R^\circ_\alpha \frac{\partial b_\alpha{}^2}{\partial a^\beta} b_\beta{}^2)$$

$$= \mathrm{diag.}\,(\hat{g}, \hat{g}). \tag{3.14}$$

In different words, the conventional, canonical symplectic structure ω can always be factorized, thus leaving all nonlinear and nonlocal terms embedded in the isotopic element $\hat{T}_2$. In turn, this implies the possibility of preserving the conventional, local-differential topology of the symplectic geometry under our isotopic representation of nonlocal-integral systems, because all geometries are insensitive to the structure of their own unit, once positive-definite[10].

An *isosymplectic manifold* is then given by isospaces (3.6) equipped with a two-isoform, and will be indicated with the symbol

$$T^*\hat{E}_2(r,\hat{g},\hat{\mathfrak{R}}), \quad \hat{g} = T_2\delta, \quad \hat{\mathfrak{R}} = \mathfrak{R}\,\hat{1}_2, \quad \hat{1}_2 = \hat{T}_2^{-1} = \mathrm{diag.}\,(\hat{g}^{-1}, \hat{g}^{-1}). \qquad (3.15)$$

The *isosymplectic geometry* in isocanonical realization is then given by the isospace $T^*\hat{E}_2(r,\hat{g},\hat{\mathfrak{R}})$ equipped with the canonical two-isoform (3.12). The reader should keep in mind the change of isometric and, consequently, of isounit in the transition from one- to two-isoforms (which is absent in the conventional case).

A fundamental property of the isosymplectic geometry is that it possess a consistent *isopoincaré lemma*, i.e.,

$$\hat{d}\,\hat{\Phi}_2 = \hat{d}\,(\hat{d}\,\hat{\Phi}_1) \equiv 0, \qquad (3.16)$$

which confirms the consistency of the isosymplectic geometry as an "isotopy" of the conventional symplectic geometry.

This is not a mere formal property, inasmuch as *property (3.16) provides the integrability conditions for the contravariant version* $\hat{\omega}^{\mu\nu}$ *of the isosymplectic tensor* $\hat{\omega}_{\mu\nu} = \omega_{\mu\alpha}\, T_2{}^{\alpha}{}_{\nu}$ *to verify the Lie algebras axioms* (see below in this section).

Systems (3.7) are then called *Hamilton-admissible vector-fields* when there exists a function $H = T + V$ on $T^*\hat{E}_2(r,\hat{g},\hat{\mathfrak{R}})$ such that (see below for explicit realizations)

$$\omega\, \hat{T}_2 \rfloor \Gamma = -\,\hat{d}\, H \qquad (3.17)$$

The isotopic lifting of all remaining properties of the conventional symplectic geometry then follows[11].

[10] These mathematical results are nontrivial from the viewpoint of the physical effectiveness of the theory because conventional integral topologies are rather complex indeed and of difficult practical use.

[11] In ref. [14], Sect. II.9 we developed the most general possible isosymplectic geometry in local coordinates, that with a factorization of the Birkhoffian, rather than the canonical tensor, in which case the associated mechanics is called *Birkhoff-isotopic* rather than Hamilton-isotopic. This broader geometry is excessively

It is evident that *the isosymplectic geometry is a bona-fide integrodifferential generalization of the conventional symplectic geometry, it admits the latter as a particular case for T_1= diag. (1, 1, 1) and it coincides with the latter at the abstract level by construction*.

ISOANALYTIC STRUCTURE (see ref. [14], Sect. II.7). It is based on the representation of systems (3.7) via the *isocanonical principle*

$$\delta \hat{A} = \delta \int_{t_1}^{t_2} dt\,(p * \dot{r} - H) = \delta \int_{t_1}^{t_2} dt\,[\, p\, T_1(t, r, p, \ldots)\, \dot{r} - H\,] =$$

$$= \delta \int_{t_1}^{t_2} dt\,(R^\circ_\mu\, \hat{T}_1{}^\mu{}_\nu\, \dot{a}^\nu - H), \quad \mathfrak{R}^\circ = (p, 0), \hat{T}_1 = \text{diag.}\,(T_1, T_1) \qquad (3.18)$$

where one recognizes in the integrand the one-isoform (3.10).

The corresponding analytic equations, called *covariant Hamilton-isotopic equations*, can be written

$$\omega_{\mu\alpha}\, \hat{T}_2{}^\alpha{}_\nu\, \dot{a}^\nu = \frac{\partial H(a)}{\partial a^\mu}, \qquad (3.19)$$

where $\omega_{\mu\alpha}$ is the canonical-symplectic tensor and $\hat{T}_2$ ($\neq \hat{T}_1$) is given by Eq.s (3.14).

The *contravariant Hamilton-isotopic equations* are then given by

$$\dot{a}^\mu = \hat{I}_2{}^\mu{}_\alpha\, \omega^{\alpha\nu} \frac{\partial H}{\partial a^\nu} \qquad \left\{ \begin{array}{ll} \dot{r}_{ia} = I_{2iaja} \dfrac{\partial H}{\partial p_{ja}}, & (3.20a) \\[2ex] \dot{p}_{ia} = - I_{2iaja} \dfrac{\partial H}{\partial r_{ja}}, & (3.20b) \end{array} \right.$$

where

$$\hat{I}_2 = \text{diag.}\,(I_2, I_2) = \text{diag.}\,(T_2^{-1}, T_2^{-1}) = \text{diag.}\,(\hat{g}^{-1}, \hat{g}^{-1}) \qquad (3.21)$$

and

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general for our needs in this paper and will be ignored hereon. Nevertheless, this broader geometry is important to establish the "direct universality" (see footnote[2]) of our isosymplectic geometry for all nonlinear, nonlocal and nonhamiltonian systems (3.7) verifying the needed continuity restrictions.

$$(\hat{1}_a{}^{\mu}{}_{\alpha}\, \omega^{\alpha\nu}) = (\omega_{\mu\alpha}\, \hat{T}_2{}^{\alpha}{}_{\nu})^{-1}, \tag{3.22}$$

The *isotopic Hamilton-Jacobi equations* are then given by

$$\frac{\partial \hat{A}}{\partial t} + H = 0, \qquad \frac{\partial \hat{A}}{\partial r_{ka}} = T_{kaia}\, p_{ia}, \qquad \frac{\partial \hat{A}}{\partial p_{ka}} = 0, \tag{3.23}$$

where one should note for the isotopic quantization of the next section that $\partial \hat{A} / \partial p \equiv 0$ [12].

An *isoanalytic representation* of a given system $\Gamma = (\Gamma^{\mu}(t, a, \dot{a}, ...))$ of Eq.s (3.7) in terms of covariant Eq.s (3.19) is achieved via the techniques of the *inverse problem of Newtonian mechanics* [5,10], and holds when one constructs a Hamiltonian $H = T + V$ and an isotopic elements T_2 in $T^*\hat{E}_2(r,\hat{g},\hat{\mathfrak{R}})$ from the given vector-field, such that the following equalities hold identically

$$\omega_{\mu\alpha}\, \hat{T}_2{}^{\alpha}{}_{\nu}(t, a, \dot{a}, ..)\ \Gamma^{\nu}(t, a, \dot{a}, ...) = \frac{\partial H(a)}{\partial a^{\mu}}, \tag{3.24}$$

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[12] This property can be understood despite the arbitrary dependence of the integrand, by nothing that, from a geometric viewpoint, there is no distinction between the conventional one-form $R^{\circ}\, da = R^{\circ}_{\mu}\, \delta^{\mu\nu}\, da_{\nu} = R^{\circ}_{\mu}\, da^{\mu}$, and its isotope $R^{\circ} \bullet da = R^{\circ}_{\mu} \hat{g}^{\mu\nu}\, da_{\nu} = R^{\circ}_{\mu}\, da^{\nu}$. This implies different meanings and expressions for the *same* covariant or contravariant quantity, evidently depending on the explicit form of the metric. On rigorous grounds, Eq.s (3.19) should be derived from the *Birkhoffian Hamilton-Jacobi equations* (ref. [10], p. 205 ff), for which isoaction (3.14) becomes

$$\hat{A} = \int_{t_1}^{t_2} dt\, (R_{\mu}\, \dot{a}^{\mu} - H), \qquad R = (pT_1, 0) \tag{a}$$

and Hamilton-Jacobi equations (3.23) take the unified form

$$\frac{\partial \hat{A}}{\partial t} + H = 0, \qquad \frac{\partial \hat{A}}{\partial a^{\mu}} = R_{\mu}. \tag{b}$$

The reader should be aware that, in their general form in which the Birkhoffian functions R_{μ} are arbitrary, $R \neq (pT, 0)$, the above equations imply in general that $\partial \hat{A} / \partial p_k \neq 0$, with consequential problems in the construction of an operator image (see next section). In reality, the achievement of a structure with $\partial \hat{A}/\partial p \equiv 0$ is the fundamental reason for the *restriction* of the general formulation of Birkhoffian mechanics of ref. [10] to the isotopic form with $R = (pT_1, 0)$ considered in this paper.

which provides the realization in local coordinates of the abstract geometric notion (3.17).

Thus, the isoanalytic representation holds when all local-potential forces are represented via the Hamiltonian H = T + V, and all nonlocal-integral and nonpotential-nonhamiltonian forces F^{NSA} and $\mathcal{F}^{NSA}$ are represented via the isotopic element $\hat{T}_2$, exactly as desired (see below for examples).

The isotopic liftings of all remaining aspects of conventional Hamiltonian mechanics then follow. Note that *the Hamilton-isotopic mechanics is a covering of the conventional one; it admits the latter as a particular case for $T_1 = T_2 = I = diag.\ (1, 1, 1)$; and it coincides with the latter at the abstract level*.

ISOLIE STRUCTURE (ref. [14], Sect. II.6). The brackets underlying the contravariant Hamilton-isotopic equations can be written

$$[A \hat{,} B] = \frac{\partial A}{\partial a^\mu} \hat{1}_2{}^\mu{}_\alpha\, \omega^{\alpha\nu} \frac{\partial B}{\partial a^\nu} =$$

$$= \frac{\partial A}{\partial r_{ia}}\, \hat{1}_{2iaja}(t, r, p, ..) \frac{\partial B}{\partial p_{ja}} - \frac{\partial B}{\partial r_{ia}}\, \hat{1}_{2iaja}(t, r, p, ..) \frac{\partial A}{\partial p_{ja}}, \quad (3.25)$$

$$\mu, \nu = 1, 2, ..., 6n, \quad i, j = 1, 2, 3, \quad a = 1, 2, ..., 6n,$$

whose verification of the Lie algebra axioms is ensured by the validity of the isopoincaré lemma, Eq.s (3.16). The emerging algebras are then called *Lie-isotopic algebras* [3].

The transformation groups as per isotopic rule (3.8) generated by algebras (3.25) has the structure

$$\hat{f}(w) = \{\textstyle\prod_k \exp [\, w_k\, \hat{1}_2{}^\mu{}_\alpha\, \omega^{\alpha\nu} (\partial X_k / \partial a^\nu) (\partial / \partial a^\mu)]\} \hat{1} \quad (3.26)$$

verify the generalized laws

$$\hat{f}(w) * \hat{f}(w') = \hat{f}(w') * \hat{f}(w) = \hat{f}(w + w'), \quad (3.27a)$$

$$\hat{f}(w) * \hat{f}(- w) = \hat{1}_2, \quad (3.27b)$$

$$\hat{f}(0) = \hat{1}_2 \quad (3.27c)$$

and are called *Lie-isotopic groups* [3].

The isotopies of the various aspects of the conventional formulation of Lie's theory then follow.

Note the explicit appearance of the isounit in the structure of the brackets of the algebras, Eq.s (3.25), and of the groups, exponentiations (3.26). Note also that conventional local-differential topologies can be preserved owing, again, to the insensitivity of Lie's theory to the structure of its unit, when positive-definite. Finally, note the embedding of all nonlocal and nonhamiltonian interactions in the isounits of the theory.

It is evident that *the above classical formulation of the Lie-isotopic theory is a covering of the conventional canonical formulation; it admits the latter as a particular case for $\hat{I}_2 = I$; and it coincides with the latter by construction at the abstract level*.

The physical applications of the classical nonrelativistic isotopies are numerous and intriguing, such as:

a) the direct representation of the shape of the particle considered (say, ellipsoid (2.8)) via the isounit $\hat{I}_2 = \text{diag.}\ (b_1^2, b_2^2, b_3^2)$ at the purely classical level;

b) the direct representation of the deformation of said shape via the isotopy of the isotopy $\hat{I}_2 \Rightarrow \hat{I}'_2 = \text{diag.}\ (b'_1{}^2, b'_2{}^2, b'_3{}^2)$, where the b'_k's are now function of an external quantity, such as pressure or intensity of an external field, etc.;

c) the direct representation of the inhomogenuity of the medium in which motion occurs, e.g., via a dependence of the b's from the local density, and of the anisotropy of the medium via the factorization in the isometric of the direction of intrinsic angular momentum of the medium considered;

d) the direct representation of nonlocal-nonhamiltonian forces via the isounits $\hat{I}_2$;

e) the reconstruction of the exact, conventional, rotational symmetry for all possible ellipsoidical deformations of the sphere; the reconstruction of the exact Galilean symmetry under nonlinear, nonlocal and nonhamiltonian interactions; and others.

As a simple example, consider an extended-deformable particle moving within a physical medium under no potential force, but experiencing a quadratic resistive force with nonlocal corrections due to its shape. This system admits the nonlinear, nonlocal and nonhamiltonian equation of motion

$$\dot{a} = \Gamma = \begin{pmatrix} p / m \\ - \gamma \ \dot{r}^2 \int_\sigma d\sigma \ \mathcal{F}(\sigma,..) \end{pmatrix} \qquad (3.28)$$

and can be represented in isospace $T^*\hat{E}_2(r,\hat{g},\hat{\mathfrak{R}})$ via the isometric

$$\hat{g} = T\delta = \hat{\delta} \exp \{ \gamma \ r \int_\sigma d\sigma \ \mathcal{F}(\sigma, ...) \} > 0, \quad \hat{\delta} = \text{diag.} (b_1^2, b_2^2, b_3^2), \quad (3.29)$$

and the Hamiltonian

$$H = p^{\hat{2}} / 2m = p \, \hat{g} \, p / 2m \qquad (3.30)$$

Note: the abstract identity of Hamiltonian (3.30) with the conventional one; the representation of the resistive forces with the isounit and *not* with the Hamiltonian; the embedding of the nonlocal terms in the isometric; the direct representation of the actual shape of the particle, say a prolate ellipsoid in the direction of motion as well as all its possible deformations, via the factor $\hat{\delta}$ in the isometric; and, finally, the restoration of the exact rotational O(3) and Galilean G(3.1) symmetry for the system considered, of course, at our isotopic level, because of the positive-definiteness of the isometric (3.29).

A virtually endless number of examples can then be constructed via any desired combination of local-Hamiltonian and nonlocal-nonhamiltonian forces (see ref.s [14.15]) via an arbitrary positive-definite isounit and the Hamiltonian on $T^*\hat{E}_2(r,\hat{g},\hat{\mathfrak{R}})$

$$\hat{1}_2 = \text{diag.} (\hat{g}^{-1}, \hat{g}^{-1}), \qquad (3.31a)$$

$$H = \textstyle\sum_a (p^{\hat{2}} / 2m_a + V(\hat{r}_{ab}) = \sum_{ija}(p_{ia} \, \hat{g}_{ij} \, p_{ja} / 2m_a + V(\hat{r}_{ab}), \quad (3.31b)$$

$$\hat{r}_{ab} = | \, (r_{ia} - r_{ib}) \, \hat{g}_{ij} \, (r_{ja} - r_{jb}) \, | ^{\frac{1}{2}} \qquad (3.31c)$$

which is manifestly invariant under the isogalilean symmetry $\hat{G}_{\hat{1}_2}(3.1)$ constructed with respect to the isounit $\hat{1}_2$ [15].

Finally, note the preservation of the original teaching by Lagrange and Hamilton. In fact, the representation of a particle in the original equations (3.1) requires the knowledge of four functions, the Hamiltonian $H = T + V$, and the three components F_k of the external force. The representation of the same particle with the Hamilton-isotopic equations (3.20) also requires four quantities, the same Hamiltonian H and the three diagonal elements $b_k{}^2$ of the isounit $\hat{g}$.

4: ISOTOPIC QUANTIZATION

The conventional naive quantization of classical Hamiltonian mechanics can be performed via the mapping of the conventional canonical action A into the expression $-i\hbar \log|\psi>$,

$$A = \int_{t_1}^{t_2} dt [R^\circ_\mu \dot{a}^\mu - H] \quad \Rightarrow \quad -i\hbar \log|\psi>, \tag{4.1}$$

under which the conventional Hamilton-Jacobi equations, say, for one particle in Euclidean space $E(r,\delta,\mathfrak{R})$

$$\frac{\partial A}{\partial t} + H = 0, \qquad \frac{\partial A}{\partial r_i} = p_i, \quad \frac{\partial A}{\partial p_i} = 0, \tag{4.2}$$

are mapped into Schrödinger's equations

$$i\hbar \frac{\partial}{\partial t}|\psi> = H|\psi>, \qquad p_k|\psi> = -\hbar \nabla_k|\psi> \tag{4.3}$$

More rigorous methods are those of symplectic quantization (see, e.g., ref. [20]).

Naive quantization (4.1) is no longer applicable to the isotopic actions (3.18), evidently because of their generalized structure. The isotopy of the above quantization submitted in ref.s [9,19] is based on the generalization of the trivial unit I of quantum mechanics into a generalized operator unit $\hat{1}$ of ref. [4] which is nonsingular, Hermitean and positive-definite, but possesses otherwise the most general known, nonlinear and nonlocal dependence in all possible or otherwise needed quantities, such as: time t, coordinates r, their derivatives $\dot{r}$ (or p), $\ddot{r}$ (or

$\dot{p}$) , the wavefunctions ψ and their conjugate $\psi^\dagger$, their derivatives $\partial\psi$ and $\partial\psi^\dagger$, as well as the local density μ, temperature τ, etc.,

$$\hbar \quad \Rightarrow \quad \hat{1} = \hat{1}(t, r, p, \dot{p}, \psi, \psi^\dagger, \partial\psi, \partial\psi^\dagger, \mu, \tau, ...), \tag{4.4a}$$

$$\det \hat{1} \neq 0, \quad \hat{1} = \hat{1}^\dagger, \quad \hat{1} > 0, \tag{4.4b}$$

Mapping (4.1) is then lifted into the *naive isotopic quantization* , also called *hadronization*,

$$\hat{A} = \int_{t_1}^{t_2} dt [R^\circ_\mu \hat{T}_1{}^\mu{}_\nu \dot{a}^\nu - H] \quad \Rightarrow \quad -i\,\hat{1}\log|\hat{\psi}>, \tag{4.5}$$

under which the isotopic Hamilton–Jacobi equations (3.23), i.e.,

$$\frac{\partial \hat{A}}{\partial t} + H = 0, \qquad \hat{1}_{ki}\frac{\partial \hat{A}}{\partial r_i} = p_k, \quad \hat{1}_{ki}\frac{\partial \hat{A}}{\partial p_i} = 0, \tag{4.6}$$

are mapped into the forms

$$i\frac{\partial}{\partial t}|\hat{\psi}> = H^{eff}\, T\,|\hat{\psi}> = H^{eff} * |\hat{\psi}> \tag{4.7a}$$

$$-i\,\hat{1}_{ki}\nabla_i\,|\hat{\psi}> = p^{eff}{}_k\, T\,|\hat{\psi}> = p^{eff}{}_k * |\psi>, \tag{4.7b}$$

where

$$H^{eff} = H - i\,[\,(\partial\hat{1}\,/\,\partial t)\log|\hat{\psi}>\,]\,\hat{1}, \tag{4.8a}$$

$$p^{eff}{}_k = p_k + i\,[\,\hat{1}_{ki}\,(\nabla_i\,\hat{1})\log|\hat{\psi}>\,)\,\hat{1}. \tag{4.8b}$$

One can therefore see in this way the appearance of an operator formulation with an essential isotopic structure, as necessary to constitute a true operator image of the classical formulations. A more rigorous mapping is provided via the isotopy of the symplectic quantization [20].

For $\hat{1} = \hbar$ one evidently recovers quantum mechanics. In fact, conventional equations (4.3) can also be written in the isotopic form

$$i \frac{\partial}{\partial t} | \psi > \; = \; H * | \psi > \; = \; H (\hbar^{-1}) | \psi >, \qquad (4.9a)$$

$$- i \nabla_k | \psi > \; = \; p_k * | \psi > \; = \; p_k (\hbar^{-1}) | \psi >. \qquad (4.9b)$$

This essentially indicates that quantum mechanics itself can be formulated in a way admitting an isotopic structure with the isotopic element $T = \hbar^{-1}$.

We should finally note the crucial role of isotopic action (3.18) and related isotopic Hamilton-Jacobi equations (3.23) for the construction of the desired operator image. In fact, the assumption of the most general possible first-order action

$$\hat{A} = \int_{t_1}^{t_2} dt \, [R_\mu(a) \dot{a}^\mu - H], \qquad R_\mu \neq (pT_1, 0), \qquad (4.10)$$

would lead to the general Birkhoffian form of Hamilton-Jacobi equations (ref. [10], p. 205 ff) for which $\partial \hat{A} / \partial p \neq 0$ (see also footnote[12]). In turn, the operator image of such a general theory would require the generalization of the conventional wave functions $\psi(t,r)$ into forms of the type $|\psi[t, r, p)$ which are mostly unknown at this writing.

We reach in this way the conclusion that, besides the capability of representing extended and deformable objects under nonlocal and nonhamiltonian interactions, *the isotopies of classical Hamiltonian mechanics have the additional fundamental role of selecting the most general possible functional dependence* $\hat{R}^\circ = (pT(t, r, p, ..), 0)$ *whose operator image implies wavefunctions* $\psi(t, r)$ *independent from the momenta*.

The above comments also illustrate the reasons why the Birkhoffian, step-by-step generalization of Hamiltonian mechanics of monograph [5,10] resulted to be basically insufficient for the achievement of a consistent isotopic quantization, by therefore calling for the the additional laborious task of constructing a further step-by-step generalization of the Birkhoffian-isotopic type [14,15].

5: ELEMENTS OF HADRONIC MECHANICS

A generalization of quantum mechanics under the name of *hadronic mechanics*, was suggested by the author [4], subsequently studied by several researchers [6], and finally reached sufficient maturity for applications in ref.s [7,8,9].

Proposal [4] is essentially that of *building hadronic mechanics as an isotopy of quantum mechanics*, in such a way to possess the Lie-isotopic structure identified in the preceding memoir [3] [13].

A primary objective of the proposal is the direct representation of extended and therefore deformable particles under conventional local, potential and Hamiltonian interactions, as well as additional, nonlinear, nonlocal and nonhamiltonian internal, short range effects.

A quantitative study of the Bose-Einstein correlation via hadronic mechanics appears recommendable to attempt a deeper understanding of the correlation itself, to illustrate the possibilities of the new mechanics, as well as to confront with experimental data the fundamental assumption of the theory: the historical legacy on the nonlocality of strong interactions.

The central notion of hadronic mechanics, from which all other aspects can be derived, is the generalization of Planck's constant $\hbar$ into the integrodifferential operator $\hat{1}(t, r, p, \dot{p}, \psi, \psi^\dagger, \partial\psi, \partial\psi^\dagger, \mu, \tau, ...)$ of Eq.s (4.4). The assumption of the this quantity as the algebraic unit of the theory requires a necessary compatible generalization of the entire structure of quantum mechanics. The emerging generalized mechanics is reducible to the following primary mathematical structures:

Â) The *universal, enveloping, isoassociative, operator algebra* $\hat{\xi}$, which coincides with envelope ξ of Eq.s (2.1) as vector space (namely, the elements of $\hat{\xi}$ are the same as those of ξ) [14], but it is now equipped with the new product

$$\hat{\xi}: \quad A*B \overset{\text{def}}{=} ATB, \quad T = \text{fixed}, \qquad \hat{1} = T^{-1}, \tag{5.1}$$

permitting $\hat{1}$ to be the correct right and left unit of the theory, i.e.,

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[13] Proposal [4] was actually that of building hadronic mechanics as a *genotopy* [3] of quantum mechanics resulting in the more general *Lie-admissible algebras* for the representation of *open-nonconservative conditions*, such as one proton in the core of a collapsing star considered as external. In this paper we study only *closed-conservative conditions*, such as isolated composite systems with nonhamiltonian internal forces, which are represented by the simpler *isotopy* and related *Lie-isotopic algebras*.

[14] This is technically due to the property that *the basis of a linear space remains unchanged under isotopies*. See ref. [14], Sect. II.3.

$$\hat{1}*A \equiv A*\hat{1} \equiv A \quad \forall \ A \in \hat{\xi}, \tag{5.2}$$

in which case $\hat{1}$ is called the *isounit*.

$\hat{B}$) The *isofields* $\hat{F}$ consisting of the *isoreals* $\hat{\mathfrak{R}}$ or *isocomplex numbers* $\hat{C}$

$$\hat{F} = \{\hat{N} \,|\, \hat{N} = N\hat{1},\ N \in F,\ \hat{1} \in \hat{\xi}\},\quad F = \mathfrak{R}, C \tag{5.3}$$

where the quantities $\hat{N}$, called *isonumbers*, verify the conventional sum but the isotopic multiplication

$$\hat{N}_1 + \hat{N}_2 = (N_1 + N_2)\hat{1}, \tag{5.4a}$$

$$\hat{N}_1 * \hat{N}_2 = (N_1 N_2)\hat{1}, \tag{5.4b}$$

Note that the isoproduct of an isonumber $\hat{N}$ by a quantity Q coincides with the conventional multiplication, $\hat{N}*Q \equiv NQ$. As a result, the final "numbers" of hadronic mechanics are the conventional ones (see later on the isoeigenvalue equations $H*|\psi> = \hat{N}*|\psi> \equiv N\,|\psi>$)[15].

$\hat{C}$) The *isohilbert space* $\hat{\mathcal{H}}$ which also coincides with $\mathcal{H}$ as vector space (except for different renormalizations), but is now equipped with the composition

$$\hat{\mathcal{H}}: \quad <\psi \,\hat{|}\, \phi> \stackrel{def}{=} \hat{1} <\psi|G|\hat{\phi}> =$$

$$= \hat{1}\int dr'\, \psi^\dagger(t, r')\, G(t, r', p', \psi, \psi^\dagger, \partial\psi, \partial\psi^\dagger, \mu, \tau, ...)\, \hat{\phi}(t, r') \in \hat{C}, \tag{5.5a}$$

$$\det G \neq 0, \quad G = G^\dagger, \quad G > 0, \tag{5.5b}$$

where G is generally different than T, and dr is a suitable invariant measure.

The above structures imply a generalization of each and every aspect of conventional quantum mechanics. As an example, we have the

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[15] The reader should be aware of the unifying power of the isotopies and of its implications. As an example, the infinite class of isoreals $\hat{\mathfrak{R}} = \mathfrak{R}\,\hat{1}$ includes as particular case all existing fields of characteristic zero (real numbers, complex numbers and quaternions) as well as all their isotopies, owing to the arbitrariness of the isounit $\hat{1}$ (ref. [14], Sect. II.2). As a result, hadronic mechanics on isoreals $\hat{\mathfrak{R}}$ implies a formulation of quantum mechanics on quaternions [14]. For simplicity, we shall use in this paper the simpler notion of isoreals $\hat{\mathfrak{R}} = \mathfrak{R}\hat{1}$ (isocomplex $\hat{C} = C\hat{1}$) in which the complex and quaternionic (real and quaternionic) realizations are excluded.

isoheisenberg's equation submitted in the original proposal [4]

$$i\dot{Q} = [Q \hat{,} H]_{\xi} = Q*H - H*Q = QTH - HTQ, \qquad (5.6)$$

which is characterized by the so-called *Lie-isotopic algebras* $\hat{L}$ [3,9,10,11] homomorphic to the antisymmetric algebra attached to ξ, $\hat{L} \approx \xi^{-1}$, with consequential Lie-isotopic theory.

The equations equivalent to (5.6) were identified subsequently [7,25], are now known under the name of *isoschrödinger's equations*, and are given by[16]

$$i \frac{\partial}{\partial t} | \psi > = H*| \psi >, \qquad i < \psi | \frac{\partial}{\partial t} = < \psi | * H. \qquad (5.7)$$

Exactly as expected [9,19], the above equations coincide with Eq.s (4.7a) constructed via naive hadronization of classical isotopic formulations (we assume hereon for simplicity, owing to the arbitrariness of the Hamiltonian, that $H^{eff} = H$). The equivalence of the above equations with Eq.s (5.6) was proved in ref. [7].

The corresponding equation for the moments were identified at a later time, owing to the initial difficulties in identifying an operator image of the Birkhoffian mechanics in its general formulation (see footnote[12]). In fact a solution of the problem was reached in ref.s [9] only following the identification of the Hamilton-isotopic particularization of Birkhoffian mechanics, and can be written

$$- i \hat{I}_{ki} \nabla_i | \hat{\psi} > = p_k * | \psi >, \qquad i < \psi | {}_i\nabla {}_{ik}\hat{I} = < \psi | * {}_k p, \qquad (5.8)$$

[16] As shown by Jannussis et al. [26], the representational capabilities of Eq.s (5.6) and (5.7) is so vast to include *discrete systems* (evidently via the embedding of the discrete part in the isounit), including *Caldirola's chronon equations* [27]. In particular, owing to their explicit form , e.g.,

$$H* | \psi > = H\, T(t, r, p, \dot{p}, \psi, \psi^{\dagger}, \partial\psi, \partial\psi^{\dagger}, \mu, \tau, n, ...) | \psi > = E | \psi >, \qquad (a)$$

the isoequations are "directly universal" (footnote[2]) for all possible nonlinear equations in operator form. Moreover, equations (a) above are *isolinear and isolocal* , that is, they verify the conditions of linearity and locality at the isotopic level, while being nonlinear and nonlocal when projected in a conventional space. This implies that *nonlinearity and nonlocality are not true geometric axioms of a theory because they can be made to disappear at the isotopic level*. These results are not merely mathematical, because they have rather subtle experimental implications, e.g., for the search of nonlinearity, which we hope to consider at some future time.

which evidently coincide with form (4.9b) under the identification $p^{eff} = p$.

Note the appearance of the isounit $\hat{1} = T^{-1}$ in the l.h.s. of ∇, originating from the structure $R = (pT, 0)$ in the isotopic action (3.18), which then appear in the coefficient of the isotopic Hamilton-Jacobi equations (4.6). This form will have important implications in the operator realization of isosymmetries.

The exponentiation must be done, for mathematical consistency, via power series expansions in the isoenvelope $\hat{\xi}$, as permitted by the *isotopic Poincaré-Birkhoff-Witt Theorem* [3],

$$e_{|\hat{\xi}}^{\alpha X} = \hat{1} + \alpha X / 1! + (\alpha X) * (\alpha X) / 2! + \ldots.., \tag{5.9}$$

which can be rewritten in the old envelope for simplicity

$$e_{|\hat{\xi}}^{\alpha X} = \hat{1}\, e_{|\xi}^{\alpha TX} = (e_{|\xi}^{XT\alpha})\, \hat{1}, \tag{5.10}$$

and characterizes the operator formulation of *Lie-isotopic groups* [4,9].

The isoexponentiated form of Eq.(5.6) is then given by [4]

$$Q(t) = \{e_{|\hat{\xi}}^{itH}\} * Q(0) * \{e_{|\hat{\xi}}^{-Hti}\} = \hat{1}\{e_{|\xi}^{itTH}\} * Q(0) * \{e_{|\xi}^{-HTti}\}\,\hat{1} =$$
$$= e_{|\xi}^{HTti}\, Q(0)\, e_{|\xi}^{-itTH}, \tag{5.11}$$

Similarly, all operations on $\mathcal{H}$ are generalized on $\hat{\mathcal{H}}$. As an example, the condition of Hermiticity becomes the following *isohermiticity* [8]

$$H^{\hat{\dagger}} = T^{-1} G\, H^{\dagger} T\, G^{-1} = H; \tag{5.12}$$

that of unitarity becomes the following condition of *isounitarity* [4,7,8]

$$U * U^{\hat{\dagger}} = U^{\hat{\dagger}} * U = 0, \qquad U^{\hat{\dagger}} = U^{-\hat{1}}; \tag{5.13}$$

and similarly for all other operations, e.g.,

$$\hat{T}r\ A = (Tr\ A)\,\hat{1}, \tag{5.14a}$$

$$\hat{T}r\ A{\times}B = (\hat{T}rA){\times}(\hat{T}rB), \tag{5.14b}$$

$$\hat{D}et\ A = \{Det\,(AT)\}\,\hat{1}, \tag{5.14c}$$

$$\hat{D}et\,(A{*}B) = (\hat{D}et\ A){*}(\hat{D}et\ B), \tag{5.14d}$$

$$\hat{D}et\,(A^{-\hat{1}}) = (\hat{D}et\ A)^{-\hat{1}}, \tag{5.14e}$$

The expectation values of an operator A now acquire the form of *isoexpectation values*

$$<\hat{A}> = <\psi_k\,|\,\odot A{*}\,|\,\psi_k>\hat{1} = \hat{1}\int dr'\ \psi^{\dagger}{}_k(r')\ G\ A\ T\ \psi_k(r'), \tag{5.15}$$

where the multiplication by the isounit can be ignored for all practical purposes.

Finally, conventional projection operators are now lifted to the following *isoprojection operators* [8]

$$\hat{P} = |\,\psi_k><\psi_k\,|\,G\,\hat{1}. \tag{5.16}$$

with intriguing new possibilities of lifting conventional Hamiltonian models into their isotopic generalization[17]

Regrettably, we cannot review here the entire new mechanics and must refer the interested reader to ref.s [7,8,9].

We are now in a position to illustrate the construction of hadronic mechanics as an isotopy of quantum mechanics. To begin, the axiomatic properties of Planck's unit $\hbar = 1$ are nonsingularity, Hermiticity and, therefore, positive-definiteness. The liftings $\hbar \Rightarrow \hat{1}$ then characterizes an infinite family of possible isotopes $\hat{1}$ of the original unit $\hbar$. The point is that *all distinctions between Planck's constant and the integrodifferential isounit $\hat{1}$ cease to exist at the abstract level by conception*.

Similarly, one can see that, under assumptions Â, B̂, and Ĉ above, the generalized structure $\hat{\xi}$ remains a universal enveloping associative

[17] As an example, Animalu [28] showed that the two-body generalized bound states submitted in the original proposal of hadronic mechanics [4] to represent internal nonlocal and nonhamiltonian effects can be obtained from the conventional Schrödinger's equation for the hydrogen atom via the use of a suitable isoprojection operator.

algebra [3], isofields $\hat{F}$ remain fields, and isospaces $\hat{\mathcal{H}}$ remain Hilbert [7] [9,12]. Thus, hadronic mechanics is based on the infinite family of isotopies $\xi \Rightarrow \hat{\xi}$, $F \Rightarrow \hat{F}$, $\mathcal{H} \Rightarrow \hat{\mathcal{H}}$, which preserve, by central assumption, the axiomatic structure of the original theory.

But,

$$\xi \approx \hat{\xi}, \quad F \approx \hat{F}, \quad \mathcal{H} \approx \hat{\mathcal{H}} \tag{5.17}$$

and all distinctions between ξ and $\hat{\xi}$, F and $\hat{F}$ and $\mathcal{H}$, $\hat{\mathcal{H}}$ cease to exist at the abstract level. We therefore have the following property.

Quantum and hadronic mechanics coincide, by construction, at the abstract, realization-free level.

In particular, from an abstract viewpoint, there is no difference between the conventional modular action $H \mid \psi >$ and its isotopic generalization $H * \mid \psi >$[18]. One obtains conventional quantum mechanic when the simplest possible realization of the structures is assumed, and hadronic mechanics when one selects lesser trivial realizations.

A further consequence is that

The mathematical consistency of hadronic mechanics is today established. Only its physical effectiveness is under consideration in this paper.

After all, criticisms on the mathematical structure of hadronic mechanics constitute in reality criticisms in the axiomatic structure of quantum mechanics.

Note that each conventional quantum mechanical model characterized by given structures ξ, F and $\mathcal{H}$ admits an infinite variety of possible isotopic generalizations characterized by $\hat{\xi}$, $\hat{F}$ and $\hat{\mathcal{H}}$ because of the infinite variety of possible quantities $\hat{1} = T^{-1}$ and G.

This means that, while a system is solely characterized in quantum mechanics by the Hamiltonian operator $H = T + V$ (because of the generally tacit assumption of the unit $\hbar$), *in hadronic mechanics a*

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[18] These properties imply a truly intriguing connection (investigated in memoirs [9] between hadronic mechanics and the so-called "hidden variables", which are in fact no longer "hidden", but turned into "explicit" degrees of freedom represented by the infinitely possible isotopic operators T in the isomodular "realization of the abstract actio on $H \mid \psi >$" = "$H * \mid \psi >$" = "$H\ T \mid \psi >$", where $T = T^\dagger > 0$ is independent from H. "Hidden variables" are then explicitly realized via the infinitely possible isotopic operators T for each given Hamiltonian [7].

system is characterized by the Hamiltonian operator $H = T + V$ plus the two independent isotopic operators T and G, as the operator version of the teaching by Lagrange and Hamilton indicated in Sect. 3.

Thus, any original quantum mechanical system of particles characterized by a Hamiltonian H admits an infinite variety of nonlocal and nonhamiltonian internal forces characterized by T and G. This is evidently due to the infinite possibilities of internal physical conditions and related nonlocal effects for each given set of particles originally interacting at large mutual distances.

To understand this crucial point, consider a proton and a neutron, from their simplest possible strong interactions in the deuteron, to high energy collisions and then in the core of a star undergoing gravitational collapse. It is evident that nonlocal internal effect are expected to be very small in the deuteron, then increase with energy in inelastic collisions, and finally reach their maximal conceivable form in the core of a collapsing star where, in addition to total mutual penetration of the wavepackets, we have their compression. All these infinitely different interior physical conditions with consequently different nonlocal and nonhamiltonian effects are represented via infinitely different isounits $\hat{1}$.

Finally, the reader should recall that, besides providing the general laws that must be verified by all Hamiltonians, *quantum mechanics cannot identify the numerical value of each Hamiltonian, which must be identified via experiments*. Along exactly the same lines, besides providing the general laws that must be verified by all quantities H, $\hat{1}$ and G, *hadronic mechanics cannot identify the numerical values of the isounits $\hat{1}$ and isotopic elements G, which must also be identified via experiments* [19], as we shall see in this paper for the Bose-Einstein correlation.

It is evident from the above outline that hadronic mechanics possesses the necessary elements for resolving Limitations 1-5 of quantum mechanics indicated in Sect. 2. In fact, the new mechanics permits the following advances:

$\hat{1}$) ***DIRECT REPRESENTATION OF THE ACTUAL SHAPE OF EXTENDED CHARGE DISTRIBUTIONS***. While for conventional formulations we have to go to the second quantization to reach an indirect representation of the extended character of particles, hadronic mechanics permits the direct representation of the actual shape

[19] In fact, *no mathematical or physical theory can identify the value of its own unit.* As a matter of fact, this is the ultimate root of hadronic mechanics and the reason why it has escaped identification for so long.

considered via a factorization of the isounit of the form

$$\hat{1} = \hat{\delta}^{-1}\,\hat{1}^{\circ}, \quad \hat{\delta} = \text{diag.}\,(b_1^{\,2},\ b_2^{\,2},\ b_2^{\,2}), \qquad b_k = \text{const.} > 0; \quad (5.18)$$

Rather than needing second quantization, the representation exists already at the classical level, and then simply persists under hadronization.

More specifically, the representation of arbitrary nonspherical shapes $\hat{\delta}$ are permitted under the exact rotational symmetry, of course, realized at our isotopic level $\hat{O}(3)$, i.e., constructed with respect to the isounit $\hat{1} = \hat{\delta}^{-1}$, which results to be locally isomorphic to O(3), $\hat{O}(3) \approx O(3)$ (see Sect. 7).

As we shall see, hadronic mechanics permits a direct representation of the actual shape of the fireball via a factorization of type (5.18), although in a suitable relativistic extension.

$\hat{2}$) *DIRECT REPRESENTATION OF THE DEFORMATIONS OF EXTENDED CHARGE DISTRIBUTIONS* . This is also permitted by the isounit via factorizations of the type more general than (5.18)

$$\hat{1} = \hat{\delta}^{-1}\,\hat{1}^{\circ}, \quad \hat{\delta} = \text{diag.}(\hat{b}_1^{\,2}, \hat{b}_2^{\,2}, \hat{b}_3^{\,2}), \quad \hat{b}_k = \hat{b}(t, r, \dot{r}, \ddot{r}, \ldots) > 0, \qquad (5.19)$$

thus permitting a direct representation of all infinitely possible evolutions and/or deformations of the original shape, while preserving the exact rotational symmetry, of course, at our isotopic level (see Sect. 7).

We begin to see in this way the first two applications of the isounit $\hat{1}$ of hadronic mechanics, the representation of the actual shape of the system considered, as well as of all possible deformations of the original shape.

Again, this representation of the deformation of shape exists at the primitive classical level, and simply persists in the operator formulations.

$\hat{3}$) *DIRECT REPRESENTATION OF NONLINEAR, NONLOCAL AND NONHAMILTONIAN INTERACTIONS* . This requirement is also readily permitted by the isounit of hadronic mechanics. As an example for the case of two extended particles with wavefunctions $\psi(r)$ and $\phi(r)$ in conditions of total mutual penetration, the isounit of hadronic mechanics admits the realization called *Animalu's isounit* [28].

$$\hat{1} = e^{\,t\,N \int dr'\,\psi(r')\,\phi(r')}_{|\xi}\ , \qquad (5.20)$$

which recovers conventional quantum mechanics whenever there is no overlapping.

The *nonlinear* character of the interactions when represented via the isounits $\hat{1}$ is evident[20].

The *nonlocal* character is illustrated in a direct way by isounit (5.20), although in its simplest possible form, i.e., *nonlocality in the wavefunctions*. The possible nonlocality in all other variables and their derivatives is then consequential.

Finally, the *nonpotential-nonhamiltonian* character is expressed by the fact that the interactions are not represented by the Hamiltonian, but rather by the isounit itself, thus providing an operator formulation of the lack of potential for contact interactions. Equivalently, one can inspect the action of the isotopic element T in the dynamical equations (5.6) and (5.7), where it *multiplies* the Hamiltonian from the right and from the left.

$\hat{4}$) *DIRECT REPRESENTATION OF THE BOSE-EINSTEIN CORRELATION FROM BASIC AXIOMS*. Consider again a system of n particles with correlated states a and uncorrelated ones b represented with $| k, a > \times | k, b >$, $k = 1, 2, ..., n$, as in Eq.s (2.9). Then, from isoinner structure (5.5) of the underlying Hilbert space, conventional expression (2.9) is generalized into the form

$$\hat{C}_n = \overbrace{\underbrace{<1,a\,|<1,b\,|\,.......<n,a\,|<n,b\,|}} \quad G \begin{pmatrix} |1,a> \\ |1,b> \\ .... \\ \\ |n,a> \end{pmatrix} \quad \hat{1} = \qquad (5.21)$$

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[20] This point is important to clarify a number of potentially misleading aspects of experiments [24] claiming lack of nonlinearity. To begin, experiments [24] were conducted within purely atomic settings and, as such, they are fundamentally inapplicable to the interior of strong interactions studied by hadronic mechanics. Second, experiments [24] considered the simplest possible form of "nonlinearity", that in the "wavefunction ψ", while the most important nonlinearity for hadronic mechanics is that in the "derivatives $\partial\psi$ of the wavefunctions", as typically the case for all drag effects. Finally, the reader should keep in mind that all nonlinearities may eventually result to be a mere first approximation of the more general nonlocality, and that our isotopies permit the elimination of all possible nonlinearities and nonlocalities at the abstract, realization-free level.

$|n,b\rangle$

which can be written

$$\hat{C}_n = \{ \sum_{k,a} \langle k,a \mid \hat{G}_{ka,kb} \mid k,b \rangle \} \hat{1} =$$

$$\{\sum_k (K_{ka}\langle k,a \mid \hat{G}(0) \mid k,a\rangle + K_{kb}\langle k,b \mid G[0) \mid k,b\rangle + K_{kab} \langle k,a \mid G(0) \mid k,b\rangle) \} \hat{1}, \quad (5.22)$$

where the K's are suitable isorenormalization coefficients. The important point is that, when compared to conventional expression (2.9), isotopic expression (5.22) exhibits precisely the presence of the cross terms responsible for the correlation (see Sect.s 8 and 9 for details).

$\hat{5}$) *RECONSTRUCTION OF EXACT SPACE-TIME SYMMETRIES AND RELATIVITIES*. One of the primary objectives for which the Lie-isotopic theory was submitted [3] is precisely the reconstruction of conventional, exact space-time symmetries when believed to be broken under nonlocal and nonhamiltonian interactions. This aspect will be investigated in the next sections in more details. At this point let us simply outline the following main aspects:

α) All nonlocal-integral effects are embedded in the isounit of the theory. This permits the preservation of the conventional topologies virtually unchanged, because, as indicated earlier, algebras and geometries are insensitive to the structure of their own unit once positive-definite;

β) The isotopic symmetries result to be locally isomorphic to the conventional ones. In fact, the property mentioned before that isotopes $\hat{O}(3)$ of $O(3)$ constructed with respect to $\hat{1} > 0$ are isomorphic to the latter, $\hat{O} \approx O(3)$, extends to all other space-time symmetries, such as the *isogalilean, isolorentz and isopoincaré symmetries*, as studied in details at the classical level in monographs [14,15], and at the operator level in memoirs [9]. (see the outline of Sect. 7). Thus, *the space-time symmetries at the foundation of contemporary theoretical physics are not lost in our approach to Bose-Einstein correlation, but actually preserved in an exact form, although realized in their most general known forms*.

γ) The relativities characterized by the isotopic space-time symmetries coincide with the corresponding, conventional relativities at the abstract, realization-free level [9,15], as expected from the abstract identity of hadronic and quantum mechanics (see also Sect. 7).

We can therefore state that:

hadronic mechanics is a covering of conventional quantum mechanics in the sense that:

i) the former is based on mathematical formulations (the Lie-isotopic theory on isohilbert spaces) which are structurally more general than those of the latter (the conventional Lie's theory on conventional Hilbert spaces);

ii) the former represents physical conditions (extended-deformable particles under nonlinear, nonlocal and nonhamiltonian forces) which are structurally more general than those of the latter (point-like particles under local-potential-Hamiltonian forces); and

iii) the former can approximate the latter as close as desired for $\hat{I} \approx I$, and recovers the latter in its entirety for $\hat{I} \equiv I$.

The formulation described above, sometimes called *general hadronic mechanics* [9], is excessively broad for the objectives of this paper. In fact, from hereon, we can effectively restrict our study to the particular case of the so-called *restricted hadronic mechanics* characterized by [9]

$$T = G , \qquad (5.23)$$

namely, when the isotopic element T of the isoenvelope ξ coincides with the isotopic element G of the isohilbert space $\hat{\mathcal{H}}$.

This is a particularly significant case in which *the operation of isohermiticity coincides with the conventional Hermiticity*, i.e., from Eq.s (5.12), we have

$$H^{\hat{\dagger}} \equiv H^{\dagger} \quad , \qquad (5.24)$$

Similarly, the isoexpectation values become

$$< \hat{A} > = < \psi_K | T A T | \psi_K > \hat{1}, \qquad (5.25)$$

the isoprojection operators formally coincide with the conventional ones

$$\hat{P} = | \psi_K > < \psi_K |, \qquad (5.26)$$

while all other expressions depending only on the isotopic element T remain unchanced.

We therefore have the following

PROPOSITION 5.1 [8,9]: Under condition (5.23), the observables of quantum mechanics remain observables for the covering hadronic mechanics.

The implications of the above property are nontrivial. In fact, contrary to a rather popular belief, we have:

PROPOSITION 5.2 [9]: A Hermitean operator $H = H^\dagger$ does not possess a unique set of real eigenvalues, but admits an infinite number of different sets of eigenvalues, each of which is real.

PROOF: Consider a given Hamiltonian H, and suppose that it has the (discrete or continuous) set of eigenvalues E° in quantum mechanics,

$$H|\phi> = E^\circ|\phi>, \quad H^\dagger = H. \tag{5.35}$$

Then, the *same* operator H remains Hermitean under isotopy (5.23) and admits *different* eigenvalues within the context of the covering hadronic mechanics, called *isoeigenvalues*, trivially, because of the presence of the isotopic element T,

$$H * |\psi> = H\, T|\psi> = \hat{E} *|\psi> \equiv E_T|\psi>, \quad E_T \neq E^\circ, \tag{5.36}$$

But the isotopic elements T are unrestricted, and can be infinite in number for each given Hamiltonian H, thus implying an infinite number of different isoeigenvalues for the same Hamiltonian H. The reality of each set of isoeigenvalues was proven in ref. [7]. QED

Similarly, we have

PROPOSITION 5.3 [9]: A Hermitean operator does not possess a unique set of real expectation values, but admits instead an infinite number of different sets of real expectation values.

PROOF: Consider again a Hermitean Hamiltonian H, and suppose that the quantum mechanical expectation values are given by the familiar form

$$<\phi|\phi> = 1, \quad <\phi|H|\phi> = E^\circ \in \Re. \tag{5.37}$$

Then, the *same* operator H within the content of the covering hadronic mechanics, say, under the normalization

$$<\hat{\psi}|*|\hat{\psi}> = 1, \qquad <\hat{\psi}|\hat{\psi}>\hat{1} = \hat{1}, \tag{5.38}$$

admits the *different* expectation values[21]

$$<\hat{\psi}|*H*|\hat{\psi}> = <\hat{\psi}|THT|\hat{\psi}> = E_T \neq E^\circ. \tag{5.39}$$

But there exists an infinite number of different isotopic element T for each given operator H, thus implying the existence of an infinite number of different sets of isoexpectation values whose reality was proved in ref. [7]. QED.

We learn in this way that, contrary to popular belief, *a given Hermitean Hamiltonian H, by no means, admits unique eigenvalues and expectation values E°, but instead an infinite number of different isoeigenvalues and isoexpectation values E_T depending on the infinitely possible internal nonlocal and nonhamiltonian effects represented by the infinitely possible isounit or isotopic element T* [9].

We should finally recall that Propositions 5.1, 5.2 and 5.3 imply that a given isotopy

$$H|\phi> = E^\circ|\phi> \Rightarrow HT|\psi> = E_T|\psi>, \tag{5.40}$$

essentially characterizes a *mutation* (in the language of ref. [4]) of given quantum mechanical characteristics, in the sense of implying an alteration of the original numerical values.

Rather than being unexpected or a mere mathematical property, ***mutations (5.40) are exactly the alterations of the predictions of***

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[21] Note that this result is nontrivial even when including the isounit. Assume T = const. Then the isoinner product does not change, because

$$<\psi|T|\psi>\hat{1} \equiv <\psi|\psi>, \qquad \hat{1} = T^{-1} = \text{cost.} > 0, \tag{a}$$

thus confirming the identity of $\mathcal{H}$ and $\hat{\mathcal{H}}$ at the abstract level. However, for the isoexpectation values we have

$$<\psi|TAT|\psi>\hat{1} = <\psi|A|\psi>T \neq <\psi|A|\psi>. \tag{b}$$

This implies that *a numerical change of Planck's constant would imply an alteration of the expectation values*.

quantum mechanics expected from the addition of internal, short range, nonlocal and nonhamiltonian interactions.

To state it differently, consider the bound state of two particles at large mutual distances under potential interactions only, Hamiltonian H and an energy spectrum E°. Suppose now that the same two particles can have a bound state one inside the other at mutual distances less than 1 Fermi , by leaving unchanged the original Hamiltonian H. The resulting internal nonlocal and nonhamiltonian effects necessarily alter (mutate) the original spectrum E° via the isotopic element T. As pointed out in the original proposal [4], the hadronic generalization of quantum mechanics was ultimately proposed exactly for the representation of such mutation[22].

In conclusion, *the predictions of quantum mechanics are exact under all physical conditions of the original conception of the theory*, for large mutual distances of particles under which nonlocal and nonhamiltonian interactions are ignorable, as it is typically the case in the atomic structure and the electromagnetic interactions at large.

However, ***quantum mechanics can well result to be approximately valid whenever nonlocal and nonhamiltonian interactions cannot be effectively ignores*** [23].

Particularly intriguing are the novel possibilities offered by hadronic mechanics which require specific additional studies, and are mentioned here on mere grounds of scientific curiosity:

I) ***A possible new, unique and unambiguous quantization of gravitation*** [9]. Recall that Einstein gravitation has an identically

[22] As shown in the original proposal [4] (and confirmed in the more recent studies [9] and Animalu's research [28]) *the isotopy of the conventional Schrödinger's equation for the positronium implies the suppression of the atomic infinite energy spectrum E^o down to only one energy level*. This is the reason why proposal [4] submitted the hypothesis that the π° is a "compressed positronium", i.e., an electron and a positron in a hadronic bound state at mutual distances of 1 Fermi. Intriguingly, the model is capable of representing with one single equation of structure *all* the characteristics of the π°, i.e.: total rest energy, charge radius, meanlife, charge, electric and magnetic moments, space and charge parity and primary decays.

[23] This aspect is better focused by a nonlocal-nonhamiltonian scattering theory proposed by Mignani [25, 29] within the context of hadronic mechanics, which indicate that *nonlocal internal effects may imply an alteration of the cross section*. In any case, such an alteration should be expected from the general principles of hadronic mechanics, such as mutation (5.40). Unfortunately, we do not have at this writing a specific re-elaboration of conventionally elaborated experiments in strong interactions to clarify this aspect. It is therefore hoped that interested phenomenologists will consider and resolve the issue *whether or not internal nonlocal-nonhamiltonian effects in strong interactions imply an alteration of the numerical value of the cross section*.

null Hamiltonian while quantum mechanics is fundamentally dependent on the existence of a nontrivial Hamiltonian. These occurrences are at the basis of the problematic aspects in the quantization of gravitation.

This conventional profile is altered by hadronic mechanics. Let $R(x,g,\Re)$ be the Riemannian space of a gravitational theory and $M(x,\eta,\Re)$ its tangent Minkowski space with familiar metric $\eta = \text{diag.}(1, 1, 1, -1)$. All gravitational metrics g(x) can be always decomposed into the form

$$g(x) = T(x)\,\eta, \quad T > 0, \tag{5.41}$$

under which the Riemannian space can be interpreted as an isotope of the Minkowski space (ref. [14], Sect. II.3)

$$R(x,g,\Re) \approx M(x,\hat{\eta},\hat{\Re}), \quad \hat{\eta} = T(x)\eta, \quad \hat{\Re} = \Re\hat{1}, \quad \hat{1} = T^{-1}, \tag{5.42}$$

The isoquantization of conventional gravitation submitted in ref. [9] is essentially given by the isotopy of conventional relativistic quantum mechanics in which the isounit represents the curved component of gravitation, i.e.,

$$\hat{1} = [T(x)]^{-1}, \qquad T\,\eta = g \in \Re(x,g,\Re) \tag{5.43}$$

In different terms, it is possible that an unambiguous quantization of gravity already exists in the current relativistic quantum theory, only "hidden" in their units.

In summary, quantum mechanics is only based on the Hamiltonian, as well known, thus leading to the indicated problematic aspects for quantum gravity. Hadronic mechanics is based instead in the Hamiltonian plus the isotopic elements T and G, thus offering the possibility of quantizing physical models which are nonhamiltonian, exactly as it is the case for gravitation.

II) ***Possibility of turning divergent perturbative series into convergent ones*** [9]. The idea is so simple to appear trivial. Consider a canonical series which is divergent, e.g.,

$$\sum = A_o + k\,[A,H]\,/\,1! + k^2[\,[A,H]\,,H]\,/\,2! + \ldots \Rightarrow \infty, \quad k \gg 1. \tag{5.44}$$

It is then evident that the divergence may be due to the excessively simplistic structure of the Lie product $[A,H] = AH - HA$. In fact, *given a divergent series (5.44), there always exists an isotopic product $[A\,\hat{,}H] =$*

ATH – HTA under which the isotopic series

$$\sum = A_0 + k\,[A\,\hat{,}\,H] / 1! + k^2[[A\,\hat{,}\,H]\,\hat{,}\,H] / 2! + \ldots = N < \infty, \ k \gg 1, \quad (5.45)$$

becomes convergent [9], as it is evidently the case, e.g., for $|T|$ sufficiently smaller than one.

In different terms, the isotopies imply a form of renormalization beginning with the classical formulations (see the classical two-body case of ref. [15], App. III.a), and this property evidently persists at the operator level. Thus, the isotopy of a given model, e.g., with a coupling constant $K > 1$ implying divergent series (5.44), can be lifted in such a way to imply a renormalized coupling constant $k \gg 1 \Rightarrow \hat{k} \ll 1$, with evident possibility of rendering convergent perturbative expansions when conventionally divergent.

III) ***Possible new "iso-grand-unification" of all interactions*** [9]. The isotopic lifting of gauge theories has been identified by Gasperini [30] precisely along the lines of hadronic mechanics, e.g., with respect to our isounit $\hat{1}$. In ref. [9] we therefore suggested *the "iso-grand-unification" essentially consisting of the isotopy of conventional unified gauge theories of weak and electromagnetic interactions in which the isounit $\hat{1}$ represents gravitation as in Eq.s (5.43), as well as the nonlocal component of the strong interactions of the historical legacy.*

Predictably, hadronic mechanics implies rather intriguing revisions of current epistemological studies of quantum mechanics in regards to causality and other aspects, only preliminarily indicated in ref.s [9] in the hope that they will be inspected by independent researchers[24].

[24] The following three epistemological aspects studied in ref. [9] are worth a mention. Their understanding requires the mind specifically set in ***interior hadronic conditions***, such as a proton in the core of a star and, under no condition should be investigated by thinking at the typical atomic setting of an electron moving in empty space for which, as stressed repeatedly, quantum mechanics (originally called for this reason *atomic mechanics*) is exact.

a) Hidden variables. We mention in footnote[18] the intriguing realization [9] of hidden variables via our isotopic eigenvalues equations $H{*}|\psi> = HT|\psi> = E_T|\psi>$. We can now complement this aspect with the property that *the celebrated von Neumann theorem on the lack of hidden variables becomes inapplicable, because of the inapplicability of its first assumption on the unicity of the eigenvalues of Hermitean operators*. In different terms, hadronic mechanics clarifies that there may be no hidden variables under the assumption of the unit of the theory $\hbar = 1$, but an infinite number of "hidden degrees of freedom" emerges whenever such an un-necessary restriction is lifted, and one assumes arbitrary isounits $\hat{1}$.

Needless to say, we are forced of necessity to leave open numerous additional aspects, only rudimentarily treated in ref..s [9], such as the problem of causality under nonlocal interactions, the measure theory for generalized isounits, etc. Nevertheless, the existence of consistent generalizations within the context of hadronic mechanics is *guaranteed* by the isotopies themselves. In fact, a generalization of the conventional causality is not isotopic unless the causality axiom itself is preserved, a generalization of the conventional measure theory is not isotopic unless the generalized theory is a measure theory, etc.

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b) Heisenberg's uncertainty .The equally celebrated Heisenberg's uncertainty may well need revisions in the interior hadronic problem because it is apparently replaced by the *principle of isouncertainty* [9]

$$\Delta r \, \Delta p \ \geqq \ \tfrac{1}{2} < \hat{1} >, \qquad \text{(a)}$$

where $< \hat{1} >$ is a suitable average of $\hat{1}$ (again recovering the conventional uncertainty $\hbar$ for $\hat{1} = \hbar$). Evidently $< \hat{1} >$ can be smaller than $\hbar$, (see the case of "isonormalization" (5.45) of divergent series (5.44)), thus showing that *a particle moving within a hadronic medium can have an uncertainty smaller than the same particle in vacuum* . Consider then the uncertainty of a particle in the core of a star undergoing gravitational collapse. But $\hat{1} = T^{-1}$ can represent the inverse of the gravitational component T of a Riemannian metric $g(x) = T(x)\,\eta$ (see Eq.s (5.41)-43), and, more particularly the isopoincaré symmetry later on for isocommutation rules (7.30) in which $\hat{g}$ is the Riemannian metric g). This implies that *in a possible operator formulation of gravitation via hadronic mechanics, the (conventional) singularities of the gravitational field $g \Rightarrow \infty$ (i.e., $T \Rightarrow \infty$), are represented by the zeros of the isounit $\hat{1} = T^{-1}$* [9]. Thus, *at the limit of a gravitational singularity ,*

$$\Delta r \, \Delta p \geqq \ \tfrac{1}{2} < \hat{1}_{\text{singularities}} > \equiv 0, \qquad \text{(b)}$$

and Heisenberg's uncertainty may well recover the conventional determinism of classical physics . After all, a star is a classical object whose center of mass can be exactly determined. But then so is any particle in its collapsed interior.

***c)* Bohr's complementarity** . In this case too we have now predictable revisions in the notion of complementarity, e.g., because the particle considered is not moving in empty space, but could be in the center of the collapsing star considered above. In this case, the identification of one of its properties does not necessarily imply a perturbation of the particle itself owing to the contact nonlocal and nonhamiltonian interactions which may force the particle to keep its position (see the notion of *isocenter* of Sect. 6). These novel physical conditions were not evidently considered by Bohr.

6. GENERALIZED HADRONIC PARTICLES AND BOUND STATES

After having identified in Part I the basic methods, in this Part II we shall study the Bose-Einstein correlation beginning with the identification of the type of state constituting the fireball.

This study will be conducted in this section via the generalized notion of composite systems of hadronic mechanics, called *closed nonhamiltonian systems*. These are systems of extended-deformable particles which are conventionally closed in the sense of verifying all conventional total conservation laws, but admit nonlinear, nonlocal and nonhamiltonian internal forces. As such, closed nonhamiltonian systems appear to represent precisely the fireball of the Bose-Einstein correlation under our hypothesis of Sect. 1.

Closed nonhamiltonian systems were submitted at both the classical and operator levels in the original proposal of hadronic mechanics [4], and subsequently studied in details at the classical level in monographs [10,15] and in operator form in memoirs [9] (see also the reviews [16,18]). In this section we can evidently review only the elements essential for the phenomenological studies of the next sections.

First, *in the transition from quantum to hadronic mechanics, the notion of particle and/or constituent of a composite system is generalized into a notion called "isoparticle"* [11,14,15]. Nonrelativistically (relativistically), we have the transition from the notion of particle as a representation of the conventional *Galilei symmetry* G(3.1) (*Poincaré symmetry* P(3.1)) with trivial unit I, to the notion of isoparticle as a representation of the infinitely possible *Galilei-isotopic symmetries* $\hat{G}(3.1)$ (*Poincaré-isotopic symmetries* $\hat{P}(3.1)$) with isounits $\hat{1}$ (see the first proposal [31] and the subsequent elaborations in ref.s [9,11-15] with an outline in Sect. 7).

This essentially implies the transition from a point-like particle which can only experience action-at-distance interactions of local-potential-Hamiltonian type, to extended-deformable particles which can experience conventional as well as contact, nonlinear, nonlocal and nonhamiltonian interactions. Since points are perennial and immutable geometric objects, the intrinsic characteristics of conventional particles are immutable. On the contrary, since extended shapes are deformable, one isoparticle can have an infinite number of different shapes and other characteristics, represented precisely by the infinitely possible isounits $\hat{1}$.

Quantum mechanical bound states are composed of a collection of point-like particles with perennial characteristics interacting at large mutual distances. On the contrary, a hadronic bound state is composed of extended-deformable particles under action-at-a-distance interactions as well as mutual penetration of their charge-distributions/wavepackets at mutual distances equal or smaller than 1Fermi. In the latter systems the total characteristics are constants and conserved, but the individual characteristics are not necessarily so. .A visual representation of the generalized hadronic bound states is presented in Fig. 2 below.

In this section we consider systems of isoparticles which are closed-isolated from the rest of the Universe and, therefore, they are expected to verify total conservation laws. The study of this issue is relevant because the recent letter [32] claims that systems of particles with different Planck's constants violate conventional total conservation laws and space-time symmetries. At any rate, the issue of the verification of total conservation laws for an ensemble of constituents in generally nonconservative conditions is evidently essential for our study of the Bose-Einstein correlation.

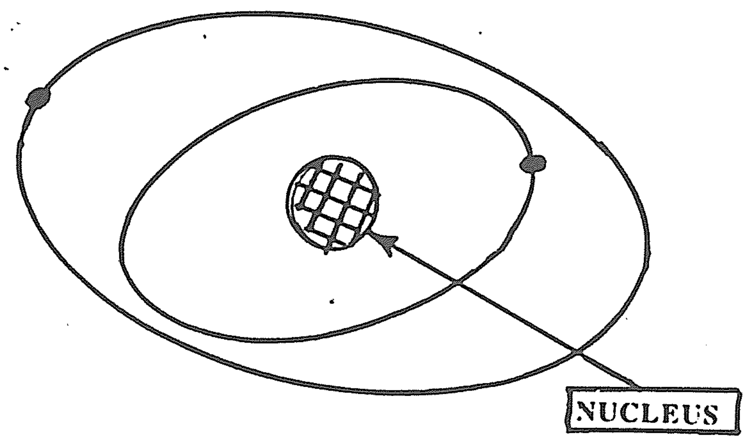
The authors of note [32] (of 1991) were apparently unaware that this author had previously submitted in ref. [31] (of 1983) the notion of particles with different units, and previously proved [9] (in 1989) the validity of total conservation laws and space-time symmetries for systems of particles with different isounits. In fact, the particles with different Planck's constants of ref. [32] are the simplest conceivable isoparticles with different integrodifferential isounits proposed in ref. [31].

In essence, as also clarified in ref. [33], *note [31] studied composite systems of particles with different units without the tensorial product of the Hilbert spaces, as necessary for conventional quantum mechanics in order not to violate linearity and other requirements* (see, e.g., [21]). The assumption of an incorrect carrier space then leads to misleading conclusions [33].

In ref.s [9], composite systems of isoparticles are characterized by a total Hilbert space given instead by the appropriate tensorial product of the individual spaces. Since the individual spaces have different isounits by assumption, the unit of the total space is then the tensorial product of the individual units. Total conventional conservation laws and space-time symmetries then follow.

To begin, let us assume the simplest possible model of a fireball

BOUND SYSTEMS IN QUANTUM MECHANICS



BOUND SYSTEMS IN HADRONIC MECHANICS

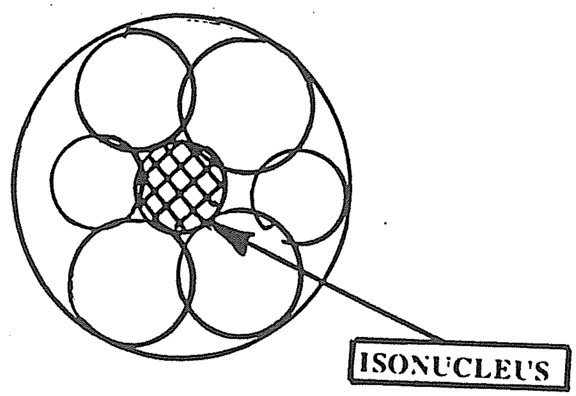


FIGURE 2: A conceptual view of the bound states characterized by quantum and hadronic mechanics. The constituents of a quantum mechanical bound state are point-like particles verifying the conventional Galilei G(3.1) and Poincaré symmetries P(3.1), resulting in a bound state having the conventional Keplerian structure. On the contrary, the hadronic particle-constituent of a bound state under strong interactions verifies the isogalilean $\hat{G}(3.1)$ and isopoincaré

symmetry $\hat{P}(3.1)$, implies generalized characteristics of the individual constituents, thus resulting in a new notion of composite system. The most effective way to see this occurrence is the following. The conventional, quantum mechanical constituents move in empty space, in which case they require a central *nucleus* which must be heavier than the peripheral constituents, as established in the atomic structure. On the contrary, in the transition to the covering hadronic mechanics, we have extended constituents in conditions of partial or total mutual overlapping, in which case the constituent at the center, called *isonucleus* [9,11,15] can have an arbitrary non-null mass, either heavier or smaller than that of the peripheral constituents, trivially, because of the physical contacts among all the constituents. From the viewpoint of space-time symmetries, the possible computerization of a composite system of particles obeying the Galilei or Poincaré symmetry is expected to yield the atomic structure, as well known. On the contrary, the possible computerization of the covering isogalilean or isopoincaré symmetries is expected to yeld precisely the nuclear structure [25]. In fact, nuclei are an aggregate of extended particles under mutual contact without a conventional central nucleus, but with our isonucleus. The proposed generalized structure of hadrons for which hadronic mechanics was proposed [4,9], is patterned along these lines, although under internal nonlocal and nonhamiltonian interactions quantitatively much bigger than the relatively small counterpart of the nuclear structure. The fireball of the Bose-Einstein correlation studied in this paper is then expected to be a limit case of these generalized structures[26].

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[25] The author would like to thank T. Gill of Howard University, Washington, D.C., for the recommending this visual computerization of the differences between the conventional Galilei or Poincaré symmetry and their isotopic generalization during recent talks at his department.

[26] Data on nuclear volumes as compared to the volumes of individual nucleons indicate that protons and neutrons are in average conditions of mutual penetration in the nuclear structure of about 10^{-3} parts of their volumes. By comparison, all massive particles are known to possess a wavepacket of the order of at least 1Fermi, which is the order of magnitude of the size of all hadrons. As a consequence, the hadronic constituents are expected to be in a state of *total* mutual penetration and overlapping of their wavepackets, thus resulting in expected nonlinear and nonhamiltonian internal effects much bigger than the nuclear ones. The limiting case is the core of a collapsing star in which, as we indicated earlier, we have not only total mutual penetration but also compression of a large number of wavepackets in a very small region of space. The very high energy p-$\bar{p}$ interactions originating the fireball, e.g., those for the UA1 experiments at CERN, are expected to produce physical conditions (e.g., densities) beyond those of the structure of hadrons, and approaching those in the core of a collapsing star.

within the context of hadronic mechanics consisting of two isoparticles of the same mass with constant isounits[27] and isotopic elements T = G

$$\hat{1}_a = T_a^{-1} = \mathrm{diag.}(b_{a1}^{-2}, b_{a2}^{-2}, b_{a3}^{-2}) = \mathrm{const.} > 0, \quad a = 1, 2, \quad (6.1)$$

with corresponding isoenvelopes $\hat{\xi}_a$, isofields $\hat{F}_a$, isohilbert spaces $\hat{\mathcal{H}}_a$, isostates $|\psi_a\rangle$ and linear momenta p_a. It should be indicated that the analysis of this section also holds for the most general possible functional dependence of the isounits, with the sole exclusion of the r-dependence, $\hat{1} = \hat{1}(t, p, \dot{p}, \psi, \psi^\dagger, \partial\psi, \partial\psi^\dagger, ...)$. In turn, the latter dependence excludes gravitational considerations from the analysis.

The eigenvalue equation of the linear momenta can now be assumed to have the generalized isotopic structure (5.8) [9]

$$p_{ak} * |\psi_a\rangle = p_{ak} T_a |\psi_a\rangle = -i \hat{1}_{ak} \frac{\partial}{\partial r_{ak}} |\psi_a\rangle, \quad (6.2)$$

where, hereon, there is no sum on the repeated Latin indices unless specifically indicated. The understanding is that realization (6.2) is, by far, non-unique because of the degrees of freedom of hadronic mechanics, and a number of other alternatives are possible (see below).

The fundamental isocommutation rules are then given by

$$[r_{ai} \hat{,} r_{bj}] * |\psi_a\rangle = [p_{ai} \hat{,} p_{bj}] * |\psi_a\rangle \equiv 0, \quad (6.3a)$$

$$[p_{ai} \hat{,} r_{aj}] * |\psi_a\rangle = -i\, b_{ia}^{-2}\, \delta_{ij} * |\psi_a\rangle \quad (6.3b)$$

$$i, j = 1, 2, 3, \quad a, b = 1, 2$$

where the isocommutator is given by

$$[A \hat{,} B]_a = [A \hat{,} B]_{\hat{\xi}_a} = A*B - B*A = A T_a B - B T_a A, \quad (6.4)$$

[27] This is generally possible for the "global" approach to a composite hadronic system, that is, its study from the outside as a whole. The explicit functional dependence of the isounit is instead needed for the "local" internal behavior, e.g., for the description of one particle in the interior of hadronic matter. At the classical level, the former isounits are given by a suitable average of the latter [15], and a similar situation is expected at the operator level.

Note the correspondence of the isoeigenvalues 0 and $- i\ b_i^{-2}\ \delta_{ij}$ of the fundamental isocommutation rules (6.3a) and (6.3b), respectively, with the corresponding classical expressions 0 and $b_i^{-2}\ \delta_{if}$, given by Eq.s (III.3.30) of ref. [15], thus confirming that the hadronization of Sect. 4 is indeed consistent and of the desired type.

This result has a number of predictable implications, from the confirmation of the isotopic character of hadronic over quantum mechanics, to the preservation of conventional space-time symmetries.

It is furthermore assumed that each isoparticle is individually free at this first stage. As such, it is represented by Hamiltonians

$$H_a = p_a^{\hat{2}} / 2m = p_{ak} T_a p_{ak} / 2m \tag{6.5}$$

with related isoschrödinger equations

$$i\,\partial_t | \psi_a > = H_a * | \psi_a > = H_a\, T_a | \psi_a > = E_a | \psi_a >. \tag{6.6}$$

Eq.s (6.6) essentially represent free but extended particles with a well identified shape described by isotopic elements (6.1), e.g., two different oblate spheroidal ellipsoids

$$T_1 = \text{diag.}\,(b_{11}^2, b_{12}^2, b_{13}^2) \neq T_2 = \text{diag.}\,(b_{21}^2, b_{22}^2, b_{23}^2). \tag{6.7}$$

We now study the two isoparticles when forming a composite system with mutual interactions of both local-potential as well as nonlocal-nonpotential internal type. This composite system is assumed to be the simplest conceivable model of fireball provided by hadronic mechanics. Numerous generalizations are evidently possible, as we shall see, both in the number of the constituents as well as in the nonlocal internal nature of the interactions.

The first and most fundamental point is the identification of the total isounit and total isotopic element which, in their simplest possible form, are given by the tensorial products

$$\hat{1}_{tot} = \hat{1}_1 \times \hat{1}_2, \qquad T_{tot} = T_1 \times T_2. \tag{6.8}$$

The understanding is that more general realizations of the total isounit with additional terms caused by the mutual interactions are, not only possible, but actually recommendable in specific models (see next sections).

This essentially implies that, according to hadronic mechanics, the rudimentary fireball here considered is based on the total isoassociative algebra of operators [9]

$$\hat{\xi}_{tot}: \qquad A*B \stackrel{def}{=} AT_{tot}B\,, \tag{6.9}$$

total isofield

$$\hat{F}_{tot}: \hat{F}_1 \times \hat{F}_2\,. \tag{6.10}$$

and total isohilbert space

$$\hat{\mathcal{H}}_{tot}: \quad \hat{\mathcal{H}}_1 \times \hat{\mathcal{H}}_2\,, \tag{6.11}$$

The total quantities of the fireball must then be properly defined in the total isospaces by following essentially the same rules as those for ordinary quantum mechanics [21]. In particular, the coordinates r_{ak} and momenta p_{ak} of the individual particles now become

$$\hat{r}_{1k} = r_{1k} \times \hat{1}_2, \qquad \hat{r}_{2k} = \hat{1}_1 \times r_{2k}, \tag{6.12a}$$

$$\hat{p}_{1k} = p_{1k} \times \hat{1}_2, \qquad \hat{p}_{2k} = \hat{1}_1 \times p_{2k}. \tag{6.12b}$$

while the total quantities become

$$\hat{P} = \hat{p}_1 + \hat{p}_2 = p_1 \times \hat{1}_2 + \hat{1}_1 \times p_2\,, \tag{6.13a}$$

$$\hat{R} = \tfrac{1}{2}(\hat{r}_1 + \hat{r}_2) = \tfrac{1}{2}(r_1 \times \hat{1}_2 + \hat{1}_1 \times r_2) \qquad M = 2m\,, \tag{6.13b}$$

with relative expressions

$$\hat{r} = \hat{r}_1 - \hat{r}_2 = r_1 \times \hat{1}_2 - \hat{1}_1 \times r_2\,, \tag{6.14a}$$

$$\hat{k} = \tfrac{1}{2}(\hat{p}_1 - \hat{p}_2) = \tfrac{1}{2}(p_1 \times \hat{1}_2 - \hat{1}_1 \times p_2), \qquad \mu = \tfrac{1}{2}m\,. \tag{6.14b}$$

The total Hamiltonian for the two interacting particles can then be written

$$H_{tot} = p_1^{\hat{2}}/2m + p_2^{\hat{2}}/2m + V(r_1 - r_2) =$$

$$= P^{\hat{2}}/2M + k^{\hat{2}}/2\mu + V(r) = (PT_{tot}P)/2M + (kT_{tot}k)/2\mu + V(\hat{r})\,. \tag{6.15}$$

By assuming $| \psi_{tot} >$ as the total isostate, the fireball is then described by the isoschrödinger equation

$$i \partial_t | \psi_{tot} > = H_{tot} * | \psi_{tot} > = H_{tot} T_{tot} | \psi_{tot} > = E_{tot} | \psi_{tot} > , \qquad (6.16)$$

where one recognizes the presence of conventional local-potential-Hamiltonian forces described by $V(\hat{r})$, as well as the additional presence of nonlocal nonhamiltonian interactions represented by $\hat{I}_{tot}$ or, equivalently, by T_{tot}

By comparison, the description of ref. [32] is based on total and relative quantities given by the conventional nontensorial sums or differences, e.g., $P = p_1 + p_2$, $r = r_1 - r_2$, etc.

Letter [32] is essentially based on the following property in conventional nontensorial spaces

$$[P_i , r_j]_\xi = -i (\hat{I}_1 - \hat{I}_2) \delta_{ij} , \qquad (6.17)$$

which consequently implies the presumed lack of conservation of the total energy

$$[H_{tot}, P]_\xi \neq 0 , \qquad (6.18)$$

in which case one would have the additional violation of the conservation law of the linear momentum and other physical quantities.

The first aspect proved in ref. [9] is the validity of the total conservation laws. In fact, one has the independence of the total linear momentum from the relative coordinates, say, for particles 1 and 2 in one space-dimension

$$[\hat{P} \hat{,} \hat{r}]_{\xi_t *} | \psi_{tot} > = \{ [p_1 \hat{,} r_1]_{\xi_{tot}} b_1^2 \hat{I}_{tot} - \hat{I}_{tot} b_2^2 [p_2 \hat{,} r_2]_{\xi_{tot}} \} T_{tot} | \psi_{tot} > \equiv 0, \qquad (6.19)$$

with similar expressions in arbitrary dimensions.

This readily establishes the following properties

$$[H_{tot} \hat{,} \hat{P}]_{\xi_{tot}} = [V(\hat{r}) \hat{,} \hat{P}]_{\xi_{tot}} \equiv 0, \qquad (6.20)$$

where the isocommutator is in ξ_{tot} under which one evidently has the *conservation of the total energy H_{tot} of the total linear momentum P_{tot} and of other physical quantities*

As we shall see in the next section, this result can be confirmed via the use of the applicable space-time symmetries within the context of the Lie-isotopic theory

7: OPERATOR ISOSYMMETRIES AND ISORELATIVITIES

No significant study of the Bose-Einstein correlation with nonlocal nonhamiltonian internal forces can be done without a knowledge of the applicable symmetries and relativities.

As indicated in Sect. 2, the historical legacy of Bogoliubov, Fermi and others on the ultimate nonlocality of strong interactions is incompatible with conventional space-time symmetries and relativities on numerous independent counts. This requires the identification of suitably generalized space-time symmetries and relativities valid under the broader interactions considered.

The operator isorotational symmetry, which is evidently fundamental per se as well as for all others, was studied in paper [34] and its content is tacitly implies hereon. We shall instead review in this section the construction of the *isogalilean* and *isopoincaré symmetries* for nonlinear, nonlocal and nonhamiltonian interactions, first identified at the classical level in ref.s [11] (see also monographs [14,15] and reviews [16–18]), and at the operator level in ref.s [9] (see also the recent ICTP preprints [35,36]).

Let us begin with a review of the classical isogalilean symmetries $\hat{G}(3.1)$. The basic carrier spaces are the isospaces $\hat{\mathfrak{R}}_t \times T^*\hat{E}(r,\hat{g},\hat{\mathfrak{R}})$, where

$$\hat{\mathfrak{R}}_t = \mathfrak{R}_t \hat{1}_t, \quad \hat{1}_t = b_4^{-2} > 0, \tag{7.1}$$

is the isofield representing time, and $T^*\hat{E}(r,\hat{g},\hat{\mathfrak{R}})$ represents isospaces (3.6).

Isosymmetry $\hat{G}(3,1)$ for a system of n particles can be defined as the Lie-isotopic group of the most general possible, nonlinear, nonlocal and noncanonical isotransformations (3.8) on $\hat{\mathfrak{R}}_t \times T^*\hat{E}(r,\hat{g},\hat{\mathfrak{R}})$ leaving invariant the isoseparations[28]

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[28] Note the uniqueness of the interior isometric for all particles. This is requested by evident geometrical needs for one single space, and it is not in conflict with the results of the preceding section owing to the arbitrariness of the metric itself.

$$t_a - t_b = \text{inv.}, \qquad (7.2a)$$

$$(r_{ak} - r_{bk})\,\hat{b}_k^{\,2}(t, p, \dot{p}, ...)\,(r_{ak} - r_{bk}) = \text{inv.} \quad \text{at } t_a = t_b, \qquad (7.2b)$$

$$t_a, t_b \in \Re_t, \qquad r_a, r_b \in \hat{E}(r,\hat{G},\hat{\Re}) \;\; k = 1, 2, 3, \qquad a, b = 1, 2, ..., n$$

The *general isogalilean transformations* are given by [11.15]

$$t' = t + t^\circ \tilde{b}_4^{-2}, \qquad \text{isotime translations} \qquad (7.3a)$$

$$r_i' = r_i + r^\circ_i \tilde{b}_i^{-2}, \qquad \text{isospace translations} \qquad (7.3b)$$

$$r_i' = r_i + t^\circ v^\circ_i \tilde{b}_i^{-2}, \qquad \text{isogalilei boosts} \qquad (7.3c)$$

$$r' = \hat{R}(\theta) * r, \qquad \text{isorotations} \qquad (7.3d)$$

$$r' = \hat{\pi} * r = -r, \; t' = \hat{\tau} * t = -t \qquad \text{isoinversions} \qquad (7.3e)$$

$$\hat{1} \Rightarrow \hat{1}^d = -\hat{1}, \qquad \text{isodual isotransformations} \qquad (7.3f)$$

where: isotransformations (7.3d) are the isorotations of ref. [34]; isotransformations (7.3f) are new, in the sense that they have no image in the conventional Galilei's relativity and characterize a new class of "isodual spaces" which can only be defined via nontrivial isounits $\hat{1} = -I$, with intriguing geometrical implications (see ref. [15], Sect. III.8 for details); the isoinversion operators are given by $\hat{\pi} = \pi\,\hat{1}$, $\hat{\tau} = \tau\, I$, where π and τ are conventional inversion operators; and the $\tilde{b}$'s, called the *characteristic functions of the medium considered* are nonlinear and nonlocal functions in all variables characterized by

$$\tilde{b}_i^{-2}(r^\circ) = \hat{b}_i^{-2} + r^\circ_j\,[\hat{b}_i^{-2} \hat{,} P_j]/2! + r^\circ_m r^\circ_n\,[[\hat{b}_i^{-2} \hat{,} P_m] \hat{,} P_n]/3! + ... \qquad (7.4a)$$

$$\tilde{b}_i^{-2}(v^\circ) = \hat{b}_i^{-2} + v^\circ_j\,[\hat{b}_i^{-2} \hat{,} G_j]/2! + v^\circ_m v^\circ_n\,[[\hat{b}_i^{-2} \hat{,} G_m] \hat{,} G_n] + ... \qquad (7.5b)$$

The *restricted isogalilean transformations* occur when the $\tilde{b}$'s are constants, in which case $\tilde{b}_k \equiv \hat{b}_k \equiv b_k = \text{cost.} > 0$, and *isotransformations (7.3) evidently become linear and local*, but still more general than the conventional ones.

Thus, being highly nonlinear, general isotransformations (7.3) are highly *noninertial*, and actually characterize a class of frames equivalent to an actual frame in our Earthly environment which, as such, is noninertial[29]. On the contrary, the restricted transformations are indeed inertial as the conventional ones.

Moreover, we should recall that the full nonlinear, nonlocal and nonhamiltonian dependence of the characteristic $\hat{b}$-functions is needed only for the local internal description, say, of one constituent at a given internal space-time point [15]. The noninertial character of the underlying symmetry then confirms the achievement of a generalized composite structure.

On the contrary, the exterior global treatment of such a generalized system as a whole requires the restricted isotransformations, owing to the need of the global behavior of the interior medium (e.g., we need the average speed of light passing through our entire atmosphere, in which case the characteristic $\hat{b}$-functions of our atmosphere are averaged to b-constants)[30].

The reader should therefore keep in mind that *only restricted and therefore inertial isotransformations are applicable for the exterior treatment of the fireball, while the general noninertial isotransformations are needed only for the study of the expected highly noninertial conditions of the individual constituents of the fireball*.

This point is important for the Bose-Einstein correlation. In fact, experimental measures are evidently external, thus requiring restricted isotransformations with consequential preservation of conventional inertial settings.

The structure of the *isogalilean algebra* $\hat{G}(3.1)$ is expressed in terms of the Lie-isotopic brackets

$$[A \hat{,} B] = \frac{\partial A}{\partial r_{ka}} \hat{b}_k^{-2}(t, r, p, ..) \frac{\partial B}{\partial p_{ka}} - \frac{\partial A}{\partial p_{ka}} \hat{b}_k^{-2}(t, r, p, ..) \frac{\partial B}{\partial r_{ka}} \quad (7.6)$$

(verifying the Lie algebra axioms because of the isopoincaré lemma of Sect. 3), possesses the conventional (ordered sets of) parameters

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[29] We should recall that, in the final analysis, *inertial reference frames are a philosophical abstraction because they do not exist in our Earthly physical reality, nor are they attainable in our Solar or Galactic systems*.

[30] As elaborated in ref. [15], the assumption of an *inertial* framework for the *individual* constituents would evidently restrict the applicable symmetries and relativities to be *linear*, thus leading to a conventional Keplerian system with *stable* individual orbits without advances. On the contrary, a necessary condition for the achievement of a generalized composite system is the assumption ab initio of a *noninertial* setting for the *individual* constituents which then permits the identification of *nonlinear* symmetries for *unstable* individual orbits, all this in a way fully compatible with a stable total system, conventional total conservation laws, and inertial exterior-global treatments.

$$w = (w_k) = (\theta_i, r^\circ_i, v^\circ_i, t^\circ), \quad k = 1,2,..10, \tag{7.7}$$

and generators

$$X = \{X_k\} = \{ J_i = \textstyle\sum_a \epsilon_{ijk}\, r_{ja} p_{ka};\ P_i = \sum_a p_{ia}; \tag{7.8a}$$

$$G_i = \textstyle\sum_a (m_a\, r_{ia} - t\, p_{ia}), \quad H = p_{ka}\, \hat{b}_k^{\,2}\, p_{ka} / 2m_a + V(r_{ab}) \} \tag{7.8b}$$

$$r_{ab} = |r_a - r_b|^{\hat{\frac{1}{2}}} = \{(r_{ka} - r_{kb})\, \hat{b}_k^{\,2}\, (r_{ka} - r_{kb})\}^{\frac{1}{2}}, \tag{7.8c}$$

although now defined in $\hat{\mathfrak{R}}_t \times T^*\hat{E}(r,\hat{g},\hat{\mathfrak{R}})$, with isocommutation rules

$$\hat{G}(3.1): \quad \begin{array}{ll} [J_i \hat{,} J_j] = \epsilon_{ijk}\, \hat{b}_k^{-2} J_k, & [J_i \hat{,} P_j] = \epsilon_{ijk}\, \hat{b}_j^{-2} P_k, \quad (7.9a) \\ [J_i \hat{,} G_j] = \epsilon_{ijk}\, \hat{b}_j^{-2} G_k, & [J_i \hat{,} H] = 0, \quad (7.9b) \\ [G_i \hat{,} P_j] = \delta_{ij}\, M\, \hat{b}_j^{-2}, & [G_i \hat{,} H] = 0, \quad (7.9c) \\ [P_i \hat{,} P_j] = [G_i \hat{,} G_j] = [P_i \hat{,} H] = 0, & \quad (7.9d) \end{array}$$

while the (local) *isocasimirs invariants* are the familiar expressions properly written in isospace

$$\hat{C}^{(o)} = \hat{1}_2, \quad \hat{C}^{(1)} = (P \hat{g} P - MH)\, \hat{1}_2, \tag{7.10a}$$

$$\hat{C}^{(2)} = (MJ - G \wedge P)^{\hat{2}} = \{ (MJ - G \wedge P) \hat{g} (MJ - G \wedge P) \}\, \hat{1}_2, \tag{7.10b}$$

The *connected isogalilean group* $\hat{G}(3.1)$ can be expressed via the structure

$$\hat{G}(3.1): \quad r' = \{ [\textstyle\prod_k e^{\,w_k\, \omega^{\mu\sigma} \times\, I_{2\sigma}^{\ \nu}\, (\partial_\nu X_k)\, (\partial_\mu)}_{|\xi}]\, \hat{1}_2 \} * r, \tag{7.11}$$

which results to be well defined owing to the consistent isotopic extension of the Baker–Campbell–Hausdorff composition theorem [3,14], and whose integrated form yields exactly isotransformations (7.3), as the reader can verify.

The proof of the local isomorphisms $\hat{G}(3.1) \approx G(3.1)$ can be done in a number of ways, e.g., via a simple redefinition of the basis under which the structure constants of $\hat{G}(3.1)$ and $G(3.1)$ coincide (see ref. [15], Sect. III.5 for details).

The infinitely possible isosymmetries $\hat{G}(3.1)$ evidently characterize an infinite number of new systems, which are precisely the closed nonhamiltonian systems of the preceding section. In fact, as recalled earlier, all isotopies $G(3.1) \Rightarrow \hat{G}(3.1)$ preserve the original generators, which are therefore conserved as in the conventional case. The systems, however, admit nonlinear, nonlocal and nonhamiltonian internal forces characterized by the isounits $\hat{1}$, thus confirming the validity of conventional total conservation laws under nonhamiltonian internal forces at the classical level.

The *classical isogalilean relativities* [3,10,11,15] can therefore be defined as *a form-invariant description of classical, closed non-Hamiltonian systems*, in a way fully parallel to the conventional relativity. Note that the generalized symmetries and relativities are not necessarily assigned a priori, as in the conventional case, but constructed from given equations of motion via the techniques of the inverse problem. Regrettably, we are forced to refer the interested reader to monographs [14,15] for further details.

The operator formulation of the isogalilean relativities was submitted, apparently for the first time, in memoirs [9]. Consider the (ordered set of) conventional parameters and generators of the quantum mechanical Galilei algebra $G(3.1)$, only properly written in the hadronic structures $\hat{\xi}$, $\hat{F}$ and $\hat{\mathcal{H}}$ of the preceding section

$$X = \{ X_k \} = \{ J = \textstyle\sum_a r_a \wedge p_a, P, \; G = \sum_a (m r_a - t p_a), H \} \qquad (7.12)$$

where the subscript "tot" has been dropped for simplicity, and the angular momentum components have the familiar form $J_k = \epsilon_{kij}\, r_i\, p_j$[31]. The isocommutation rules are then given by Eq.s (6.4) with $T = \mathrm{diag.}\,(b_1^2, b_2^2, b_3^2)$, $b_k = b_k(t, p, \dot{p}, \psi, \partial\psi, \psi^\dagger, \partial\psi^\dagger, ...) > 0$.

The use of the fundamental isocommutators rules (6.3) in $\hat{\xi}$ $(= \hat{\xi}_{tot})$

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[31] The noninitiated reader should be aware that conventional products of operators or "squares", such as $p^2 = pp$, are mathematically and physically inconsistent in hadronic mechanics (e.g., because they violate linearity and transitivity), and must be replaced by the isoproduct" or "isosquares" $p^{\hat{2}} = p{*}p$. The angular momentum components are expressed in terms of the product of a variable $r \in \mathfrak{R}$ and an operator p and, for this reason, they are expressed in terms of the ordinary product rp because of the identities $\hat{r}{*}p \equiv r\,p$, $\hat{r} \in \hat{\mathfrak{R}}$.

implies the Lie-isotopic structure[32]

$$\hat{G}(3.1): \quad \begin{aligned} &[J_i \hat{,} J_j] = -i\,\epsilon_{ijk}\,\hat{b}_k^{-2} J_k, \quad [J_i \hat{,} P_j] = -i\,\epsilon_{ijk}\,b_k^{-2} * P_k\,, && (7.13a)\\ &[J_i \hat{,} G_j] = -i\,\epsilon_{ijk}\,b_j^{-2}\,G_k, \quad [J_i \hat{,} H] = 0\,, && (7.13b)\\ &[G_i \hat{,} P_j] = \delta_{ij}\,b_j^{-2}\,M, \quad [G_i \hat{,} H] = 0\,, && (7.13c)\\ &[P_i \hat{,} P_j] = [G_i \hat{,} G_j] = [P_i \hat{,} H] = 0, \quad i, j = 1, 2, 3\ (= x, y, z)\,, && (7.13d) \end{aligned}$$

whose algebraic equivalence to rules (7.9) is evident.

The *isocasimir invariants* of $\hat{G}(3.1)$ are then given by[33]

$$\hat{C}^o = \hat{1}, \qquad \hat{C}^1 = P^{\hat{2}} - MH, \qquad \hat{C}^3 = (MJ - G \wedge P)^{\hat{2}}\,. \qquad (7.14)$$

The construction of the isoexponentiation of rules (7.13) to the operator *isogalilean group* $\hat{G}(3.1)$ is then left as an exercise for the interested reader, jointly with the verification of isocommutation rules (7.13) and isocasimirs (7.14).

An inspection of the structure of isocasimirs (7.14) illustrates the generalized notion of particles characterized by the isogalilean symmetries indicated in preceding sections.

The above results can be readily extended to an arbitrary number of constituents, as well as to more complex isounits, under a judicious use of the various rules and structures of hadronic mechanics. Nevertheless, as indicated earlier, the exterior-global case with characteristic b-constants is amply sufficient for the exterior treatment of the Bose-Einstein correlation.

It is easy to see that the above isogalilean symmetry is indeed the symmetry of the operator, two-body, composite system of the preceding section. This establishes another proof of the validity of total

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32 The reader not familiar with hadronic mechanics should keep in mind the *isotopic differential rule* used in deriving Eq.s (7.13)

$$[A*B \hat{,} C] = A*[B \hat{,} C] + [A \hat{,} C]*B \qquad (a)$$

identified the first time in ref. [7].

33 Note that in the classical realization, only the functions multiplying the isounits are the isocasimir invariants, while in the operator case the entire structures are isocasimirs, including the isounits. This is evidently due to the fact that the isocommutation rules are computed among functions in the former case, and among matrices in the latter case.

conservation laws, because the generators of the isosymmetry, which are evidently conserved, are total quantities.

This also proves the *exact validity of the Galilei symmetry for the nonrelativistic treatment of the fireball under nonlocal and nonhamiltonian internal interactions,* while the same symmetry is evidently violated in the treatment of ref. [32].

The relativistic generalization of the above results is important for this paper because the correlation is essentially relativistic in nature. Let $M(x,\eta,\Re)$ be a conventional Minkowski space with metric η = diag. (1, 1, 1, −1) over the reals $\Re$. The classical isorelativistic formulations are based on the isotopies of $M(x,\eta,\Re)$, called *isominkowski spaces* which were first introduced in ref. [31] and can be written

$$\hat{M}^{I}(x,\hat{g},\hat{\Re}): \quad x = (r, x^4) = (r, c_o t), \quad r \in \hat{E}_2(r,\hat{\delta},\hat{\Re}) \tag{7.15a}$$

$$\hat{g} = T_2\, \eta, \tag{7.15b}$$

$$\eta = \text{diag.}\ (1,1,1,-1) \in M(x,\eta,\Re), \tag{7.15c}$$

$$x^{\hat{2}} = x^\mu\, \hat{g}_{\mu\nu}\, x^\nu = x^1\, \hat{b}_1{}^2\, x^1 + x^2\, \hat{b}_2{}^2\, x^2 + x^3\, \hat{b}_3{}^2\, x^3 - x^4\, \hat{b}_4{}^2\, x^4 =$$

$$= x^1 \frac{1}{\hat{n}_1{}^2} x^1 + x^2 \frac{1}{\hat{n}_2{}^2} x^2 + x^3 \frac{1}{\hat{n}_3{}^2} x^3 - t \frac{c_o{}^2}{\hat{n}_4{}^2} t, \tag{7.15d}$$

$$T_2 = \text{diag.}\ (\hat{b}_1{}^2, \hat{b}_2{}^2, \hat{b}_3{}^2, \hat{b}_4{}^2) > 0, \quad \hat{\Re} = \Re\, \hat{I}_2, \quad \hat{I}_2 = T_2{}^{-1}, \tag{7.15e}$$

where: c_o is the speed of light in vacuum; the invariant quantity s is defined by

$$ds^{\hat{2}} = -\,dx^\mu\, \hat{g}_{\mu\nu}\, dx^\nu = -\,dr^k \hat{b}_k{}^2\, dr^k - dt\, \hat{c}^2\, dt, \tag{7.16}$$

the $\hat{b}$'s have the functional dependence

$$\hat{b}_\alpha = 1 / \hat{n}_\alpha = \hat{b}_\alpha(\dot{x}, \ddot{x}, \mu, \tau, n, ...) > 0, \quad \alpha = 1, 2, 3, 4, \tag{9.17}$$

and the quantity $\hat{c}$ is given by

$$\hat{c} = c_o \hat{b}_4 = c_o/\hat{n}_4, \tag{7.18}$$

The *general isopoincaré symmetries* [11,15] can be defined as *the Lie-isotopic group of the most general possible nonlinear, nonlocal and noncanonical isotransformations on $\hat{M}(x,\hat{g},\hat{\Re})$ leaving invariant the isoseparation*

$$x_{ab}^{\hat{2}} = (x_a^{\mu} - x_b^{\mu})\, \hat{g}_{\mu\nu}(\dot{x}, \ddot{x}, \mu, \tau, n, ...)\, (x_a^{\nu} - x_b^{\nu}), \qquad (7.19)$$

The *general isopoincaré transformations* can be written [11,15]

$$x'^{\mu} = x^{\mu} + x^{\circ\mu}\, \tilde{b}^{\mu}(s, x, \dot{x}, ...), \quad \text{isotranslations} \qquad (7.20a)$$

$$x' = \hat{\Lambda} * x \qquad \text{isolorentz transformations} \qquad (7.20b)$$

$$x' = \hat{\pi} * x = (-r, x^4), \quad x' = \hat{\tau} * x = (r, -x^4), \quad \text{isoinversions} \qquad (7.20c)$$

$$\hat{1} \Rightarrow \hat{1}^d = -\hat{1} \qquad \text{isodual isotransformations} \qquad (7.20d)$$

where: the functions $\tilde{b}^{\mu}$ are given by

$$\tilde{b}_{\mu}^{-2} = \hat{b}_{\mu}^{-2} + a^{\alpha}\, [\hat{b}_{\mu}^{-2} \hat{,} P_{\alpha}]/2! + x^{\circ\alpha} x^{\circ\beta}\, [[\hat{b}_{\mu}^{-2} \hat{,} P_{\alpha}] \hat{,} P_{\beta}]/3! + \ldots\ldots ; \qquad (7.21)$$

the isolorentz transformations are characterized by the conditions [31]

$$\hat{O}(3.1): \qquad x' = \hat{\Lambda} * x = \hat{\Lambda}\, T_2\, x, \qquad \hat{\Lambda}^t \hat{g} \hat{\Lambda} = \Lambda \hat{g} \hat{\Lambda}^t = \hat{g}^{-1}, \qquad (7.22)$$

and are explicitly given by the isotopic space–rotations $\hat{O}(3)$ of ref. [34], plus the isolorentz boosts of ref. [31],

$$x'^1 = x^1, \qquad (7.23a)$$

$$x'^2 = x^2, \qquad (7.23b)$$

$$x'^3 = \hat{\gamma}\, (x^3 - \beta x^4), \qquad (7.23c)$$

$$x'^4 = \hat{\gamma}\, (x^4 - \hat{\beta} x^3), \qquad (7.23d)$$

where[34]

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[34] The author would like to thank E. Ferrari of the Phys. Dept. "G. Marconi" of the Univ. "La Sapienza" in Rome, Italy, for bringing to his attention the original insufficient form $\hat{\gamma} = (1 - \hat{\beta}^2)^{-\frac{1}{2}}$ during a seminar delivered at the Math. Dept. "G. Castelnuovo" of the same University. In fact, certain physical media imply $\hat{\beta}^2 > 1$ even for speeds $v \ll c_o$ (see later on in this section), which would imply imaginary values of $\hat{\gamma}$. This aspect was resolved during discussions with R. Mignani of the same Phys. Dept. following the seminar, and resulted in the expression $\hat{\gamma} = | 1 - \hat{\beta}^2 |^{-\frac{1}{2}}$ based on paper [38]. Prior to these discussions, the general form of the isolorentz transformations was thought to be form (5.28) of ref. [37]. These discussions essentially implied that all possible isolorentz transformations can be cast in form (7.23) above, including those with $\hat{\beta}^2 > 1$. In turn, this has important physical implications owing to

$$\beta = v/c_o, \quad \hat{\beta} = \frac{v b_3}{c_o b_4}, \quad \hat{\beta}^2 = \beta^{\hat{2}} = \frac{v^k b_k{}^2 v^k}{c_o b_4{}^2 c_o}, \qquad (7.24a)$$

$$\cosh (v\, b_3\, b_4) = \hat{\gamma} = |1 - \beta^{\hat{2}}|^{-\frac{1}{2}}, \qquad (7.24b)$$

$$\sinh (v\, b_3\, b_4) = \hat{\beta}\,\hat{\gamma}; \qquad (7.24c)$$

the isoinversions operators are the same as in Eq.s (7.4); and the isodual transformations (7.20d) are particularly intriguing, inasmuch as they allow the identification of the *isodual isominkowski spaces*

$$\hat{M}^d(x,\hat{g}^d,\hat{\Re}^d), \quad \hat{g}^d = -\hat{g}, \quad \hat{\Re}^d = \Re\,\hat{1}^d, \quad \hat{1}^d = -\hat{1}. \qquad (7.25)$$

This includes the isodual of the conventional Minkowski space $\hat{M}^d(x,\eta^d,\hat{\Re}^d)$, $\eta^d = -\eta$, $\hat{\Re}^d = \Re\,\hat{1}^d$, $\hat{1}^d = -I$, which has remained undetected in contemporary relativistic (and gravitational) theories because it necessarily requires the isotopic structures for its very identification[35].

The *restricted isopoincaré transformations* [15] occur when the characteristic quantities of the medium are constants, $\tilde{b}_\mu \equiv \hat{b}_\mu \equiv b_\mu$ = const. > 0; they hold for the "global" exterior representation of a system as a whole; and they imply the full preservation of the inertial character of conventional relativities because isotransformations (7.20) return to be linear and local.

The *isopoincaré algebra* $\hat{P}(3.1)$ admits the decomposition as in the conventional case

$$\hat{P}(3.1) = \hat{O}(3.1) \oplus \hat{T}(3.1), \qquad (7.26)$$

with conventional (ordered sets of) parameters

$$w = \{w_k\} = \{\theta, u, x^\circ\}, \quad k = 1, 2, ..., 10 \qquad (7.27)$$

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the abstract identity of isotransformations (7.23) with the conventional ones [15].

[35] This isodual isospace is nontrivial and fundamentally different from the *conventional* Minkowski space $M(x,\eta^d,\Re)$ with metric $\eta^d = -\eta$ but on an ordinary field $\Re$. In fact, *isospace $\hat{M}^d(x,\eta^d,\hat{\Re}^d)$ is a space with a negative-valued unit* $-I$. As a consequence, $\hat{M}^d(x,\eta^d,\hat{\Re}^d)$ is a sort of "mirror image" of our space $M(x,\eta,\Re)$ in which all numerical values are negative, including absolute values, e.g., $|\,3\,| = -3$. Intriguingly, physical events are admitted jointly by a space and its isodual, that is, the equations of motion coincide for $M(x,\eta,\Re)$ and $M^d(x,\eta^d,\Re^d)$, thus yielding a new universal invariance law under isoduality (see ref. [15], Ch. V) for details).

and generators

$$X = \{ X_k \} = \{ J_{\mu\nu} = \sum_a (x_{a\mu} p_{\nu a} - x_{\nu a}\, p_{\mu a};\ P_\mu = \sum_a p_{\mu a} \}, \quad (7.28)$$

with relativistic isocommutators

$$[A,B] = \frac{\partial A}{\partial x_\mu}\, \hat{g}_{\mu\nu} \frac{\partial B}{\partial p_\nu} - \frac{\partial B}{\partial x_\mu}\, \hat{g}_{\mu\nu} \frac{\partial A}{\partial p_\nu}$$

$$= \frac{\partial A}{\partial x^\mu}\, \hat{g}^{\mu\nu} \frac{\partial B}{\partial p^\nu} - \frac{\partial B}{\partial x^\mu}\, \hat{g}^{\mu\nu} \frac{\partial A}{\partial p^\nu}, \quad (7.29a)$$

$$(\hat{g}^{\mu\nu}) = (\hat{1}_2{}^{\mu}{}_{\alpha} \eta^{\alpha\nu}) = (\hat{g}_{\mu\nu})^{-1} = (\eta_{\mu\alpha}\, T_2{}^{\alpha}{}_{\nu})^{-1}, \quad (7.29b)$$

yielding the *isocommutation rules of the classical isopoincaré algebra* [11,15]

$$[J_{\mu\nu} \hat{,} J_{\alpha\beta}] = \hat{g}_{\nu\alpha} J_{\beta\mu} - \hat{g}_{\mu\alpha} J_{\beta\nu} - \hat{g}_{\nu\beta} J_{\alpha\mu} + \hat{g}_{\mu\beta} J_{\alpha\nu}, \quad (7.30a)$$

$$\hat{P}(3.1): \quad [J_{\mu\nu} \hat{,} P_\alpha] = \hat{g}_{\mu\alpha} P_\nu - \hat{g}_{\nu\alpha} P_\mu, \quad (7.30b)$$

$$[J_\mu \hat{,} P_\nu] = 0, \qquad \mu, \nu = 1, 2, 3, 4 \quad (7.30c)$$

As one can see, isoalgebras (7.30) formally coincide with the conventional Poincaré algebra, although the quantities $\hat{g}_{\mu\nu}$ are now the components of the isometrics and, as such, possess a nontrivial functional dependence on all possible quantities.

Despite that, it is easy to prove that all possible isoalgebras $\hat{P}(3.1)$ are locally isomorphic to the conventional algebra $P(3.1)$ for all positive-definite isounits $\hat{1}$ (see ref. [15], Sect. IV.6).

The isocasimir invariants are given by

$$\hat{C}^{(o)} = \hat{1}_2 = \hat{T}_2^{-1}, \quad (7.31a)$$

$$\hat{C}^{(1)} = p^{\hat{2}} = (p^\mu\, \hat{g}_{\mu\nu}\, p^\nu)\, \hat{1}_2 = (p^\mu\, p_\mu)\, \hat{1}_2, \quad (7.31b)$$

$$\hat{C}^{(2)} = W^{\hat{2}} = (W^\mu\, \hat{g}_{\mu\nu}\, W^\mu)\, \hat{1}_2, \quad W_\mu = \epsilon_{\mu\alpha\beta\rho}\, J^{\alpha\beta}\, P^\rho, \quad (7.31c)$$

and they illustrate better the notion of *isoparticles* as constituents of a hadronic composite system.

Finally the *connected isopoincaré groups* $\hat{P}(3.1)$ can be expressed via the exponentiations

$$a' = \{\hat{\Lambda}, \hat{T}\} * a = \{ [e_{|\xi}^{\, w_k \, \omega^{ir} \, I_{2r}{}^{j} (\partial_j X_k) (\partial_i)}] \, \hat{I}_2 \} * a =$$

$$= \{ S_{\hat{g}(\theta,u)} , T_{\hat{g}}(x^\circ) \} \, a, \tag{7.32}$$

which confirm the (connected components of) isotransformations (7.20), as explicitly given for the case of isotranslations by

$$\hat{T}_a(3.1): \quad x'^{\mu} = \hat{T}(x^\circ) * x^{\mu} = x^{\mu} + x^{\circ\mu} \, \hat{b}^{\mu \, -2}(\dot{x}, \ddot{x}, \mu, \tau, n, ...), \tag{7.33a}$$

$$p'^{\mu} = \hat{T}(x^\circ) * p^{\mu} = p^{\mu}, \tag{7.33b}$$

while the isolorentz components are those of ref. [31].

Note as an incidental comment that, except for the positive-definiteness, the isometrics $\hat{g} = T\,\eta$ are unrestricted and, as such, they can indeed be conventional Riemannian metrics $g(x) = T\,\eta$. Thus, *the isopoincaré symmetry $\hat{P}(3.1)$ is the general isometry of a Riemannian space $R(x,g,\mathfrak{R})$*, i.e., the methods here outlined allow the explicit construction of the symmetry transformations of any given gravitational line element, such as the Schwartzschild's element[36].

It is useful for Sect. 9 to briefly outline the *isorelativistic kinematics* on $\hat{M}(x,\hat{g},\hat{\mathfrak{R}})$ (ref. [15], Sect. IV.7). The *isofourvelocity* $u^{\mu} = dx^{\mu}/ds$ can be defined by

$$u^{\hat{2}} = u^{\mu} \, \hat{g}_{\mu\nu} \, u^{\nu} = -1, \tag{7.34}$$

with components

$$u^4 = \frac{dx^4}{ds} = \frac{dt}{ds} = \hat{\gamma}\,\hat{c} = \hat{\gamma} c_o \hat{b}_4, \tag{7.35a}$$

$$u^k = \frac{du^k}{ds} = \frac{dx^4}{ds}\,\frac{dx^k}{dx^4} = \hat{\gamma}\,\hat{c}\,v^k = \hat{\gamma} c_o b_4 v^k, \tag{7.35b}$$

$$\hat{\gamma} = | 1 - \beta^{\hat{2}} |^{-\frac{1}{2}}, \quad \hat{\beta}^2 = (v^k b_k{}^2 v_k) / (c_o \hat{b}_4{}^2 c_o), \tag{7.35c}$$

where v is the velocity in Euclidean isospace $\hat{E}_2(r,\hat{\delta},\hat{\mathfrak{R}})$.

[36] As it was the case for the isodual isospaces and isosymmetries, these general symmetries of conventional gravitational models require the necessary lifting of the unit of the theory, and this is the reason for their lack of detection until recently.

We now introduce the *isofourmomentum* as the isofourvector in $\hat{M}(x,\hat{g},\hat{\mathfrak{R}})$

$$p = (p^{\mu}) = (\hat{m}u^{\mu}) = (m_o \hat{\gamma}\, \hat{c}\, v^k,\ m_o\, \hat{\gamma}\, \hat{c}), \tag{7.36a}$$

$$\hat{m} = m_o\, \hat{\gamma}, \tag{7.36b}$$

The isocasimir (7.31b) then implies the following *fundamental isoinvariant of the isopoincaré symmetries*

$$p^{\hat{2}} = p^{\mu}\, \hat{g}_{\mu\nu}\, p^{\nu} = p^k\, \hat{b}_k{}^2\, p^k - p^4 \hat{c}^2\, p^4$$

$$= m_o{}^2\, \hat{\gamma}^2\, c^2\, v^k\, b_k{}^2\, v^k - m_o{}^2\, \hat{\gamma}^2\, c^4 =$$

$$= - m_o{}^2 \hat{\gamma}^2\, c^4\, (1 - \beta^{\hat{2}}) = - m_o{}^2\, c^4 = - m_o{}^2\, c_o{}^4\, b_4{}^4, \tag{7.37}$$

or, equivalently,

$$(p^{\mu}\, \hat{g}_{\mu\nu}\, p^{\nu}) / m_o{}^2 c^4 = -1, \tag{7.38}$$

where the reader can see the automatic "renormalization" of the mass, energy and other physical quantities permitted by our isotopies already at the classical level.

Much along the conventional case, *the isospecial relativities [31,15] are a form-invariant description of closed-isolated relativistic systems verifying all conventional total conservation laws, while admitting nonlinear, nonlocal and nonhamiltonian internal forces.*

It should be recalled that the isogalilean and isospecial relativities share the same mutual compatibility as in the conventional case, e.g., because of the existence of a consistent isotopy of the Inonü-Wigner contraction which maps the isopoincaré symmetries into the isogalilean ones (ref. [15], Sect. VI.3)[37].

The *operator formulation of the isopoincaré symmetry* has been investigated in ref.s [9,35] via two equivalent methods. The carrier spaces remain isominkowski spaces (7.15) in both cases. Consider then the conventional parameters and generators of the operator P(3.1) symmetry, as in Eq.s (7.27) and (7.28), respectively.

The first method consists in assuming the conventional, regular (also called fundamental) matrix representation of generators (7.28)

[37] In turn, these occurrences are a particular case of the *isogeneral relativities* on isoriemannian spaces $\hat{R}(x,\hat{g},\hat{\mathfrak{R}})$ of ref. [15] which are not considered in this paper.

and subject their commutation rules to the lifting

$$[A,B] = AB - BA \quad \Rightarrow \quad [A \hat{,} B] = ATB - BTA, \qquad (7.39)$$

where T is the diagonal matrix (7.15e). The isocommutators of said conventional generators then formally yield the same the same rules (7.30) and isocasimir invariants (7.30), with reformulation of group structure (7.32) in the operator envelope $\hat{\xi}$.

The second method, evidently equivalent to the preceding one, is based on assuming a representation of the conventional generators (7.28) via isodifferential operators. A relativistic generalization of the hadronization of Sect. 4 then leads to the following operator iso-four-momentum

$$p_{a\mu} * | \psi_a > \; = \; p_{a\mu} T_a | \psi_a > \; = \; -i \, \hat{1}_{a\mu,}{}^{a\nu} \frac{\partial}{\partial x^{a\nu}} | \psi_a >, \qquad (7.40)$$

$$\mu = 1, 2, 3, 4, \qquad a = 1, 2, 3, \ldots, n$$

Under the assumption of an isotopic element T independent of x or constant, the fundamental isocommutation rules are given by the following isorelativistic extension of rules (6.3)

$$[x_{a\mu} \hat{,} x_{b\nu}] * | \psi_a > \; = \; [p_{a\mu} \hat{,} p_{b\nu} * | \psi_a > \; \equiv \; 0, \qquad (7.41a)$$

$$[p_{a\mu} \hat{,} x_{a\nu}] * | \psi_a > \; = \; - i \, \eta_{\mu\nu} * | \psi_a > \qquad (7.41b)$$

namely, the isoeigenvalues do not exhibit b-terms any more as in Eq.s (6.3), by coinciding with the corresponding conventional eigenvalues[38]

As it happened in the operator isogalilean case, the isopoincaré algebra is then given by

$$[J_{\mu\nu} \hat{,} J_{\alpha\beta}] * | \psi >= i \, (\eta_{\nu\alpha} J_{\beta\mu} - \eta_{\mu\alpha} J_{\beta\nu} - \eta_{\nu\beta} J_{\alpha\mu} + \eta_{\mu\beta} J_{\alpha\nu}) * | \psi > \qquad (7.42a)$$

$$\hat{P}(3.1): \quad [J_{\mu\nu} \hat{,} P_\alpha] * | \psi > \; = \; i \, (\eta_{\mu\alpha} P_\nu - \eta_{\nu\alpha} P_\mu) * | \psi >, \qquad (7.42b)$$

$$[J_\mu \hat{,} P_\nu] * | \psi > \; = \; 0, \qquad \mu, \nu = 1, 2, 3, 4 \qquad (7.42c)$$

---

[38] This is due to the fact that the quantity $x_{a\mu}$ in Eq.s (7.41b) is covariant and thus given by

$$x_{a\nu} \; = \; \hat{g}_{\nu\alpha} \, x^{a\nu} \; \equiv \; b_\nu^{\,2} \, \eta_{\nu\alpha} x^{a\alpha}, \quad x^a \equiv x_a \qquad (a)$$

which eliminates the b^{-2} terms originating from the isocommutators $[p_\mu \hat{,} x^\nu]$.

namely, the structure constants of $\hat{P}(3.1)$ formally coincide with those of P(3.1), thus confirming not only the local isomorphisms $\hat{P}(3.1) \approx P(3.1)$, but also the identity at the abstract level of the conventional and isotopic symmetries. The identity at the abstract level of the corresponding relativities is then consequential.

The interested reader may verify that the isocasimirs under realization (7.41) remain those of Eq.s (7.31), build the corresponding connected component of $\hat{P}(3.1)$, and work out the operator formulation of the isorelativistic kinematics as an operator image of Eq.s (7.34)–7.38).

The *isofield theory* , i.e., the isotopy of conventional field equations in a way to be invariant under $\hat{P}(3.1)$ has been studied with preliminary applications in ref.s [9] and preprint [36].

We close this section with an indication of the primary function of the isominkowski spaces (7.15): a relativistic geometrization of interior physical media (such as water, gases, conductors, super-conductors, nuclei, hadrons, stars, etc.), which has some similarities with the Riemannian characterization of gravity, although it is independent of it because it holds for flat spaces, and then persists in the presence of curvature.

When the medium is transparent (e.g., water or atmospheres), the function $\hat{c} = c_o/\hat{n}_4$ represents the *local* speed of light, with $\hat{n}_4$ being the local index of refraction. When considering their averages into constants, the quantity $c = c_o/n_4$ represents the *average* speed of light through the medium considered (e.g., the average speed of light when passing through our entire atmosphere), and n_4 is the average index of refraction.

When the medium is not transparent, the local (global) quantity $\hat{c}$ (c) is a purely geometrical quantity which does not necessarily represent a physical speed (much along the term $-g_{44}$ of the Riemannian metric) but merely expresses the alteration of the geometry of space due to the presence of matter.

Assume for simplicity that all space components of the characteristic functions are equal

$$b_1 = b_2 = b_3, \quad n_1 = n_2 = n_3 \tag{7.43a}$$

$$x^{\hat{2}} = \frac{1}{n_3^2} x_k x_k - \frac{1}{n_4^2} x^4 x^4, \tag{7.43b}$$

to remove the isorotational component, and focus the attention on the isolorentz contributions. Also, assume for simplicity the exterior global case with constant b's.

In examining isoseparation (7.43b), one can see the existence of a first class of isotopies with $n_3 = n_4$, resulting in the scalar isotopies of the Minkowskian separation

$$x^2 = \frac{1}{n_4^2} x^2, \qquad (7.44)$$

which evidently represent the simplest possible cases of homogeneous and isotropic physical media, such as water[39].

However, in general, $n_3 \neq n_4$, in which case isospaces $\hat{M}(x,\hat{g},\hat{\mathfrak{R}})$ represent genuine inhomogeneous and anisotropic physical media.

A study of all geometrically significant cases conducted in ref. [11] (see also ref. [15], Sect. IV.10) has identified the following *nine different types of (flat) isominkowski spaces* depending on the relative values of 1, n_3 and n_4

$$\text{TYPE 1:} \quad n_3 = n_4, \; n_4 = 1; \quad \hat{\beta} \equiv \beta, \quad \hat{\gamma} \equiv \gamma; \qquad (7.45a)$$

$$\text{TYPE 2:} \quad n_3 = n_4, \; n_4 > 1; \quad \beta \equiv \beta, \quad \hat{\gamma} \equiv \gamma; \qquad (7.45b)$$

$$\text{TYPE 3:} \quad n_3 = n_4, \; n_4 < 1; \quad \hat{\beta} \equiv \beta, \quad \hat{\gamma} \equiv \gamma; \qquad (7.45c)$$

$$\text{TYPE 4:} \quad n_3 < n_4, \; n_4 > 1; \quad \hat{\beta} > \beta, \quad \hat{\gamma} < \gamma; \qquad (7.45d)$$

$$\text{TYPE 5:} \quad n_3 < n_4, \; n_4 = 1; \quad \hat{\beta} > \beta, \quad \hat{\gamma} < \gamma; \qquad (7.45e)$$

$$\text{TYPE 6:} \quad n_3 < n_4, \; n_4 < 1; \quad \hat{\beta} > \beta, \quad \hat{\gamma} < \gamma; \qquad (7.45f)$$

$$\text{TYPE 7:} \quad n_3 > n_4, \; n_4 > 1; \quad \hat{\beta} < \beta, \quad \hat{\gamma} > \gamma; \qquad (7.45g)$$

---

[39] The isospecial relativity, which is specifically and solely conceived for relativistic motion within physical media, dispel a number of rather popular beliefs in conventional special relativities. As an example, ***the universal invariant speed results to be the maximal causal speed*** $V_{Max} = c_o\, n_3 / n_4$ ***and not the speed of light.*** In fact, ***the sum of two speeds of light in water is not equal to the local speed of light***, as one can easily verify. On the contrary, the maximal causal speed in water is $V_{Max} = c_o > c$ (as for the electrons in the Cherenkov light, which travel at local speeds bigger than the speed of light $c = c/n_4$). Then, ***the isorelativistic sum of two*** V_{Max} ***does indeed produce*** V_{Max} Also, the maximal causal speed is not the speed of light, evidently because electrons can travel in water faster than light. Of course, for the conventional special relativity $V_{Max} \equiv c_o$ because motion is in vacuum . The point however persists that ***the speed of light is an invariant only in vacuum*** and, therefore, it is not a "universal" invariant. The interested reader may consult ref. [15] Chap. V (and Chap. VII for experimental verifications) on these and several other intriguing aspects on the isotopy of the special relativity.

TYPE 8: $n_3 > n_4$, $n_4 = 1$; $\hat{\beta} < \beta$, $\hat{\gamma} > \gamma$; (7.45h)

TYPE 9: $n_3 > n_4$, $n_4 < 1$; $\hat{\beta} < \beta$, $\hat{\gamma} > \gamma$; (7.45i)

By and large, the above classification represents media of increasing geometrical complexities, e.g., in regard to inhomogenuity and anisotropy. Also, while the media of the first types represent ordinary matter, the achievement of the media of higher Types 7, 8 and 9 requires increased densities which can be only achieved in the interior of hadrons and stars.

In fact, media of Type 1 represent the conventional Minkowski space (i.e., the empty space-time); those of Type 2 can represent water, or any homogeneous and isotropic physical medium in which the speed of light is $c < c_0$; media of Type 3 are under investigation to represent superconductors [28]; media of Type 4 can apparently represent planetary or astrophysical atmospheres [38,39]; media of Type 5 and 6 can apparently represent nuclei [15]; and, finally, media of Type 7, 8 and 9 can apparently represent the interior of hadrons and stars [39,40].

An important objective of this paper is ***to identify the type of physical media we expect in the fireball of the Bose-Einstein correlation*** .

This is important because it can provide information suitable for possible additional tests of the nonlocality of the correlation. As an example, media of Type 4 imply a redshift of light [11] (because $\hat{\beta} > \beta$ and $\hat{\gamma} < \hat{\gamma}$), that is, *the isorelativistic theories predict that light passing through an inhomogeneous and anisotropic atmosphere of Type 4 is redshifted* (see the presentation of ref. [15]), with implications for quasars redshifts [38] which are apparently suitable for independent experimental verification [39].

By contrast, *media of Type 7, 8 and 9 imply a blueshift* [11] (because $\hat{\beta} < \beta$ and $\hat{\gamma} > \gamma$). These latter media are apparently useful for a unified description of the behavior of the meanlife of unstable hadrons with speeds [39,40].

As we indicated earlier, owing to the very high energies involved, the fireball of the Bose-Einstein correlation is expected to have a density bigger than that of hadrons. We therefore expected that the correlation fireball is a physical medium of type 7, or 8 or 9. This knowledge is evidently useful for comparative purposes with other settings [38-40].

8. INTERIOR ISORELATIVISTIC TREATMENT OF THE BOSE–EINSTEIN CORRELATION

We are now equipped to conduct a direct study of the interior of the fireball of Bose–Einstein correlation based on the following main[40]

> *HYPOTHESIS 8.1: The Bose–Einstein correlation is created by nonlinear, nonlocal and nonhamiltonian interactions caused by the total mutual overlapping of the wavepackets and charge distributions of the original constituents at the initiation of the fireball.*

Once created, a number of theoretical arguments imply that the correlation persists at all subsequent stages the process.

In this section we shall study the origin of correlation inside the fireball immediately after its creation, e.g., immediately after the annihilation of the original proton–antiproton.

Our central assumption is that the nonlocal internal interactions imply a generalization of the conventional Minkowski space

$$M(x,\eta,\Re): \; x^2 = x^\mu \eta_{\mu\nu} x^\nu = x_1x_1 + x_2x_2 + x_3x_3 - x_4x_4 , \quad (8.1a)$$

$$x = (x^\mu) = (\vec{x}, x_4), \quad x_4 = c_o t, \quad \mu = 1, 2, 3, 4, \quad (8.1b)$$

into isominkowski space (7.15), now assumed for the interior problem,

$$\hat{M}_{Int}(x,\hat{g},\hat{\Re}): \; x^{\hat{2}} = x^\mu \hat{g}_{\mu\nu} x = x_1 \hat{b}_1{}^2 x_2 + x_2 \hat{b}_2{}^2 x_2 + x_3 \hat{b}_3{}^2 x_3 - x_4 \hat{b}_4{}^2 x_4, \quad (8..2a)$$

$$T = T^\dagger > 0, \quad \hat{g} = T\eta, \quad \hat{\Re} = R\,\hat{1}, \quad \hat{1} = T^{-1}, \quad (8.2b)$$

where the isotopic element T has the most general possible, nonlinear

[40] As stressed in footnote[3] no quantitative treatment of Hypothesis 8.1 is permitted by relativistic quantum mechanics and the special relativities because of: the nonlocal and nonhamiltonian character of the interactions; their integral nature; the inhomogeneous and anisotropic structure of the medium; and other reasons. The identification of which covering mechanics is suitable for Hypothesis 8.1 is evidently debatable at this writing, and *the treatment via hadronic mechanics is not necessarily presented as unique*. The point however persists that *the insufficiency of relativistic quantum mechanics and related special relativity for the treatment of Hypothesis 8.1 remains out of scientific doubts*.

and nonlocal dependence on all variables, wavefunctions and their derivatives, as well as additional quantities as requested by the physical conditions considered,

$$T = T(x, \dot{x}, \psi, \psi^{\dagger}, \partial\psi, \partial\psi^{\dagger}, ...) . \qquad (8.3)$$

For clarity, let us first review the conventional treatment of correlation [1] and, then present our isotopic generalization. Suppose, for simplicity, that the fireball is made-up of two unspecified particles a and b, which can each exist in two separate states denoted 1 and 2 at the location r_1 and r_2, as detected from the observer's position x. The individual states are then given by

$$|\psi_1> = |\psi_1{}^1(x-r_1)> + |\psi_1{}^2(x-r_2)>, \qquad (8.4a)$$

$$|\psi_2> = |\psi_2{}^1(x-r_1)> + |\psi_2{}^2(x-r_2)>, \qquad (8.4b)$$

For the case of ordinary plane waves in Minkowski space, the above structure acquires the familiar realization [1]

$$\psi_a = f_{a1}\, e^{ip(x-r_1)} + f_{a2}\, e^{ip(x-r_2)}, \qquad (8.5a)$$

$$\psi_b = f_{b1}\, e^{ip(x-r_1)} + f_{b2}\, e^{ip(x-r_2)}, \qquad (8.5b)$$

The probability of observing two particles with momenta p_1 and p_2 is then given in the generic case by

$$P(p_1, p_2) = \begin{pmatrix} <\psi_{b1}| & <\psi_{b2}| \end{pmatrix} \begin{pmatrix} |\psi_{a1}> \\ |\psi_{b2}> \end{pmatrix} = <\psi_{a1}|\psi_{a1}> + <\psi_{b2}| + \psi_{b2}>, \qquad (8.6)$$

which, in realization (8.5), becomes

$$P(p_1, p_2) = (f_{a1})^2 + (f_{b2})^2. \qquad (8.7)$$

In the above case we have: A) a strictly local theory, B) full coherence of the two states, and C) complete absence of correlation.

We now introduce the isotopic generalization of the above setting for the specific intent of treating nonlocal internal interactions. It is evident that these interactions cannot affect the coherent states which can be at best renormalized. The nonlocal interactions are therefore expected to affect in a primary way the remaining components of the states.

This physical setting can be well represented as follows. First, we introduce the isostates of isohilbert space of Eq.s (5.5) here interpreted as the internal one $\hat{\mathcal{H}}_{int}$

$$|\hat{\psi}_1> \; = \; |\hat{\psi}_{11}(x-r_1)> + |\hat{\psi}_{12}(x-r_2)>, \qquad (8.8a)$$

$$|\hat{\psi}_2> \; = \; |\hat{\psi}_{21}(x-r_1)> + |\hat{\psi}_{22}(x-r_2)>, \qquad (8.8b)$$

where the second subscript indicates a new state, e.g., because of the new normalization in an internal isospace $\hat{\mathcal{H}}_{Int}$

The realization of the above states corresponding to Eq.s (8.5) is now given by the generalized notion of *isoplanewaves* [11,15] representing conventional plane-waves while traveling within an inhomogeneous and anisotropic physical medium geometrized via isominkowski spaces (8.2), and can be written

$$\hat{\psi}_a \; = \; \hat{f}_{a1}\, e^{ip*(x-r_1)} + \hat{f}_{a2}\, e^{ip*(x-r_2)}, \qquad (8.9a)$$

$$\hat{\psi}_b \; = \; \hat{f}_{b1}\, e^{ip*(x-r_1)} + \hat{f}_{b2}\, e^{ip*(x-r_2)}, \qquad (8.9b)$$

where the composition are now in the isospace (8.2), i.e., $p*x = pT\eta x = p_\mu \hat{g}^{\mu\nu} x_\nu = p_\mu\, x^\mu$.

In order to proceed, we have to select the appropriate, explicit form of the isotopic element. That suggested for the simplest possible case under consideration is given by the the (2×2)-form acting in the tensorial product of the total state (Sect. 6)

$$T = T(0)\,\mathcal{P}(0) \; = \; \hat{T}(0) \begin{pmatrix} K_{a1} & K_{a2}\,[1 - e^{N_1\int d^4x'\, \psi^\dagger{}_{b2}(x')\, \psi_{a1}(x')}] \\ K_{b1}[1 - e^{N_2\int d^4x'\, \psi^\dagger{}_{a2}(x')\, \psi_{b1}(x')}] & K_{b2} \end{pmatrix}$$

(8.10)

where: the exponents characterize *Animalu's isounit* [28] and the K's and N's quantities are non-null real constants.

By using the above expression, we reach the isoexpectation value (Sect. 4)

$$\hat{P}(p_1, p_2) = \{ K_{a1} < \hat{\psi}_{a1} | T(0) | \psi_{a2} > + K_{b2} < \hat{\psi}_{b2} | \hat{T}(0) | \hat{\psi}_b{}^2 > \} + \{ K_{a2} [1 - e^{iC_1 \int d^4x' \, \psi^\dagger{}_{b2}(x') \, \psi_{a1}(x')}] < \psi_{b2} | T(0) | \psi_{a1} > + K_{b1} [1 - e^{iC_2 \int d^4x' \, \psi^\dagger{}_{a2}(x') \, \psi_{b1}(x')}] < \psi_{a2} | T(0) | \psi_{b1} > \}. \quad (8.11)$$

By using isoplanewaves (8.9) and via suitable selection of the element T(0), it is possible to reduce the above expression to simpler forms with the structure

$$| \Psi (p_1, p_2) |^{\hat{2}} = (f_{a1} K_{a1} + f_{b2} K_{b2})^2 + \quad (8.12$$

$$K_{a2} [1 - e^{iC_1 \int d^4x' \, \psi^\dagger{}_{b2}(x') \, \psi_{a1}(x')}] \; K_{b1} [1 - e^{iC_2 \int d^4x' \, \psi^\dagger{}_{a2}(x') \, \psi_{b1}(x')}] \times$$

$\cos (\Delta p * \Delta r)$,

Expression [8.12] can be easily made to coincide with Eq. (19) of ref. [1] via a suitable choice of the K-coefficients.

The above results show quite clearly the nonlocal origin of the correlation because of the following properties:

I) the integrals in the exponents of Eq.s (8.12) show clearly their dependence on the superposition of the indicated wavepackets, exactly along the notion of nonlocality of hadronic mechanics (Sect. 4);

II) The above superposition occurs for the uncorrelated states, exactly as desired; and

III) At the limit when the wavefunctions are no longer superimposed, i.e., disjoint in space-time, the exponents of the integrals in Eq.s (8.12) are identically null, the coefficients of the correlation are also identically null, thus recovering the original uncorrelated structure identically.

We can therefore conclude by saying that *hadronic mechanics provides a direct interpretation of the Bose-Einstein correlation, which can be essentially reduced to the identification and suitable treatment of the internal nonlocal interactions. In particular, the correlation is directly linked to the isotopy of Planck's constant into a nondiagonal isounit*

$$\hbar\, I \;\Rightarrow\; \hbar \hat{I}(x, \dot{x}, \hat{\psi}, \hat{\psi}^{\dagger}, \partial\hat{\psi}, \partial\hat{\psi}^{\dagger}, ...), \qquad (8.13)$$

Finally, such nondiagonality emerges from the tensorial product of the individual isounits of particles under short-range nonlocal interactions.

9. EXTERIOR ISORELATIVISTIC TREATMENT

We now pass to the study of a further aspect, the exterior isorelativistic description of the formation and decay of the fireball. Its use for a better fit of available experimental data will be considered in the next section.

For this purpose, we introduce an external isominkowski space

$$\hat{M}_{Ext}(x,\hat{g},\hat{\mathfrak{R}}): \quad x^{\hat{2}} = x^{\mu}\hat{g}_{\mu\nu}x^{\nu}, \quad \hat{\mathfrak{R}} = \mathfrak{R}\,\hat{1}, \;\; \hat{g} = \hat{T}\eta, \;\; \hat{1} = T^{-1}, \quad (9.1)$$

which carries a physical interpretation different than that of the preceding section. In fact, as indicated earlier, in the exterior problem the isotopic element $\hat{T}$ can be assumed to have a structure of the type

$$\hat{T} = \hat{\mathcal{P}}(0)\, F(r,...)\, T, \qquad (9.2)$$

where: $\hat{\mathcal{P}}(0)$ is the isoprojection operator to coherent states of the preceding section; $F(r,...)$ provides the representation of the original nonlocality (i.e., that prior to the emission of free particles); the term T is of the type

$$T = \text{diag.}(b_1^2, b_2^2, b_3^2, b_4^2), \quad b_{\mu} = \text{constants} > 0, \qquad (9.3)$$

whose space component

$$\hat{\delta} = \text{diag.}\,(b_1^2, b_2^2, b_3^2), \qquad (9.4)$$

represents the shape of the fireball to be determined later on from the experimental data. Finally, the fourth component b_4 represents a further degree of freedom of the theory to be clarified below and also computed from the experimental data.

For the case of n isoparticles $a = 1, 2, \ldots, n$, the internal nonlocal interactions preceding the emission can be assumed as having been averaged into Gaussians, and the originally nonlocal term F(r, ...) can be assumed to have expressions of the type

$$F(r_1, r_2, \ldots) = F(r_1)\, F(r_2) \ldots\ldots F(r_n), \qquad (9.5a)$$

$$F(r_a) = R_a{}^6/(4\pi^2)\; e^{-\frac{1}{2}\, r_a{}^{\hat{2}}\, R_a{}^2}, \qquad (9.5b)$$

where: the the R's are the widths of the Gaussian; the increase of their power from the value R^4 of current use [1] to R^6 will be evident shortly; all compositions of vectors are of isotopic type, while those of numbers such as the R's are ordinary squares.

In different terms, in the preceding section we showed that, at the time of the creation of the fireball, we have interactions which are *nonlocal* and *nonseparable*, (i.e., not factorizable into individual local terms).

After the completion of the internal process and the production of the particles, all interactions can be effectively assumed as being *local* and *separable* (i.e., factorizable into individual local terms). This illustrates the lack of nonlocal terms in structure (9.5). Moreover, we can also effectively assume the separability of the isotopic element into terms depending on individual particles, as done in Eq.s (9.5). However, the particles cannot be effectively approximated as being point-like, and this illustrates the Gaussian forms, on one side, and the local setting on the other.

Needless to say, numerous additional expressions can be identified for $F(r_a)$ via Bessel and other functions, which are here left to the interested reader for brevity.

Assume the isohilbert space $\hat{\mathcal{H}}_{ext}$ with isostates: $\hat{\psi}(t, r_1, r_2, \ldots, r_n)$. Then the isoprobability for the production of two correlated bosons with four-momenta p_1 and p_2 originally produced at r_1 and r_2 to be detected at x_1 and x_2, is given by

$$\hat{P}(p_1, p_2) = \int d^4r_1\, d^4r_2\, \hat{\psi}^{\dagger}{}_{12}(x_1, x_2; r_1, r_2)\, T\, \hat{\psi}_{12}(x_1, x_2; r_1, r_2)\;\; F(r_1)\, F(r_2)\,, \qquad (9.6)$$

where the correlation has been assumed to be implemented, thus allowing us to ignore the projection component $\hat{\mathcal{P}}(0)$ hereon, and the symmetrization requested by the Bose–Einstein statistics is implied.

For all practical purposes, the isominkowski space can therefore be reduced to the form

$$\hat{M}_{Ext}(x,\hat{\eta},\hat{\Re}): \; x^{\hat{2}} = x*x = x^{\mu}\hat{\eta}_{\mu\nu}\, x^{\nu}, \;\; \hat{\eta} = T\eta, \;\; \hat{\Re} = \Re\hat{1}, \;\; \hat{1} = T^{-1}, \quad (9.7a)$$

$$\hat{\eta} = \text{diag}\,(b_1{}^2, b_2{}^2, b_3{}^2, -b_4{}^2), \quad b_{\mu} = \text{const.} > 0. \qquad (9.7b)$$

The enhanced isoprobability for the production of two identical correlated bosons can then be expressed via the *isocorrelation function*

$$\hat{C}_{(2)} = \frac{\hat{P}(p_1, p_2)}{\hat{P}(p_1) * \hat{P}(p_2)}, \qquad (9.8)$$

with symmetrized isostate

$$\hat{\psi}_{12}(x_1, x_2; r_1, r_2) =$$
$$= \{e^{ip_1*(x_1 - r_1)}\, e^{ip_2*(x_2 - r_2)} + e^{ip_1*(x_1 - r_2)}\, e^{ip_2*(x_2 - r_1)}\}\,(1/\sqrt{2}) \qquad (9.9)$$

where one should keep in mind that the compositions are in the isominkowski space (9.7).

In a way much similar to, and jointly with the symmetrization of the isostate requested by the Bose–Einstein statistics, the Gaussian component (9.5b) of the isotopic element T must also be symmetrized over all directions in space–time, resulting in the form

$$F(r_1, r_2) = (b_1{}^6/4\pi^2)\, e^{-\frac{1}{2} r^{\hat{2}} b_1{}^2} + (b_2{}^6/\,4\pi^2)\, e^{-\frac{1}{2} r^{\hat{2}} b_2{}^2} +$$
$$+ (b_3{}^6/4\pi^2)\, e^{-\frac{1}{2} r^{\hat{2}}} - (b_4{}^6/4\pi^2)\, e^{-\frac{1}{2} r^{\hat{2}} b_4{}^2} \qquad (9.10)$$

where: one should note the replacement of the generic Gaussian width R_{μ}, with the element $\hat{\eta}_{\mu\mu}$ of the isometric; two of the space components of the b's are expected to be equal owing to a conceivable cylindrical symmetry of the shape of the ellipsoid (see below); the term representing symmetrization with respect to the time component is evidently negative; and the expression can be written in a unified notation

$$F(r_1, r_2) = \sum_\mu \eta_{\mu\mu} (b_\mu^2 / 4\pi^2)\, e^{-\frac{1}{2} r^{\hat{2}} b_\mu^2}. \tag{9.11}$$

For future needs, one should keep in mind that exactly the same symmetrization can be done with conventional quantum mechanics, resulting in Eq.s (9.11) with $b_\mu = 1$ (up to normalization factors). This conventional relativistic formulation has been apparently derived in this paper for the first time.

One can now understand the extra power b_μ^2 in Gaussians (9.5b). In essence, by repeating the same argument in a conventional Minkowski space, one can see that the normalization of the Gaussian for the space components are +1 and that for the time component is −1, that is, the normalization is done with respect to the element of the metric $\eta_{\mu\mu}$. It is then immediate to see that the repetition of the argument under our lifting necessarily requires the normalization of the individual Gaussian to the diagonal elements of the isometric $\hat{\eta}_{\mu\mu}$, thus requiring the extra power b_μ^2.

By repeating the various passages and integrations as in the conventional case [1], we reach the following expression for the isocorrelation function

$$\hat{C}_{(2)} = 1 + b_1^2 e^{-Q_{12}^{\hat{2}}/ b_1^2} + b_2^2 e^{-Q_{12}^{\hat{2}}/ b_2^2} +$$

$$+ b_3^2 e^{-Q_{12}^{\hat{2}}/b_3^2} - b_4^2 e^{-Q_{12}^{\hat{2}}/ b_4^2}, \tag{9.12}$$

where the square is isotopic in $\hat{M}_{Ext}(x,\hat{\eta},\hat{\mathfrak{R}})$, the quantity Q_{12} is given by

$$Q_{12} = p_1 - p_2, \tag{9.13}$$

and expression (9.12) can be written

$$\hat{C}_{(2)} = 1 + \sum_{\mu} \hat{\eta}_{\mu\mu}\, e^{-Q_{12}^{\hat{2}}/\, b_\mu^{\,2}}. \qquad (9.14)$$

(where, again, we have an isocomposition for the fourvector Q_{12}, and the ordinary square for the number b_μ), which directly shows the normalization coefficients $\hat{\eta}_{\mu\mu}$ as factors of the exponentials.

By repeating the same procedure for the isocorrelation function for three bosons, one get an expression of the type

$$\hat{C}_{(3)} = 1 + \sum_{\mu ij} \hat{\eta}_{\mu\mu} e^{-Q_{ij}^{\hat{2}}/\, b_\mu^{\,2}} + 2\sum_{\mu\mu}\hat{\eta}_{\mu\mu} e^{-\frac{1}{2}\sum_{ij} Q_{ij}^{\hat{2}}/b_\mu^{\,2}}, \qquad (9.15)$$

Let us now study in more details expression (9.14). The terms in the exponents are explicitly given by

$$Q^{\hat{2}}/\, b_\mu^{\,2} = (Q_1 b_1^{\,2} Q_1 + Q_2 b_2^{\,2} Q_2 + Q_3^{\,2} b_3 Q^2 - Q_4^{\,2} b_4^{\,2} Q_4) \,/\, b_\mu^{\,2}, \qquad (9.16a)$$

$$Q = Q_{12} = p_1 - p_2, \qquad (9.16b)$$

Let us suppose that the conventional relativistic limit, prior to isotopies/correlations, is based on the perfectly spherical and rigid fireball

$$R^2 = 1 + 1 + 1 = 3 = \text{const.} \qquad (9.17)$$

as directly represented by the conventional Euclidean metric δ = diag. (1, 1, 1).

However, experimental evidence indicates that the fireball is a prolate spheroidal ellipsoid oriented in the direction of the p-$\bar{p}$ collision. We can therefore assume that *the geometry of the process implies a deformation of the above sphere into an ellipsoid*

$$\hat{R}^2 = b_1^{\,2} + b_2^{\,2} + b_3^{\,2} \neq 3, \quad b_k^{\,2} \neq 1, \qquad k = 1, 2, 3. \quad (9.18)$$

As we shall see, experimental data can indeed provide the numerical value of the relative shape of the fireball expressed in a way invariant under expansions.

Our nonrelativistic representation of the fireball is outlined in Fig. 3. Its relativistic extension is straightforward. In fact, the Poincaré symmetry P(3.1) of the Minkowski metric η, by no means, is lost within the context of relativistic hadronic mechanics, but it is instead assumed at the foundation of the theory, although in the isotopic form $\hat{P}(3.1)$ of Sect. 7 which leaves invariant the isoseparations $x^t\hat{\eta}x$; the isosymmetry can smoothly interconnect the transition from the original Minkowski structure to our isopoincaré structure

$$P(3.1): \; x^2 = x^{\mu}\eta_{\mu\nu}x^{\nu} \quad \Rightarrow \quad \hat{P}(3.1): \; x^{\hat{2}} = x^{\mu}\hat{\eta}_{\mu\nu}x^{\nu}; \qquad (9.19)$$

and, last but not least, all possible isosymmetries $\hat{P}(3.1)$ are locally isomorphic to P(3.1) for positive-definite isotopic elements T, as per our basic assumption (8.2b).

REPRESENTATION OF FIREBALL
IN HADRONIC MECHANICS

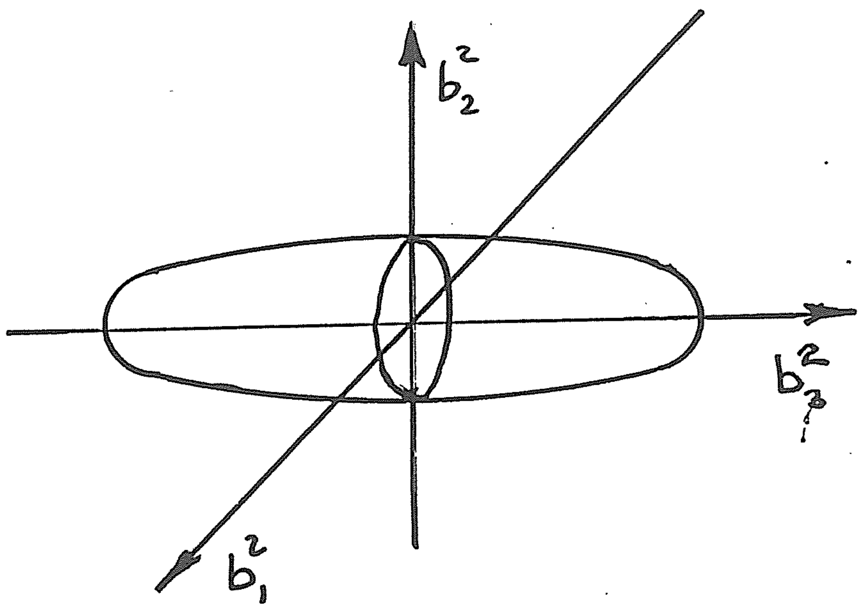

FIGURE 3: A schematic view of the fireball as represented in hadronic mechanics. The *physical context* is the following. Experimental data indicate that: 1) the fireball is not spherical but has instead the shape of a prolate spheroidal ellipsoid; 2) the prolateness is oriented in the original direction of p-$\bar{p}$ flight, and 3) once created, the fireball expands rapidly by essentially preserving the prolate ellipsoidical shape. By assuming that the original p-$\bar{p}$ direction is along the x^3 axis,

we can therefore represent the fireball with the prolate spheroidal ellipsoid

$$\hat{R}^2 = \hat{b}_1(t)^2 + \hat{b}_2(t)^2 + \hat{b}_3(t)^2 \neq \text{cost.}, \quad \hat{b}_1 = \hat{b}_2, \quad \hat{b}_3 > \hat{b}_1, \hat{b}_2 \quad \text{(a)}$$

The condition of preservation of the original prolate ellipsoidical character can be essentially represented via a time dependence of the characteristic $\hat{b}$-functions which is the same for all, i.e.

$$\hat{b}_k(t) = f(t)\, b_k, \quad b_k = \text{constants}, \quad k = 1, 2, 3, \qquad \text{(b1)}$$

$$\hat{R}^2 = \hat{b}_1(t)^2 + \hat{b}_2(t)^2 + \hat{b}_3(t)^2 = f(t)\ (b_1^2 + b_2^2 + b_3^2), \qquad \text{(b2)}$$

$$b_1^2 + b_2^2 + b_3^2 = K^2 = \text{constant}. \qquad \text{(b3)}$$

We reach in this way the conclusion, rather important for phenomenological fits (see later on)

$$\hat{b}_k(t)^2 / [\, \hat{b}_1(t)^2 + \hat{b}_2(t)^2 + \hat{b}_3(t)^2\,] = \text{constant}, \quad k = 1, 2, 3. \quad \text{(c)}$$

Hadronic mechanics permits the direct and quantitative representation of spheroidal ellipsoid (a) via its fundamental symmetries, the *isorotational symmetries* $\hat{O}(3)$ originally identified in ref. [34], and then studied in more details in memoirs [9]. These isosymmetries are constructed with respect to the isounit

$$\hat{1} = \text{diag.}\{\hat{b}_1(t)^{-2}, \hat{b}_2(t)^{-2}, \hat{b}_3(t)^{-2}\}, \qquad \text{(d)}$$

isotopic element $T = \hat{1}^{-1} = \text{diag.}\{\hat{b}_1(t)^2, \hat{b}_2(t)^2, \hat{b}_3(t)^2\}$, Lie-isotopic product $[A \hat{,} B] = A*B - B*A = ATB - BTA$, conventional parameters θ_k (the Euler's angles), and conventional generators, resulting in the isocommutation rules and invariant

$$J_k = \epsilon_{kij}\, r_i\, p_j, \qquad \text{(e1)}$$

$$[J_i \hat{,} J_j] = -i\, \epsilon_{ijk}\, \hat{b}_k^{\,2} J_k, \qquad \text{(e2)}$$

$$J^{\hat{2}} = JTJ = \text{inv.}, \qquad \text{(e3)}$$

where we have evidently used the fundamental isocommutation rules (6.3). For the equivalent matrix representation, see ref. [34]. As one can see, structure (e) is fully equivalent to the corresponding classical framework presented in Sect. III.3 of ref. [15]. The above representation of the fireball is incorporated in the isorelativistic representation provided in the text. In summary, our isorotational symmetries $\hat{O}(3)$ provide:

1) the groups of isometries of all possible ellipsoids (a), that is, a theory of *deformable bodies* ;

2) a smooth interconnection in the transition from the sphere to the ellipsoids,

$$O(3): r_1 r_1 + r_2 r_2 + r_3 r_3 \Rightarrow \hat{O}(3): r_1 \hat{b}_1^2 r_1 + r_2 \hat{b}_2^2 r_2 + r_3 \hat{b}_3^2 r_3, \quad \text{(f)}$$

and

3) the reconstruction of the exact rotational symmetry for the deformed ellipsoids (a), because of the local isomorphisms $\hat{O}(3) \approx O(3)$.

The full understanding of the above results requires the treatment within the context of the *isoriemannian geometry* [15], with related *isoparallel transport* and *isogeodesic motion* . In fact, the geometric reason why, contrary to a rather popular belief, the *rotational* symmetry results to be *exact* for *ellipsoids* is that ellipses on the surface of the ellipsoids are indeed geodesics, of course, after abandoning the trivial unit I = diag. (1, 1, 1) of the current Lie's theory in favor of less trivial units of type (d).

Note finally that the quantities $Q_{12}^{\hat{2}} / b_\mu^2$ in exponents (8.16) are invariant under scalar renormalization of the metric, namely, they are invariant under the scale transformations

$$x^{\hat{2}} \Rightarrow Nx^{\hat{2}} = x^t \hat{\eta}' x, \quad \hat{\eta}' = N\hat{\eta}, \quad N \in \Re. \tag{9.20}$$

evidently because of the appearance of the isotopic element b_μ^2 in the denominator.

As a result, the interpretation of the experimental data is insensitive to the assumed normalization or, equivalently, to the evolution of the fireball, because its expansion is characterized by a multiplicative function of time appearing identically in the numerator and denominator.

We can therefore summarize our analysis with the following primary results.

Isocorrelation function (9.14) can be reached from the corresponding, conventional, relativistic expression

$$C_{(2)} = 1 + \sum_{\mu}\eta_{\mu\mu}\, e^{-Q_{12}^{\hat{2}}}. \qquad (9.21)$$

via a smooth deformation of the Minkowski space $M(x,\eta,\Re)$ into the isominkowski space $\hat{M}_{Ext}(x,\hat{\eta},\hat{\Re})$, $\hat{\eta} = T\eta$, $\hat{\Re} = \Re\, \hat{I}$, $\hat{I} = T^{-1}$, where T is symmetric, real-valued, positive-definite and, therefore, always diagonalizable to form (9.7). This formulation permits: 1) the treatment of nonlocal internal effects of the Bose-Einstein correlation; 2) the derivation of the actual shape of the fireball from the experimental data; and 3) a quantitative representation of available experimental data derived from a basic theory which is not provided by current empirical models (Sect. 2).

Point 1) and 2) have been proved in the preceding sections. To verify Point 3), we proceed as follows. First, we note that *the results reached until now are essentially exact, with the sole approximation used in the calculation of the isocorrelation functions being based on a Gaussian representation of the extended character of the particles, Eq. (5.5b).*

From now on a number of approximations have to be made in order to reach an expression of the isocorrelation function which is suitable for experimental verification.

Next, exponent (9.16) must be reduced to a form solely dependent on the best quantity usually derived from the measured tracks, the *transverse momentum difference of boson pairs* q_t in $M(x,\hbar,\Re)$ with corresponding form $\hat{q}_t$ in $\hat{M}_{Ext}(x,\hat{\eta},\hat{\Re})$.

This approximation is here done exactly along conventional lines. We assume that, in first approximation and for the case of equal masses, the total four-momentum $P = p_1 + p_2$ is isoperpendicular to Q_{12}, i.e. (see the isorelativistic kinematics of Sect. 7 for details)

$$P*Q = P_{\mu}\, \hat{\eta}^{\mu\nu} Q_{\nu} = (p_1 + p_2)*(p_1 - p_2) \approx 0, \qquad (9.22)$$

This is the case if and only if

$$q_o = Q_{12}^{\ o} = p_1^{\ o} - p_2^{\ o} \approx 0. \qquad (9.23)$$

The use of the isorelativistic kinematics then leads to the value for the longitudinal momentum transfer $q_l \approx 0$, exactly as in the conventional case.

Thus, in the above indicated first approximation, the relativistic isokinematics implies that, under the conventional approximation recalled above,

$$Q^{\hat{2}} / b_\mu{}^2 = \{q_t{}^2(b_1{}^2 + b_2{}^2 + b_3{}^2) + q_l{}^2(b_1{}^2 + b_2{}^2 + b_3{}^2) - q_o{}^2 b_4{}^2\} / b_\mu{}^2$$

$$\approx q_t{}^2 (q_1{}^2 + q_2{}^2 + q_3{}^2) / b_\mu{}^2 = q_t{}^2 / b'_\mu{}^2, \qquad (9.24a)$$

$$b'_\mu = b_\mu / (q_1{}^2 + q_2{}^2 + q_3{}^2). \qquad (9.24b)$$

The implementation of the above approximation in the isocorrelation function (9.14) then yields our final expression[41]

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[41] The author would like to thank F. Cardone and R. Mignani of the Dept of Physics of the Univ. "La Sapienza" in Rome, Italy, for a careful reading of this manuscript in one of its earlier versions, for a number of invaluable comments and, more specifically, for suggesting the writing of Eq.s (9.25) in terms of $\hat{q}_t$, rather the author's formulation in terms of q_t. It may be useful for the interested phenomenologist to review this alternative. By recalling that $q_o \approx 0$ and $q_l \approx 0$, the correct exponent of Eq.s (9.25) is

$$q_t{}^2 (b_1{}^2 + b_2{}^2 + b_3{}^2) / b_\mu{}^2 \qquad (a)$$

But the model is invariant under scale transformation (9.20), thus leading to the property of Figure 3

$$b_\mu{}^2 / (b_1{}^2 + b_2{}^2 + b_3{}^2) = K^2 = \text{constant} \qquad (b)$$

which implies the equivalences

$$q_t{}^2 (b_1{}^2 + b_2{}^2 + b_3{}^2) / b_\mu{}^2 \approx \hat{q}_t{}^2 / b_\mu{}^2 \approx q_t{}^2 / b'_\mu{}^2, \quad b'_\mu = b_\mu / K. \qquad (c)$$

which, however, have different implications for the normalization of the model (see later on Eq.s (10.9)). The model under study has been conceived to be invariant under scale transformations of the isometric, i.e., to be insensitive to the actual expanding shape of the fireball, and only sensitive to the "normalized shape" to a given ellipsoid. *The author prefers expression (9.25) in terms of $q_t \in M(x,\eta,\Re)$ and $b'_\mu{}^2 \in \hat{M}_{Ext}(x,\hat{\eta},\hat{\Re})$, rather than in terms of $\hat{q}_t$, and $b_\mu{}^2 \in \hat{M}_{Ext}(x,\hat{\eta},\hat{\Re})$ because the isominkowski space $\hat{M}_{Ext}(x,\hat{\eta},\hat{\Re})$ is a geometrical space useful for a quantitative treatment of the correlation, with the understanding that the physical space in which the measures are made is the conventional space $M(x,\eta,\Re)$. But the quantity measured is $q_t \in M(x,\eta,\Re)$ and not $\hat{q}_t \in \hat{M}_{Ext}(x,\hat{\eta},\hat{\Re})$. This leads to structure (9.25).*

$$\hat{C}_{(2)} = 1 + \sum_\mu \hat{\eta}_{\mu\mu}\, e^{-q_t^2/\, b'_\mu{}^2} =$$

$$= 1 + b_1{}^2 e^{-q_t^2/\, b'_1{}^2} + b_2{}^2 e^{-q_t^2/\, b'_2{}^2} + b_3{}^2 e^{-q_t^2/\, b'_3{}^2} - b_4{}^2 e^{-q_t^2/\, b'_4{}^2}, \qquad (9.25)$$

which still non-renormalized, but nevertheless in a form already suitable for plotting with experimental data, as shown in the next section.

10. VERIFICATION WITH EXPERIMENTAL DATA

For the sake of clarity, let us review the main phenomenological steps of the conventional treatment of the Bose-Einstein correlation. This will provide us the elements necessary for the final normalization of Eq.s (9.25).

The first step is the conventional Gaussian relativistic treatment which results in Eq. (9) of ref. [1], i.e.,

$$C_{(2)} = 1 + e^{-Q_{12}{}^2 R^2}, \qquad (10.1)$$

with the following conventional limits at $Q_{12} = 0$ and $Q_{12} \Rightarrow \infty$, respectively,

$$C_{(2)}{}^{Max} = 2, \qquad C_{(2)}{}^{Min} = 1. \qquad (10.2)$$

However, as stressed in ref. [1], the above model is not verified by available experimental data forcing the introduction by hand of the coefficient λ, called the *caoticity*, as per form [1]

$$C_{(2)} = 1 + \lambda\, e^{-Q_{12}{}^2 R^2}, \qquad (10.3)$$

which has the primary motivation of obtaining a maximum value lower than 2, while keeping the minimum value one.

Despite the addition of the theoretically unknown caoticity, expression (10.3) remains unsatisfactory, owing to the insufficient number of free parameters, their unknown origin and other reasons. The interested reader may consult ref. [1] and quoted paper for additional aspects, as well as models essentially patterned along the above lines. One of the current plots is shown in Fig. 4.

A SAMPLE OF EXPERIMENTAL DATA IN BOSE-EINSTEIN CORRELATION

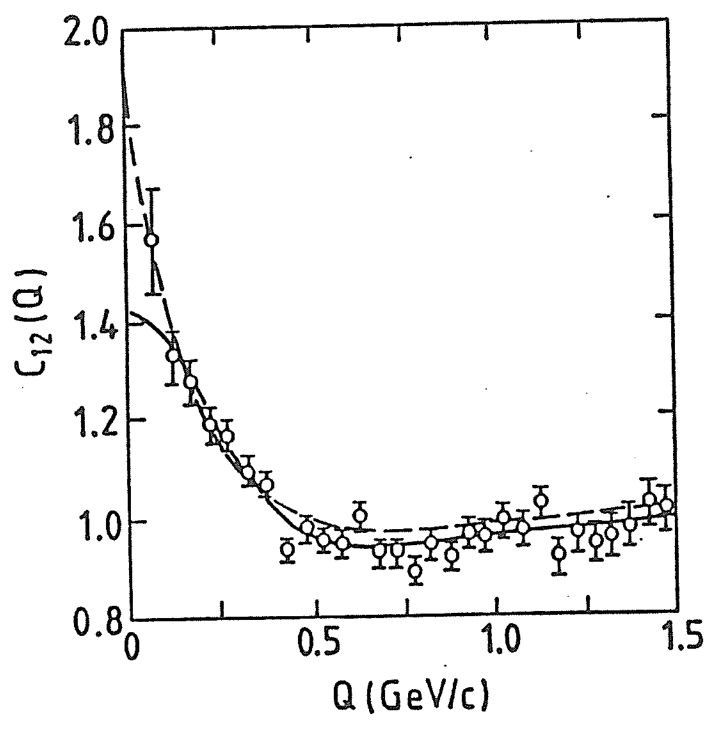

FIGURE 4: The correlation function of two identical pions from ref. [1] originating in $e^- e^+$ annihilation experiments. The full line is a fit to a Gaussian, while the dashed line represents a string model. Note the insufficiency of the latter for a representation of the data.

By observing the preceding expressions, the most recommendable normalization of the radius of the perfect sphere is to one. This suggests the replacement of normalization (9.17) with the following one

$$R^2 = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1 = \text{const.}. \tag{10.4}$$

As a result, our limit relativistic model in the absence of correlation and nonlocal-nonhamiltonian internal effects, Eq.s (9.21), acquires the final, properly normalized form

$$C_{(2)} = 1 + \frac{1}{3} \sum_{\mu=1,2,3,4} \eta_{\mu\mu} \; e^{-q_t^2}, \tag{10.5}$$

A direct consequence of the above model is the lowering of the maximum value (10.2) without introducing theoretically unmotivated, empirical parameters by hand, according to the new expression below, while keeping the minimum value the same as that of Eq. (10.1),

$$C_{(2)}^{Max} = 1 + (1/3 + 1/3 + 1/3) - 1/3 = 1.67, \quad C_{(2)}^{Min} = 1. \tag{10.6}$$

The above expressions are the first numerical results of our analysis. To our best knowledge, ***limits (10.6) are verified by all available experimental data*** (see, e.g., those of ref. [1]).

Evidently, expression (10.5) is inadequate to reproduce the experimental data, and remains unsatisfactory even if we add by hand the caoticity, because of the lack of sufficient parameters.

Model (10.5) is however a limit model to be recovered by our isotopic model at the limit $\hat{\eta} \Rightarrow \eta$. Thus, model (10.5) suggests the normalization coefficients 1/3 of our isotopic generalization (9.25), by reaching in this way our *final, properly normalized, two-particles, isocorrelation function*

$$\hat{C}_{(2)} = 1 + \frac{1}{3} \sum_{\mu} \hat{\eta}_{\mu\mu} \left(e^{-q_t^2 K^2 / b_\mu^2} \right) =$$

$$= 1 + \frac{b_1^2}{3} e^{-q_t^2 K^2 / b_1^2} + \frac{b_2^2}{3} e^{-q_t^2 K^2 / b_2^2} +$$

$$+ \frac{b_3^2}{3} e^{-q_t^2 K^2 / b_3^2} - \frac{b_4^2}{3} e^{-q_t^2 K^2 / b_4^2} \tag{10.7.a}$$

$$K^2 = b_1^2 + b_2^2 + b_3^2 , \tag{10.7b}$$

which can be equivalently written in the form

$$\hat{C}_{(2)} = 1 + \frac{K^2}{3} \sum_\mu \hat{\eta}_{\mu\mu} (e^{-q_t^2 / b_\mu^2} =$$

$$= 1 + \frac{b_1^2 K^2}{3} e^{-q_t^2 / b_1^2} + \frac{b_2^2 K^2}{3} e^{-q_t^2 / b_2^2} +$$

$$+ \frac{b_3^2 K^2}{3} e^{-q_t^2 / b_3^2} - \frac{b_4^2 K^2}{3} e^{-q_t^2 / b_4^2} \qquad (10.8)$$

or in the still equivalent form

$$\hat{C}_2 = 1 + \frac{S^2}{3} \sum_\mu \hat{\eta}_{\mu\mu} (e^{-q_t^2 n_\mu^2}$$

$$= 1 + \frac{S^2}{3n_1^2} e^{-q_t^2 n_1^2} + \frac{S^2}{3n_2^2} e^{-q_t^2 n_2^2} +$$

$$+ \frac{S^2}{3 n_3^2} e^{-q_t^2 n_3^2} - \frac{S^2}{3 n_4^2} e^{-q_t^2 n_4^2} \qquad (10.9a)$$

$$S^2 = \frac{1}{n_1^2} + \frac{1}{n_2^2} + \frac{1}{n_3^2} \qquad (10.9b)$$

where we have introduced the redefinition of the characteristic b-constants of the fireball as per definition (7.15c)

$$b_\mu^2 = 1 / n_\mu^2. \qquad (10.10)$$

As we shall see shortly, reformulation (10.9) is preferable for the identification of the type of isominkowskian geometrization holding in the interior of the fireball among the nine possible cases (7.45).

Model (10.9) is now ready for experimental plots (see later on Fig. 5). The reader should remember that ***model (10.9) is exact, i.e., deduced from first principles, with the sole approximation of $q_o \approx 0$ and $q_l \approx 0$, as in the conventional treatment***

Our next objective is that of reaching a preliminary estimate of the characteristics n-constants of model (10.9) also from first principles, and prior to any plot of the model with experimental data.

The geometrization of the physical medium in the interior of the fireball via our isospecial relativity outlined in Sect. 7 (see ref. [15], Sect. IV.10 for details) predicts that, at the very high energy of the UA1 experiments at CERN, the fireball is a medium of Type 9, Eq.s (7.45i) at the time of its formation. This is due to the fact that the density of the fireball is much higher than any known heavy hadron, by approaching the density in the core of collapsing stars, as elaborated in Sect. 7.

We have indicated earlier that, immediately after its formation, the fireball rapidly expands and decomposes itself into correlated bosons. This implies average densities not necessarily higher than those in the interior of hadrons. Nevertheless, the final maximal dimension at the time of decomposition of the fireball must remain of hadronic type.

This identifies the open problem of the *average density of the fireball* which, as we shall soon see, affects the numerical value n_4.

In summary, ***a first, fundamental theoretical prediction of the isospecial relativity is that the characteristic n-constants of the fireball must have values verifying the following conditions, i.e.,***

FIREBALLS ARE MEDIA OF TYPE 9:

$$n_3 > n_4, \qquad n_4 < 1, \quad \hat{\beta} < \beta, \ \hat{\gamma} > \gamma \tag{10.11}$$

where[42]

$$\hat{\beta} = vn_4 / c_o n_3, \quad \beta = v / c_o, \quad \hat{\gamma} = |1 - \hat{\beta}^2|^{-\frac{1}{2}}, \ \gamma = |1 - \beta^2|^{-\frac{1}{2}}, \tag{10.12}$$

This yields the second theoretical prediction of our model, namely, that ***photons emitted in the interior of the fireball are "blueshifted" while propagating within the inhomogeneous and anisotropic medium inside the fireball*** according to a prediction first made in ref. [11] via the generalized Doppler's law of the isospecial relativity (see ref. [15], Eq.s (9.48) for a detailed treatment)

$$\hat{\omega} = \omega \hat{\gamma} (1 - \hat{\beta} \cos \alpha). \tag{10.13}$$

[42] The attentive reader may have noted that we have used absolute values for the conventional γ too. This is due to the currently considered speeds of quasars reviewed later in this section which are bigger than c_o, thus implying imaginary γ if conventionally expressed.

which, for $\hat{\beta} < \beta$, $\hat{\gamma} > \gamma$, yields the ***fireball isoblueshift*** $\hat{\omega} > \omega$.

This essentially means that *the photons detected in laboratory which are emitted in the interior of the fireball have original frequencies lower than those actually measured* Stated differently, the occurrence is one way of representing the extremely energetic fireball medium and the (classical) expectation that, as such, the medium transfers energy to electromagnetic waves propagating in it.

This occurrence is intriguing because it is complementary to that under study for ***quasars isoredshift***. In fact, it was conjectured in ref. [11] that *light experiences a natural redshift while propagating within the inhomogeneous and anisotropic atmospheres of the quasars according also to isodoppler's law (10.13).* Stated differently, the prediction expresses the expectation that electromagnetic waves lose energy while propagating in inhomogeneous and anisotropic planetary atmospheres, whose density is much lower than that of the fireball.

It should be noted that *the hypothesis was submitted as a correction to the current interpretation of the quasars speeds to prevent a violation of Einsteinian laws under Einsteinian conditions (motion in vacuum at speeds higher than c_o)*. In fact, redshifts have considerably increased to such values to require speeds v of the order of ten time and more the speed of light in vacuum, $v \approx 10c_o$. Proposal [11] essentially consists in using isodoppler's law (10.13) to bring quasars speeds below c_o, while maintaining the expansion of the Universe (that of the associated galaxies).

Numerical calculations along this proposal were done in ref. [38] showing that isodoppler's law (10.13) can indeed reduce the speed of the quasars all the way to that of the associated galaxies. Evidently, this was submitted as a limiting case in which the quasars are at rest with respect to the associated galaxies and the difference in the measured redshift (between galaxies and quasars) is entirely of isotopic nature, that is, due to the particular isominkowskian geometry of quasars' atmospheres. It is understood that various intermediate cases are possible in which the quasars are indeed expelled from their associated galaxies, but at Einsteinian speeds $v < c_o$.

The results of ref. [38] were examined in ref. [39] by showing that

QUASARS' ATMOSPHERES ARE A MEDIUM OF TYPE 4:

$$n_3 < n_4, \quad n_4 > 1, \quad \hat{\beta} > \beta, \quad \hat{\gamma} < \gamma \tag{10.14}$$

namely, they are media which are transparent, thus admitting a local speed of light $c = c_o n_4 < c_o$, and which predict a natural redshift $\hat{\omega} < \omega$. due to their inhomogenuity and anisotropy In different terms, according to isominkowskian geometrization (10.14), the frequency of the light measured outside the quasars' atmospheres is smaller than that at the time of its emission in the interior of the medium.

In particular, the average value of the ratio n_3 / n_4 for all quasars's isoredshift measured in ref. [38) is given by [39]

$$< n_3 / n_4 |_{quasars} > \cong 1.37 \times 10^{-2}. \qquad (10.15)$$

Additional numerical data in the characteristic n-constants, which are particularly useful for this paper as a comparison, are those of ref. [40] on light hadrons, such as the K^o. In essence the behavior of the meanlife of the K^o with speed is noneinsteinian between 35 and 100 GeV according to experiments [41] while it is Einsteinian between 100 to 400 GeV according to the experiments [42]. In papers [40] we showed that, rather than being incompatible, these two experiments are fully compatible, provided that the K^o is interpreted as an isominkowskian medium, exactly along the lines used in this paper for the fireball.

In particular, the use of the combined experimental data [41,42] yields the following numerical values [40]

$$b_3{}^2 = 0.909080 \pm 0.00004, \qquad n_3{}^2 \cong 1.1 \qquad (10.16a)$$

$$b_4{}^2 = 1.002 \pm 0.007, \qquad n_4{}^2 \cong 0.998, \qquad (10.16b)$$

$$b_3{}^2 / b_4{}^3 = 0.907 \qquad n_3{}^2/n_4{}^2 = 1.102. \qquad (10.16c)$$

These values were re-examined in ref. [39] by showing that experimental measures [41,42] imply that ***the K^o is a medium of Type 9***, Eq.s (10.14)).

The above results are confirmed by the low energy studies of ref. [43] in the Higgs sector of spontaneous symmetry breakdown for light mesons, resulting in the identification of the generalized metric for the interior of the particles

$$\hat{\eta} = \text{diag.} \{ (1 - \alpha/3), (1 - \alpha/3), (1 - \alpha/3), - (1 + \alpha) \} \qquad (10.17)$$

with the following values for the pions

$$\alpha = (-3.79) + 1.37) \times 10^{-3} \qquad (10.18)$$

and the different values for the kaons

$$\alpha = (+0.61 + 0.17) \times 10^{-3} \qquad (10.19)$$

But generalized metric (10.17) is exactly our isometric as submitted in ref. [31] and, as such, it can be reinterpreted in terms of our isominkowski space, yielding the values [39]

$$n_3{}^2{}_{\pi^\circ} \approx 0.998, \quad n_4{}^2{}_{\pi^\circ} \approx 1.004, \quad n_3{}^2{}_{\pi^\circ} / n_4{}^2{}_{\pi^\circ} = 249.5 \qquad (10.20a)$$

$$n_3{}^2{}_{K^\circ} \approx 1.002, \quad n_4{}^2{}_{K^\circ} \approx 0.998, \quad n_3{}^2{}_{K^\circ} / n_4{}^2{}_{K^\circ} = 1.004, \qquad (10.20b)$$

that is, ***according to measures [43],***

THE π°-MESONS ARE MEDIA OF TYPE 7:

$$n_3 > n_4, \quad n_4 > 1, \quad \hat{\beta} < \beta, \quad \hat{\gamma} > \gamma \qquad (10.21a)$$

THE K°-MESONS ARE MEDIA OF TYPE 9:

$$n_3 > n_4, \quad n_4 < 1, \quad \hat{\beta} < \beta, \quad \hat{\gamma} > \gamma. \qquad (10.21b)$$

This confirms the experimental plots of ref. [40] for the kaons, Eq.s (10.16), as well as the expectation that all hadronic media with a density heavier than that of the K^o are media of Type 9 .

We are now equipped to present our prediction of the numerical values of the characteristic n-constants of the fireball which can be identified via the following:

CONDITION 1: Prolate spheroidal character of the fireball

$$n_1{}^2 \equiv n_2{}^2; \qquad (10.22)$$

CONDITION 2: Normalization of the ellipsoid

$K^2 = b_1^2 + b_2^2 + b_3^2$ to 3, i.e.,

$$S^2 = \frac{1}{n_1^2} + \frac{1}{n_2^2} + \frac{1}{n_3^2} = 3 \qquad (10.23)$$

CONDITION 3: Prolateness proportional to the energy. Specifically, if the energies of the UA1 experiments are of the order of 30 times those of experiments [41,42] we assume

$$n_1^2 = n_2^2 \approx 30\, n_3^2 \qquad (10.24)$$

The above three conditions evidently yield the numerical values

$$n_1^2 = 32/3 = 10.666, \quad n_2^2 \; 32/3 = 10.666, \quad n_3^2 = 32/90 = 0.355 \qquad (10.25)$$

We finally remain with the evaluation of n_4^2. This is readily achieved via the following additional

CONDITION 4: Model (10.9) yields conventional relativistic value (10.6) at $q_t = 0$. In different terms, this hypothesis assumes that at $q_t = 0$, we have no correlation, thus resulting in the condition easily derived from isocorrelation (10.9)

$$1 + \frac{S^4}{3} - \frac{S^2}{3\, n_4^2} = 1.67 \qquad (10.26)$$

from which we obtain the numerical value

$$b_4^2 = 1/2.33 = 0.429 \qquad (10.27)$$

Summarizing, the above conditions imply the following, evidently preliminary numerical values for the geometrization of the fireball medium (see Fig. 5) for a preliminary analysis)

$$n_1^2 = 10.666, \quad n_2^2 = 10.666, \quad n_3^2 = 0.355, \quad n_4^2 = 0.429, \qquad (10.27a)$$

$$n_3^2 / n_4^2 = 0.827, \quad < n_k^2 > / n_4^2 = 16.850. \qquad (10.28b)$$

As one can see, the above values verify in full Conditions (10.11) for the geometrization of the fireball as a medium of Type 9.

EXPERIMENTAL VERIFICATION

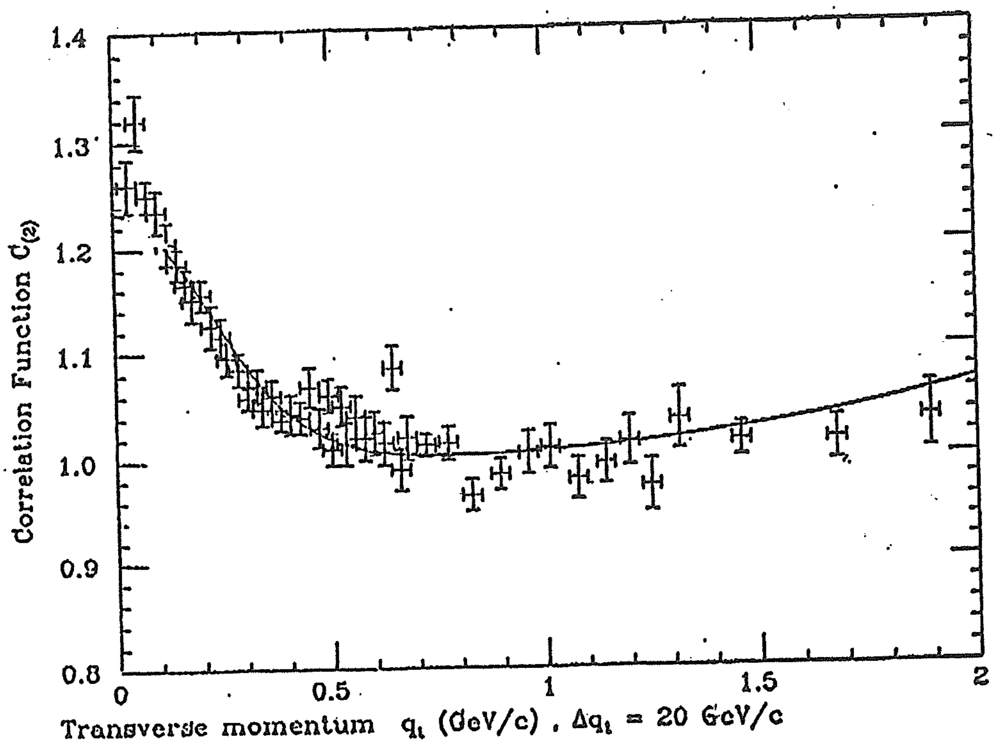

FIGURE 5: The first experimental plot of the model of nonlocal Bose-Einstein correlation of this paper conducted by F. Cardone and R. Mignani of the Department of Physics "G. Marconi" of the University "La Sapienza", Rome, Italy (private communication). The plot was conducted for model (9.25) via the use of the UA1 experimental data at CERN and resulted in the following numerical values of the characteristic quantities of the fireball:

$$b_1^2 = 0.0925,\ b_2^2 = 0.0971,\ b_3^2 = 3.463,\ b_4^2 = 3.616 \qquad \text{(a)}$$

or, equivalently, for definitions (10.10),

$$n_1^2 = 10.801,\ n_2^2 = 10.309,\ n_3^2 = 0.288,\ n_4^2 = 0.276. \qquad \text{(b)}$$

As one can see, the plot confirms the main hypotheses of this paper, such as: 1) the nonlocal origin of the fireball represented via an averaged deviation from the Minkowskian description; 2) the behaviour of the two-points correlation function; 3) the maximum value 1.67 of the two-points correlation function; 4) the high prolateness of the fireball with a ratio $n_2^2 / n_3^2 \cong 36$ vs the value 30 assumed in the text; and 5) the central prediction that the fireball is an isominkowskian medium of Type 9, Eq.s (7.45i).

The plot is however preliminary, because conducted for our unrenormalized isocorrelation function (9.25) prior to the availability of our fully normalized model (10.8) or (10.9). In fact, for numerical data (a), the K^2-value of Eq. (10.7b) is not normalized to the value $K^2 = 3$, but rather to the value

$$K^2 = b_1^2 + b_2^2 + b_3^2 = 3.653. \qquad \text{(c)}$$

It is our knowledge that the final plot of renormalized model (10.8) or (10.9) by Cardone and Mignani is being published elsewhere.

For the reader interested in conducting additional, independent plots, we note that the fully remornalized model of two-bosons isocorrelation function is model (10.8) or (10.9) under maximum and minimum values (10.6) and normalization (10.23), such as the expression

$$\hat{C}_{(2)} = 1 + \frac{1}{n_1^2}\, e^{-q_t^2 n_1^2} + \frac{1}{n_2^2}\, e^{-q_t^2 \times n_2^2} + \frac{1}{n_3^2}\, e^{-q_t^2 \times n_3^2} - \frac{1}{0.429}\, e^{-q_t^2 \times 0.429} \qquad \text{(d)}$$

$$K^2 = n_1^{-2} + n_2^{-2} = n_3^{-2} = 3. \qquad \text{(e)}$$

which verifies by construction the maximum and minimum values

$$\hat{C}_{(2)\, q_t \Rightarrow 0} = 1.67, \qquad \hat{C}_{(2),\, q_t \Rightarrow \infty} = 1, \qquad \text{(f)}$$

We should also indicate for clarity that numerical values (10.28) are merely indicational. In fact families of different values are permitted by isorelativistic hadronic mechanics for the following reasons:

1) The value of n_4^2 can differ from value 0.429 for various reasons, including the possibility that the actual maximal value of the isocorrelation function is lower than the relativistic limit 1.67 (e.g., because there is still some residual correlation at that value), or because the average density of the fireball, from its creation to its disintegration, is lower than what expected;

2) The prolateness of the fireball, condition (10.24), is empirical, to begin with, and also dependent on the energy, thus clearly admitting values in a broader range, say, from 20 to 40;

3) The normalization $K^2 = b_1^2 + b_2^2 + b_3^2 = 3$, condition (10.22), is also arbitrary, and different normalizations may be needed, depending on the case at hand. It is evident that different normalization of K^2 directly imply different values for all n_μ^2;

and other reasons. It is hoped that interested phenomenologists will resolve these alternatives in the only possible way, via the plotting of model (10.9) with experimental data.

11: CONCLUDING REMARKS

In this paper we have shown that quantum mechanics is unable to treat the expected nonlocal and nonpotential character of the Bose-Einstein correlation via conventional basic axioms because of their inherent local-potential structure.

We have then reviewed the covering nonrelativistic and relativistic hadronic mechanics and shown that the correlation is fully interpretable from their first principles under the hypothesis that it is due to nonlinear, nonlocal and nonhamiltonian effects in the interior of the fireball caused by the deep mutual overlapping of the charge distributions and wavepackets of the original proton and antiproton collisions at high energy.

The emerging hadronic model (10.9) preserves the exponential structure of conventional quantum models, but exhibits four free parameters, the characteristic n_μ-constants, $\mu = 1, 2, 3, 4$, which are an average of the internal nonlocal effects as seen from the outside, and imply an isominkowskian geometrization of the medium inside the fireball apparently shared by all (heavy) hadrons. As such, model (10.9) is definitely an improvement for experimental plots over existing models with a smaller number of theoretically unmotivated parameters.

To our best knowledge at this writing, model (10.9) can :

1) Explain the origin of the correlation via nonlocal and nonhamiltonian effects in the interior of the fireball;

2) Represent in a satisfactory way available experimental data from first principle, such as the isoexpectation values;

3) Verify the currently available maximal value 1.67 of the two-point correlation function and its minimum value 1;

4) Directly represent the actual non-spherical shape of the fireball as a highly prolate spheroidal ellipsoid along the original particle-antiparticle flight, as well as its rapid expansion in time; and

5) Reconstruct the exact Poincaré symmetry at the isotopic level under nonlinear, nonlocal and nonhamiltonian effects in the interior of the fireball.

However, in the opinion of this author, ***the most significant implication of the studies is that, if confirmed by additional independent plots, the Bose-Einstein correlation could result to be the first experimental evidence of the historical legacy by Bogoliubov, Fermi and others, on the ultimate nonlocality of the strong interactions, with far reaching implications, such as the need for a new generation of relativities specifically built for the most general conceivable interactions in the Universe.***

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