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**ISO-, GENO-, HYPER-MECHANICS FOR MATTER, THEIR  
ISODUALS, FOR ANTIMATTER, AND THEIR NOVEL  
APPLICATIONS IN PHYSICS, CHEMISTRY AND BIOLOGY**

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## ABSTRACT

Pre-existing numbers, related mathematics and consequential physical theories are generally used for the treatment of new scientific problems. In this memoir we outline the research conducted by various mathematicians, physicists and chemists over the past two decades who have shown that the inverse approach, the construction of new numbers, related new mathematics and consequential new physical theories from open physical, chemical and biological problems, leads to new intriguing formulations of increasing complexity called *iso-, geno- and hyper-mathematics* for the treatment of matter in reversible, irreversible and multi-valued conditions, respectively, plus anti-isomorphic images called *isodual mathematics* for the treatment of antimatter.

These novel formulations are based on new numbers characterized by the lifting of the multiplicative unit of ordinary fields (with characteristic zero) from its traditional value +1 to: (1) invertible, Hermitean and single-valued units for isomathematics; (2) invertible, non-Hermitean and single-valued units for genomathematics; and (3) invertible, non-Hermitean and multi-valued units for hypermathematics; with corresponding liftings of the conventional associative product and consequential lifting of all branches of mathematics admitting a (left and right) multiplicative unit. An anti-Hermitean conjugation applied to the totality of quantities and their operation of the preceding mathematics characterizes the isodual mathematics.

The above new mathematics are then used for corresponding liftings of Newtonian, Hamiltonian and quantum mechanics, today known as *iso-, geno-, hyper-mechanics for the description of matter and their isoduals for antimatter*, with compatible liftings of geometries and symmetries, and, inevitable of contemporary relativities. The above new body of knowledge is also known as *hadronic mechanics, superconductivity and chemistry*, wherein conventional Hamiltonians represent conventional, linear, local and potential interactions among point particles, while generalized units provide an invariant representation of extended, nonspherical and deformable particles under additional nonlinear, nonlocal and nonpotential interactions due to deep mutual penetration of wavepackets at short distances. Whenever the latter effects are ignorable due to large distances, conventional units, mathematics and relativities are recovered identically.

We finally outline the novel scientific and industrial verifications and applications permitted by the new mathematics and relativities in physics, chemistry and biology, including numerous experimental verifications, and applications for new clean energies and fuels that are prohibited by contemporary mechanics, special relativity and related mathematics.

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## 1. INTRODUCTION

Customarily, new problems in physics, chemistry, biology and other quantitative sciences are treated via pre-existing mathematics. Such an approach is certainly valuable at the initiation of new studies. However, with the advancement of scientific knowledge such an approach historically lead to serious limitations and controversies due to the insufficiency of the used mathematics for the problem at hand. On historical grounds, the above occurrence is illustrated by the fact that the mathematics so effective for the study of planetary systems (Hamiltonian vector fields, Hamilton-Jacobi equations, etc.) resulted to be inadequate for the study of the atomic structure. In fact, the latter mandated the use of a *new mathematics*, that based on infinite dimensional Hilbert spaces over a field of complex numbers, that has no application for planetary mechanics.

Similar occurrences exist in contemporary science due to the continued use for new scientific problems of pre-existing mathematics proved to be so effective in preceding scientific problems. This is the case for:

- (1) the lack of a *classical* formulation of antimatter, due to the inapplicability of conventional mathematics so effective for the classical treatment of matter;
- (2) the lack of quantitative studies of nonlocal-integral interactions as occurring in chemical valence bonds, due to the inapplicability of conventional mathematics because of its strictly local-differential character;
- (3) the lack of representation of the irreversible and multi-valued nature of biological systems, due to insufficiencies of both conventional mathematics and hyper-mathematics as currently formulated; and other cases.

In this memoir we outline research conducted by numerous mathematicians, physicists and chemists over the past two decades showing that the construction of new mathematics from open scientific problems does indeed permit new, intriguing scientific horizons with far reaching implications in mathematics as well as quantitative science in general.

As we shall see, the emerging new mathematics are based on progressive generalizations of the multiplicative unit  $+1$  into everywhere invertible and sufficiently smooth, but otherwise arbitrary quantities (such as numbers, matrices or integro-differential operators) with corresponding generalizations of the associative product, thus implying generalizations of all branches of conventional mathematics (hereinafter defined as the mathematics based on the multiplicative left and right unit  $+1$  over a field of characteristic zero).

In this memoir we also show that the above new mathematics imply certain liftings of Newtonian, Hamiltonian and quantum mechanics with corresponding liftings of geometries and symmetries and, inevitably, of contemporary relativities. In fact, all efforts outlined in this memoir were aimed at the construction of new mechanics today known as *iso-, geno-, and hyper-mechanics* for the description of matter in conditions of increasing complexity, and corresponding *isodual mechanics* for the description of antimatter. All new scientific and industrial applications can then be reduced to a few primitive mechanical or, more properly, relativistic axioms. Alternatively, an objective of this memoir is to show that no truly novel scientific advance is possible without truly novel mechanics. In turn, no mechanics can be considered as truly new without new mathematics. Finally, no mathematics can possibly be truly new without new numbers. This illustrates the reason why, out of the rather vast scientific studies over three decades reviewed in this memoir, primary

efforts were devoted to the search of *new numbers* from which new mathematics, new relativities and new scientific and industrial applications uniquely follow.

The reader should be aware that the literature in the topic of this memoir is rather vast because it encompasses numerous studies in pure mathematics, applications in various quantitative sciences, several experimental verifications as well as rapidly expanding new industrial applications. As a result, in this paper we can only review the most fundamental aspects of the new formulations. A technical knowledge of the new advances can only be achieved via the study of the quoted literature. To avoid a prohibitive length, references have been restricted to contributions specifically based on the lifting of the unit with a compatible lifting of the product. Regrettably, we have to defer to the specialized literature the treatment of connections with numerous other studies. References are grouped by main fields indicated with square brackets (e.g., [5]), while individual references are indicated with curved brackets (e.g., (201)). Except for monographs, proceedings and reprint volumes, the titles of the individual contributions are not provide to avoid a prohibitive length, as well as because of their lack of general availability in the physics literature without an extensive library search.

The reader should be aware that the new mathematics and their applications are still in their initial stages and so much remains to be done. The author would be grateful for any comment, as well as for the indication by interested colleagues of mathematical or other references in the *origination* of the new formulations that have escaped his knowledge.

## 2. CONSTRUCTION OF ISODUAL MECHANICS FROM CLASSICAL ANTIMATTER

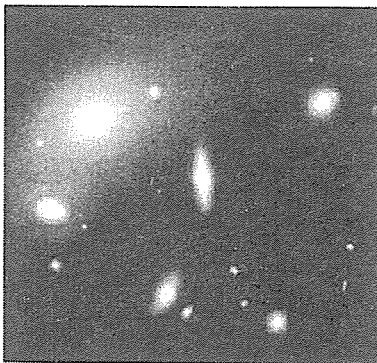
**2.1: The scientific unbalance caused by antimatter.** One of the biggest scientific unbalances of the 20-th century has been the treatment of *matter* at all possible levels, from Newtonian to quantum mechanics, while *antimatter* was solely treated at the level of *second quantization*. In particular, the lack of a consistent *classical* treatment of antimatter left fundamental open problems, such as the inability to study whether a far away galaxy or quasar is made up of matter or of antimatter.

It should be indicated that classical studies of antimatter simply cannot be done by merely reversing the sign of the charge, because of inconsistencies due to the existence of only one quantization channel. In fact, the quantization of a classical particle with the reversed sign of the charge leads to a particle (rather than a charge conjugated antiparticle) with the wrong sign of the charge.

The origin of this scientific imbalance was not of physical nature, and was instead due to the *lack of a mathematics suitable for the classical treatment of antimatter in such a way to be compatible with charge conjugation at the quantum level*. Charge conjugation is an *anti-homomorphism*. Therefore, a necessary condition for a mathematics to be applicable for the classical treatment of antimatter is that of being anti-homomorphic, or, better, anti-isomorphic to conventional mathematics.

The absence of the needed mathematics is confirmed by the fact that classical treatments of antimatter require *fields, functional analysis, differential calculus, topology, geometries, algebras, groups, etc. that are anti-isomorphic to conventional formulations*. The absence in the mathematics of the 20-th century of such a

FIGURE 1. An illustration of the first scientific imbalance of the 20-th century, the inability due to the lack of adequate mathematics to conduct quantitative studies as to whether far away galaxies and quasars are made up of matter or of antimatter.



mathematics then mandated its construction as requested by the physical reality here considered (rather than adapting physical reality to pre-existing mathematics). **2.2: Elements of isodual mathematics.** A novel mathematics verifying the above conditions was proposed by R. M. Santilli in Ref. (11) of 1985 and then developed in various works (see Refs. (12, 14, 15, 54, 55, 160, 161) and [3]).

The fundamental idea is the assumption of a *negative-definite, left and right multiplicative unit*, called *isodual unit*, and denoted  $I^d$ , where  $I$  denotes the conventional positive-definite unit,  $I > 0$ ,

$$(2.1) \quad I^d = -I < 0,$$

with corresponding reformulation of the conventional associative product  $A \times B$  among generic quantities  $A, B$  (such as numbers, vector fields, operators, etc.) into the form

$$(2.2) \quad A \times^d B = A \times (I^d)^{-1} \times B,$$

under which  $I^d$  is the correct left and right multiplicative unit of the theory,

$$(2.3) \quad A \times^d I^d = I^d \times^d A = A,$$

for all elements  $A$  of the considered set.

More generally, *isodual mathematics* is given by the image of a given mathematics admitting a *left and right multiplicative unit* under the following isodual map

$$(2.4) \quad A(x, \dots) \rightarrow A^d(x^d, \dots) = -A^\dagger(-x^\dagger, \dots).$$

when applied to the totality of conventional quantities and their operations, with no exception of any type. In this paper we cannot possibly review the entire isodual mathematics, and must restrict ourselves to an elementary review of only the foundations.

DEFINITION 2.1: Let  $F = F(a, +, \times)$  be a field of characteristic zero representing real numbers  $F = R(n, +, \times), a = n$ , complex numbers  $F = C(c, +, \times), a = c$ , or quaternionic numbers  $F = Q(q, +, \times), a = q$ , with conventional associative, distributive and commutative sum  $a + b = c \in F$ , associative and distributive product  $a \times b = c \in F$ , left and right additive unit  $0, a + 0 = 0 + a = a \in F$ , and left and right multiplicative unit  $I > 0, a \times I = I \times a = a, \forall a, b \in F$ . The *isodual fields* (first introduced in Refs. (11,12)) are rings  $F^d = F^d(a^d, +^d, \times^d)$  with *isodual numbers*

$$a^d = -a^\dagger, \tag{2.5}$$

associative, distributive and commutative *isodual sum*

$$a^d +^d b^d = -(a + b)^\dagger = c^d \in F^d, \tag{2.6}$$

associative and distributive *isodual product*

$$a^d \times^d b^d = a^d \times (I^d)^{-1} \times b^d = c^d \in F^d, \tag{2.7}$$

*additive isodual unit*

$$0^d = 0, a^d +^d 0^d = 0^d +^d a^d = a^d, \tag{2.8}$$

and *isodual multiplicative unit*

$$I^d = -I^\dagger, a^d \times^d I^d = I^d \times^d a^d = a^d, \forall a^d, b^d \in F^d. \tag{2.9}$$

LEMMA 2.1 (12): Isodual fields are fields (namely, isodual field verify all axioms of a field with characteristic zero).

The above lemma establishes the property (first identified in Refs. (11,12)) that the axioms of a field *do not* require that the multiplicative unit be necessary positive-definite, because it can also be negative-definite. The proof of the following property is equally simple.

LEMMA 2.2 (12): Fields (of characteristic zero) and their isodual images are anti-isomorphic to each other.

Lemmas 2.1 and 2.2 illustrate the origin of the name "isodual mathematics." In fact, the needed mathematics must constitute a "dual" image of conventional mathematics, while the prefix "iso" is used in its Greek meaning of preserving the original axioms.

It is evident that for real numbers we have  $n^d = -n$ , while for complex numbers we have  $c^d = (n_1 + i \times n_2)^d = -n_1 + i \times n_2 = -\bar{c}$ , with a similar formulation for quaternions.

DEFINITION 2.2 (12): A quantity is called *isoselfdual* when it coincides with its isodual.

It is easy to verify that the imaginary unit is isoselfdual because

$$(2.10) \quad i^d = -i = -i^d$$

As we shall see, isoselfduality is a new symmetry with rather profound implications, e.g., in cosmology.

It is evident that, for consistency, all operations of numbers must be subjected to *isosuality*. This implies: the *isosual powers*.

$$(2.11) \quad (a^d)^n = a^d \times a^d \times a^d \times \dots$$

(n times with n an integer); the *isosual square root*

$$(2.12) \quad \sqrt{a^d} = -\sqrt{-a^d}; \quad a^d \times a^d = a^d, \quad I^d \times I^d = -i;$$

the *isosual quotient*

$$(2.13) \quad a^d/b^d = -(a^d/b^d) = c^d, \quad b^d \times c^d = a^d; \text{ etc.}$$

An important property for the characterization of antimatter is that *isosual fields*

have a *negative-definite norm*, called *isosual norm* (I2)

$$(2.14) \quad |a^d| = |a^d| \times I^d = -(a^d)^{1/2} > 0,$$

where  $|\dots|$  denotes the conventional norm. For isosual real numbers  $n^d$  we have the isosual norm  $|n^d| = -|n| < 0$ , the isosual norm for isosual complex numbers

$$|c^d| = -(n_1^2 + n_2^2)^{1/2}, \text{ etc.}$$

Recall that functional analysis is defined over a field. Therefore, the lifting of fields into isosual fields requires, for necessary condition of consistency, the formulation of the *isosual functional analysis* (54). We here merely recall that

$$(2.15a) \quad \sin^d \theta = -\sin(-\theta), \quad \cos^d \theta = -\cos(-\theta),$$

$$(2.15b) \quad \cos^d \theta = \sin^d \theta, \quad \sin^d \theta = -\cos^d \theta;$$

the *isosual hyperbolic functions*

$$(2.16a) \quad \sinh^d w = -\sinh(-w), \quad \cosh^d w = -\cosh(-w),$$

$$(2.16b) \quad \cosh^d w = \sinh^d w, \quad \sinh^d w = -\cosh^d w;$$

the *isosual logarithm*

$$(2.17) \quad \log^d n = -\log(-n).$$

Particularly important is the *isosual exponentiation* which can be written

$$(2.18) \quad e^d = I^d + A^d/I^d + A^d/I^d + A^d \times A^d/I^d + \dots = -e^d,$$

Other properties of the isosual functional analysis can be easily derived by the

interested reader (see also Refs. (14,21,22)).

It is little known that the differential and integral calculi are indeed dependent

on the assumed basic unit. In fact, the lifting of  $I$  into  $I^d$  and of  $F$  into  $F^d$  implies the *isosual differential calculus*, first introduced in Ref. (14), which is characterized

$$(2.19) \quad d^d x = dx,$$

by the *isosual differentials*

with corresponding *isosual derivatives*

$$(2.20) \quad \partial^d f(x) = -\partial f(x)/\partial(-x),$$

and other isosual properties interested readers can easily derive. Note that the differential is isoselfdual.



**2.3: Isodual spaces and geometries.** Conventional vector and metric spaces are defined over a field. It is then evident that the isoduality of fields requires, for consistency, a corresponding isoduality of vector, metric and all other) spaces.

DEFINITION 2.3: Let  $S = S(x, g, R)$  be an  $N$ -dimensional metric space with real-valued local coordinates  $x = \{x^k\}$ ,  $k = 1, 2, \dots, N$ , nowhere degenerate, sufficiently smooth, real-valued and symmetric metric  $g(x, \dots)$  and related line element  $x^2 = (x^t \times g \times x) \times I = (x^i \times g_{ij} \times x^j) \times I \in R$ . The *isodual spaces*, first introduced in Refs. (11,14), are vector spaces  $S^d(x^d, g^d, R^d)$  with *isodual coordinates*  $x^d = -x^t$  where  $t$  denotes transposed, *isodual metric*  $g^d(x^d, \dots) = -g^t(-x^t, \dots)$ , and *isodual line element*

$$\begin{aligned} (x^d)^{2^d} &= (x^2)^d = (-x^t)^{2^d} = (x^d) \times^d (g^d) \times^d (x^{td}) \times^d I^d = \\ &= [(-x^j)(-\times)(-g_{ji})(-\times)(-x^i)](-\times)(-I) = -x^2 \in R^d. \end{aligned} \quad (2.21)$$

The *isodual Euclidean space*  $E^d(x^d, \delta^d, R^d)$  is a particular case of  $S^d$  when  $g^d_{ij} = \delta^d_{ij}$ . The *isodual distance* on  $E^d$  is negative definite and it is given by  $D^d = -D$ , where  $D$  is the conventional (positive-definite) distance on  $E$ . The *isodual sphere* on a 3-dimensional isodual space  $E^d$  is the perfect sphere with negative radius and expression  $r^{d2^d} = [(x_1^{d2^d} +^d x_2^{d2^d} +^d x_3^{d2^d}) \times^d I^d = -r^2 \in R^d$ . The *isodual Minkowskian, isodual Riemannian and isodual symplectic geometries can be defined accordingly* (14,15).

**2.4: Isodual Lie theory.** Lie's theory in its conventional formulation in mathematics or physics can only characterize matter at the classical level, thus preventing the study of antimatter via fundamental tools so familiar for matter, such as symmetries and conservation laws.

To overcome such an imbalance, R. M. Santilli proposed in the *isodual Lie theory* in the original proposal of isoduality (11), whose explicit form is left to the interested reader. We merely indicate that the isoduality of Lie's theory were proposed also for the classification of all possible realizations of abstract simple Lie algebras.

From the above rudiments interested readers can construct the rest of the isodual mathematics, including: isodual topologies, isodual manifolds, etc. Particularly important for physical applications is the *isodual Lie theory* (first introduced in Ref. (11) (see also (14,22)), including *isodual universal enveloping associative algebras, isodual Lie algebras, isodual Lie groups, isodual symmetries, and isodual representation theory*, which we cannot review here for brevity.

The main physical theories characterized by isodual mathematics can be outlined as follows.

**2.5: Isodual Newtonian Mechanics.** To resolve the scientific imbalance between matter and antimatter indicated earlier, the isodual mathematics has first permitted a *Newtonian* characterization of antimatter consistent with all available experimental data (14,22). Then, isodual mathematics has identified a new quantization channel (which is distinct from conventional symplectic quantization) leading to an operator formulation which is equivalent to charge conjugation (14,16,21).

We first have the *isodual Newton equations*

$$m^d \times^d \frac{d^d v^d}{d^d t^d} = F^d(t^d, x^d, v^d), v^d = \frac{d^d x^d}{d^d t^d}. \quad (2.22)$$

and related formulations known under the name of *isodual Newtonian Mechanics*.

**2.6: Isodual Hamiltonian Mechanics.** The direct analytic representation of the above equations are permitted by the following *isodual action functional* (14) (for the case when the Newtonian force  $F$  is representable with a potential, see below for nonhamiltonian interactions)

$$(2.23) \quad \delta_d A_d(t_d, x_d) = \delta_d \int_{d'} (p_d \times_d p_d x_d -_d H_d \times_d p_d p_d t_d) = 0,$$

which characterizes the following the *isodual Hamilton equations* (14)

$$(2.24) \quad \frac{\partial_d x_d}{\partial_d H_d} = p_d, \quad \frac{\partial_d p_d}{\partial_d H_d} = -p_d, \quad \frac{\partial_d t_d}{\partial_d H_d} = -\frac{\partial_d x_d}{\partial_d H_d},$$

with corresponding *isodual Hamilton-Jacobi equations*

$$(2.25) \quad \frac{\partial_d A_d}{\partial_d t_d} +_d H_d = 0, \quad \frac{\partial_d A_d}{\partial_d x_d k} -_d p_d^k = 0, \quad k = 1, 2, 3.$$

Again, interested readers are encouraged to verify that the above *isodual Hamiltonian Mechanics* does indeed represent correctly all *classical* experimental data on antimatter.

**2.7: Isodual Quantum Mechanics.** The isoduality of the naive (or symplectic) quantization can be expressed via the elementary map (14,16,21,55) based on the *isodual Planck's constant*  $\hbar_s = I_d = -1$

$$(2.26) \quad A_d(t_d, x_d) \rightarrow -i_d \hbar_s I_d \times_d I_d \times_d L_m^d \psi_d(t_d, x_d),$$

which can be applied to the isodual Hamilton-Jacobi equations yielding the expressions

$$(2.27a) \quad \frac{\partial_d A_d}{\partial_d t_d} +_d H_d = 0 \rightarrow -i_d \hbar_s I_d \times_d I_d \times_d \frac{\partial_d \psi_d}{\partial_d t_d} +_d \psi_d \times_d H_d = 0,$$

$$(2.27b) \quad \frac{\partial_d A_d}{\partial_d x_d k} -_d p_d^k = 0 \rightarrow -i_d \hbar_s I_d \times_d I_d \times_d \frac{\partial_d \psi_d}{\partial_d x_d k} -_d \psi_d \times_d p_d^k = 0,$$

Therefore, isodual mathematics characterizes the novel *isodual quantum mechanics*, also known as the *isodual branch of hadronic mechanics*, which can be expressed on the *isodual Hilbert space*  $H_d$  with *isodual states*  $|\psi_d\rangle = - \langle \psi_d|$  and *isodual inner product*  $\langle \psi_d| \times_d \psi_d \rangle$  over the isodual field  $C^d$  with basic *isodual Schroedinger equations* (for a Hermitian Hamiltonians  $H$ )

$$(2.28a) \quad i_d \times_d \langle \psi_d| \times_d \frac{\partial_d \psi_d}{\partial_d t_d} -_d \langle \psi_d| \times_d H_d = - \langle \psi_d| \times_d E_d,$$

$$(2.28b) \quad - \langle \psi_d| \times_d \frac{\partial_d \psi_d}{\partial_d x_d k} +_d \langle \psi_d| \times_d p_d^k > = -i_d \hbar_s I_d \times_d \langle \psi_d| \times_d \frac{\partial_d \psi_d}{\partial_d x_d k},$$

and corresponding *isodual Heisenberg equations* (for a Hermitian observable  $A$ )

$$(2.29a) \quad i_d \times_d \langle \psi_d| \times_d \frac{\partial_d A_d}{\partial_d t_d} -_d \langle \psi_d| \times_d [A_d, H_d] = - \langle \psi_d| \times_d \frac{\partial_d A_d}{\partial_d x_d k} +_d \langle \psi_d| \times_d p_d^k,$$

$$(2.29b) \quad [p_d^i, x_d] = -i_d \hbar_s I_d \times_d \langle \psi_d| \times_d \frac{\partial_d \psi_d}{\partial_d x_d k}, \quad [x_d^i, p_d^j] = 0.$$

The equivalence of the above operator formulation of antimatter with charge conjugation has been proved in Refs. (16,21).

**2.8: Isodual special relativity.** The vast scientific literature on special relativity throughout the 20-th century is silent on the fact that, classically, it can solely

represent matter, under the belief that classical antimatter can be represented via a mere change of the sign of the charge without an inspection of the various consequential inconsistencies identified earlier.

To resolve this impasse, R. M. Santilli proposed in Refs. [3] the *isodual special relativity* which is based on the isodual topology (14), the isodual Minkowski space (15), the isodual Poincaré symmetry (29) and the isodualities of relativistic dynamics and physical laws (55). The explicit form of these structures can be easily constructed by the interested reader via the isodual map (2.4).

We only note that that the change of the sign of elementary charges or, more appropriately, charge conjugation, are anti-homomorphic maps, while isoduality is an anti-isomorphic map. Therefore, *according to special relativity antiparticles exist in the same spacetime of particles, while, according to isodual special relativity, antiparticles exist in the isodual spacetime which coexists with, yet it is independent from our spacetime.*

**2.9: Origin of the isodual theory in Dirac's equation.** On historical grounds it should be indicated that the isodual mathematics and related theory of antimatter originated from an inspection of the celebrated Dirac equation (6). In fact, its basic unit displays precisely the negative-definite unit for the antiparticle component, namely, the isodual unit,

$$\gamma^o = i \times \begin{pmatrix} I_{2 \times 2} & 0_{2 \times 2} \\ 0_{2 \times 2} & -I_{2 \times 2} \end{pmatrix} = i \times \begin{pmatrix} I_{2 \times 2} & 0_{2 \times 2} \\ 0_{2 \times 2} & I_{2 \times 2}^d \end{pmatrix} \quad (2.31)$$

Similarly, by recalling that Pauli's matrices are Hermitean, the space components of Dirac's gamma matrices exhibit precisely the isodual Pauli's matrices of antimatter precisely for the antimatter component of the equation,

$$(\gamma^k) = \begin{pmatrix} 0_{2 \times 2} & \sigma_{2 \times 2} \\ -\sigma_{2 \times 2} & 0_{2 \times 2} \end{pmatrix} = \begin{pmatrix} 0_{2 \times 2} & -\sigma_{2 \times 2}^d \\ -\sigma_{2 \times 2} & 0_{2 \times 2} \end{pmatrix}. \quad (2.32)$$

It then follows that *Dirac's gamma matrices have the new symmetry of being isoselfdual* (Definition 2.2)

$$\gamma^\mu = -\gamma^{\mu\dagger} = \gamma^{\mu d}, \quad (2.33)$$

and, when interpreted via the isosual mathematics, *Dirac's equation directly describes the Kronecker product of an electron and a positron*

$$\begin{aligned} & \{ \gamma^\mu [p_\mu - 2 \times A_\mu] x / c_o + i \times m \} \times d\phi(x) = \\ & = \left[ \begin{pmatrix} 0 & -\sigma^{kd} \\ -\sigma^k & 0 \end{pmatrix} \times (p_k - e \times A_k / c_o) - \right. \\ & \left. -i \times \begin{pmatrix} I & 0 \\ 0 & I^d \end{pmatrix} \times (p_4 - e \times A_4 / c_o) + i \times m \right] \times \begin{pmatrix} \phi(x) \\ \phi^d(x) \end{pmatrix} = 0. \end{aligned} \quad (2.34)$$

Finally, *the true invariance of Dirac's equation is not that under the Poincaré symmetry alone, as popularly believed until now, but rather under the Poincaré symmetry and its isodual*

$$S^{Tot} = P(3.1) \times P^d(3.1). \quad (2.35)$$

Note the elimination of the controversial "hole theory" and second quantization, since the isodual theory of antimatter holds at the *classical* level, let alone that in first quantization, without excluding, of course, its applicability to second quantization. In particular, as indicated earlier, negative-energy solutions of Dirac's

equations mandated the "hole theory" because referred to *positive units*, thus being unphysical, while the negative-energy solutions of Eq. (2.34) are referred in isodual mathematics to *negative units*, thus being fully physical.

Note also the elimination of the additional controversy on the "four-dimensional irreducible representation of spin 1/2" because, under the proper interpretation, Dirac's gamma matrices solely characterize a *two-dimensional representation of spin 1/2*, as a Kronecker product of one representation for matter and its isodual for antimatter. Note the need to eliminate second quantization to admit only two-dimensional irreducible representations of spin 1/2, as mandated by Lie's theory.

Regrettably, Dirac was unaware of the fact that a negative unit can indeed be the correct unit of an appropriate mathematics and, as a consequence, he developed the "hole theory" restricting the treatment of antiparticles to the sole level of second quantization. It is an easy prediction that, in the event the isodual numbers had been discovered prior at the beginning of the 20-th century (rather than in 1985 (11)), Dirac would never have proposed his "hole theory.

**2.10: Experimental verifications and applications.** It is easy to see that the isodual theory represents correctly all available classical experimental evidence on antimatter. For instance, the Coulomb laws for matter with charge  $q$  and antimatter with charge  $q^d$  at mutual distance  $r$  are given by

$$(2.30a) \quad F_{\text{MatterObserver}} = q \times q^d / r^2, \quad F_{\text{AntimatterObserver}} = q^d \times q^d / d^2 r^2,$$

$$(2.30b) \quad F_{\text{MatterObserver}} = q \times q / r^2, \quad F_{\text{AntimatterObserver}} = q^d \times q^d / d^2 r^2,$$

and they correctly represent mutual attraction (mutual repulsion) for

antimatter-antimatter, (matter-matter and antimatter-antimatter) for both matter and antimatter observers, where  $F$  is attractive when having negative value for a matter observer on  $R^d$ , as conventional, and a positive value for an antimatter observer on  $R^d$ , the opposite occurring for attraction. For additional details, interested reader can inspect Ref. (22).

The equivalence of the isoduality on Hilbert spaces with charge conjugation proved in Refs. (16,21) establishes that the isodual theory of antimatter with available experimental data at the operator level too.

Despite its simplicity, the physical, astrophysical and cosmological implications of isodual mathematics are rather deep. To begin, the isodual map (2.4) implies the change of the sign not only of the charge, but also of all other physical quantities of matter, including mass, energy, time, etc. For instance, the energy eigenvalue  $E$  of Eqs. (2.9) has *negative values* since it is positive in Eq. (2.9), yet computed on  $R^d$ . Note that the measurement of physical quantities with respect to *negative definite units* resolves the traditional inconsistencies for negative mass and energy.

In particular, the isodual theory of antimatter recovers the old hypothesis that *antiparticles move backward in time* (since they have a negative-definite time) by resolving the inherent violation of causality which lead to its abandonment in the second half of the 20-th century. In fact, *motion backward in time measured with respect to a negative unit of time is as causal as the conventional motion forward in time referred to a positive unit of time.*

Most importantly, the isodual theory of antimatter mandates the existence of *antigravity defined as a gravitational repulsion experienced by antimatter in the field of matter and vice-versa* (16,21,22), while resolving the historical objections