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DISSIPATIVITY AND LIE-ADMISSIBLE ALGEBRAS *

Ruggero M. Santilli **

SOMMARIO: Si richiama la validità della Meccanica Hamiltoniana e l'invarianza secondo le algebre di Lie per campi liberi, insieme alle loro difficoltà per campi interagenti. Si discute il ruolo della dissipatività dei campi interpolati come un indirizzo che permette l'introduzione per le regioni di interazione di più larghe strutture analitiche ed algebriche connesse da parentesi di Poisson generalizzate. Si introduce un criterio di lavoro espresso in termini di algebre Lie-ammissibili, il quale da luogo alla Meccanica Pseudo-Hamiltoniana introdotta da R. J. Duffin nel 1962 per sistemi classici, discreti e dissipativi. Come esempio di applicazione fisica si costruisce un modello di plasma dissipativo e si mostra come i parametri della formulazione influenzano il vettore corrente elettrica ed il tensore di conduttività. Inoltre la procedura viene estesa a sistemi classici, continui e dissipativi costruendo, come applicazione, alcuni esempi di equazioni differenziali per campi interpolati indotte da strutture di tipo Lie-ammissibile.

SUMMARY: The validity of Hamiltonian mechanics and Lie algebra invariance for free fields and their difficulties for interacting fields are recalled. The role of dissipativity is discussed as an approach allowing the introduction for interacting regions of larger analytical dynamics and algebraic formulations related by the enlarged bracket. A working criterion is introduced in terms of Lie-admissible algebras and the pseudo-Hamiltonian mechanics introduced by R. J. Duffin for discrete dissipative systems is considered as an explicit choice able to reduce to the Hamiltonian mechanics when the systems become conservative. An example of dissipative plasma is explicitly investigated. Furthermore the procedure is extended to continuous systems and classical interpolating dissipative fields induced by Lie-admissible structures are constructed.

1. Introduction.

As is well known, *Hamiltonian mechanics* is a fundamental mathematical tool for elementary particle physics both from a field theoretical and an algebraic viewpoints. Indeed the field equations for free fields have been constructed by performing quantization of the Hamilton equations for continuous systems [1]. Furthermore the validity of *Lie algebras* as invariance algebras of free fields is guaranteed by Hamiltonian mechanics since already at a classical level the Poisson bracket satisfies the defining Lie algebra identities and after transition into the Lie product under quantization the algebraic structure remains unchanged [2].

However is well known too that for interacting fields Hamiltonian mechanics presents so many different problematic aspects which at the present time do not seem to be soluble [3]. Indeed the Hamilton field equations should satisfy the asymptotic conditions, that is the interpolating fields should reduce to the free fields for $t \to \pm \infty$; at least in principle the spectrum of the mass operator should contain a discrete spectrum of singlets, a continuous spectrum and the vacuum all in compatible form with the boundary conditions; the uniqueness of the Hamiltonian operator cannot be generally stated in the presence of highly singular interactions while even the selfadjointness of the Hamiltonian on the basic Hilbert space must be proved [4]; the renormalization procedure opens many problems for the transition from the interpolating fields to the free fields; and finally the canonical commutation rules may be violated in the presence of divergent terms in the systems of correlation functions (e. g. for the excessively singular structure of the propagators), in which case there is the same practical impossibility of constructing the field equations by means of the quantized Hamilton equations.

The loss of the Hamiltonian mechanics for interacting fields means the loss of an analytical dynamics procedure for justifying the Lie product and the Lie algebra invariance generally becomes formal.

As a consequence of the above problematic aspects the Heisenberg S matrix idea has been preferred to the Hamiltonian mechanics for interacting fields and has been developed up to a satisfactory level. However the S matrix is a bridge between the initial and final states. Thus a satisfactory mathematical representation of the interpolating fields from both an analytical dynamics and an algebraic viewpoints is still an open problem.

In this connection from an analytical dynamics viewpoint we note that at a classical level too Hamiltonian mechanics possesses well defined limits of validity. Indeed it can be applied in its general formulation to conservative (and holonomic) systems only and can be extended to only a few specific dissipative (or non-holonomic) systems [5]. Consequently *dissipativity* may have a role in the problematic aspects of the Hamiltonian mechanics for interacting fields. Indeed if, instead of considering a process on the whole, we consider partial regions of globally conservative systems with internal exchange of energy or momentum

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** Boston University, Physics Department, Boston, Massachusetts, USA.

or angular momentum (e. g. we consider one or more lines of a Feynman diagram as external), then dissipativity becomes the essential feature of the transition region, the Hamiltonian mechanics is not longer generally valid (e. g. when the dissipative forces are not derivable from a generalized potential) and the way is open to the investigation of a generalization of the Hamiltonian procedure. On the contrary, if a particle process is considered on the whole, then it is always conservative. By the variational principe, the only allowed procedure is the Hamiltonian mechanics and the above problems cannot be avoided. Thus dissipativity is an approach in quantum field theory (not yet deeply investigated) allowing the introduction of a generalization of the Hamiltonian mechanics for interpolating fields.

From an algebraic viewpoint we note that this generalization implies as a first consequence the enlargement of the Lie algebra invariance since the abstract product of the algebra is the generalized bracket of the analytic formulation. Moreover the algebraic viewpoint may itself give a criterion for selecting an analytical dynamics procedure for dissipative regions since the enlarged formulation must be able to regenerate Lie algebra invariance when the fields become conservative. In this connection the interest of a generalization of the Lie algebra structure by means of *Lie-admissible algebras* (1) [8] has been recently pointed out for physical regions where Lie algebra invariance no longer holds. Indeed it has been shown that [6], [7]:

i) Lie-admissible algebras possess a well defined content of Lie algebras. Consequently, the generalization of the Lie algebra invariance by means of Lie-admissible structure for an interpolating field is able to preserve a defined content of the original free field invariance.

ii) Lie-admissible algebras are able to reduce to Lie algebras when anticommutativity of the product is requested (see [7] footnote 10). Consequently they are able to satisfy the asymptotic condition in the sense that the generalized invariance algebra of an interpolating field is able to reduce to a Lie algebra when the field becomes conservative.

iii) The only simple trace-admissible power-associative and normed Lie-admissible algebras able to satisfy requests i) and ii) are the *mutation algebras* $A(\lambda, \mu)$ of quasiassociative type, i. e. the (λ, μ)-mutation of an associative algebra A in terms of the product [6], [8]

$$(a, b) = \lambda ab + \mu ba = \varrho[a, b] + \sigma\{a, b\} \qquad (1.1)$$

where $\lambda = \sigma + \varrho$ and $\mu = \sigma - \varrho$ are free scalars and ab is the associative product in A. By noting that a Lie algebra is a $(1, -1)$-mutation of A, the investigations on a possible dynamical meaning of the "deviations" $(\lambda - 1)$ and $(\mu + 1)$ in connection with an interpolating field seem to be interesting, while the asymptotic conditions can be simply obtained by requiring that $\sigma \to 0$ for conservative regions.

In the present paper we introduce a working criterion for selecting analytical dynamics procedures for dissipative systems in terms of Lie-admissible algebras and we discuss an explicit choice.

Let us consider a conservative (and holonomic) system described in terms of Hamiltonian mechanics and possessing a given symmetry Lie algebra L. We suppose that during a given period of time and in a given region of space the system becomes dissipative under the action of an external system according to Fig. 1. For an analytical dynamics representation of the dissipative region we require that:

A) The formulation must be able to regenerate Hamiltonian mechanics and Lie algebra invariance at the boundaries without loss of continuity.

B) The formulation must be able to induce a (nonassociative) algebra U by means of a generalization of the Poisson bracket.

C) The algebra U must be Lie-admissible. Then a well defined content of the original invariance is preserved if an imbedding of the Lie algebra L into the Lie-admissible algebra U holds according to [6]

$$L \xrightarrow{\text{Isomorphism}} U^- \xrightarrow{\text{Imbedding}} U \qquad (1.2)$$

that is when the attached algebra U^- is isomorphic to L.

Since a Lie algebra is always Lie-admissible [6], [7] the first procedure which trivially satisfies our requirements is the same Hamiltonian mechanics. Similarly the above requirements are satisfied by any generalization of the Poisson bracket satisfying the Lie algebra identities, as for instance the Dirac bracket [9], since the Jacobi iden-

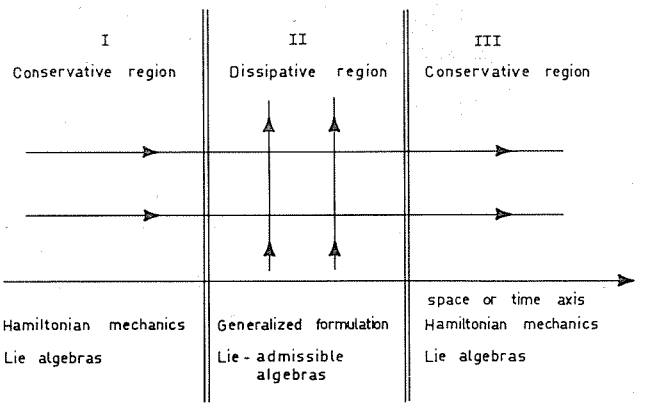

Fig. 1 - The figure refers to an interacting system where some processes are considered as external. The horizontal and vertical lines represent respectively the considered states and the external states. The transition between the conservative initial and final regions I and III occurs by means of an interacting region II which is essentially dissipative. The Hamiltonian mechanics and Lie algebra invariance occur for the asymptotic regions only, while for the dissipative region a generalization of the Hamilton procedure and a Lie-admissible algebra able to satisfy the asymptotic conditions are requested.

(1) The non-trivial Lie-admissible algebras are algebras U with product $a \odot b$ neither totally antisymmetric nor totally symmetric such that the attached algebra U^-, which is the same vector space as U but with the new product $[a, b] = a \odot b - b \odot a$, is a Lie algebra [6], [7].

tity can be considered as the Lie-admissibility condition for anticommutative algebra [7].

In the present paper as a generalization of the Hamiltonian formalism for a dissipative region satisfying our requests we consider the pseudo-Hamiltonian mechanics introduced by R. J. Duffin for discrete systems according to the Eqs. [10]

$$\dot{q}_k = \frac{\partial H}{\partial p_k}, \qquad k = 1, 2, ..., n, \tag{1.3}$$

$$\dot{p}_k = \frac{1}{\varepsilon}\frac{\partial H}{\partial q_k},$$

where ε is a free scalar ($\neq 0$) and f_k denotes the external force components. Then the Poisson bracket is generalized according to the Duffin expression

$$(A, B)_D = \frac{\partial A}{\partial q_k}\frac{\partial B}{\partial p_k} + \frac{1}{\varepsilon}\frac{\partial B}{\partial q_k}\frac{\partial A}{\partial p_k} \tag{1.4}$$

while the total derivative of any function $F(p, q, t)$ (for $f_k = 0$) can be written

$$\frac{dF}{dt} = (F, H)_D + \frac{\partial F}{\partial t}. \tag{1.5}$$

Clearly the pseudo-Hamiltonian mechanics satisfies our requirements. Indeed when $\varepsilon = -1$ the formulation reduces to the usual Hamiltonian mechanics; the generalization of the Poisson bracket according to (1.4) induces a (nonassociative) algebra $U(\varepsilon)$ which is not longer a Lie algebra; the algebra $U(\varepsilon)$ is Lie-admissible since the product $(A, B)_D$ satisfies the general Lie-admissiblity condition for nonassociative algebras [6], [7]

$$[A, B, C] + [B, C, A] + [C, A, B] = \\ = [C, B, A] + [B, A, C] + [A, C, B], \tag{1.6}$$

where

$$[A, B, C] = ((A, B)_D, C)_D - (A, (B, C)_D)_D. \tag{1.7}$$

The above statement can be more directly seen by noting that the attached algebra $[U(\varepsilon)]^-$ of $U(\varepsilon)$ is characterized by the Poisson bracket, being

$$(A, B)_D - (B, A)_D = (1 - \varepsilon)\left(\frac{\partial A}{\partial q_k}\frac{\partial B}{\partial p_k} - \frac{\partial B}{\partial q_k}\frac{\partial A}{\partial p_k}\right). \tag{1.8}$$

Pseudo-Hamiltonian mechanics has been introduced in order to represent some dissipative electrical networks and to describe small motions of dissipative systems about a position of static equilibrium [10]. However the procedure may be useful not only for dissipative systems in general but also, because of its Lie-admissibility content, for partial regions of globally conservative systems with internal exchange of energy or momentum or angular momentum. Thus the procedure may be interesting for particle physic too, i. e. for some non-elastic processes represented in terms of Feynman diagrams where one or more lines are considered as external.

In the present paper we introduce a two-parameter formulation of the procedure and we perform an extension to continuous systems. Some classical examples are discussed. In a subsequent paper we should like to attempt the direct construction of a particle model by investigating the quantization of the procedure on the basis of the property that the Lie algebra content (1.8) can be quantized. In this case the proposed imbedding of Lie algebras (1.2) might possess analytical dynamics procedures useful for deriving field equations for both classical and quantum mechanical systems given by the Hamiltonian mechanics for conservative regions characterized by Lie algebras and pseudo-Hamiltonian mechanics for dissipative regions characterized by Lie-admissible mutation algebras. Finally it is interesting to note that at a quantum mechanical level this procedure may be proved to be the only possible generalization of the Hamiltonian formalism possessing simple trace-admissible power-associative and normed algebraic structure [6].

2. Two-parameter pseudo-hamiltonian mechanics for discrete systems.

We consider a discrete conservative system corresponding to region I of Fig. 1 characterized by the generalized coordinates q_k ($k = 1, 2, ..., n$) possessing time derivatives $\dot{q}_k$ and the Lagrangian density $L(q, \dot{q})$ expressed in terms of kinetic $T(\dot{q})$ and potential $V(q)$ energies

$$L = T(\dot{q}) - V(q). \tag{2.1}$$

We suppose that in the transition from region I to region II the system becomes dissipative under the action of an external system characterized by the components f_k, and that the new expression L_D for the Lagrangian density for region II at a given instant of time can be written

$$L_D = T'(\dot{q}) - V'(q), \tag{2.2}$$

where

$$T'(\dot{q}) = \lambda T(\dot{q}), \\ V'(q) = -\mu V(q) \tag{2.3}$$

with λ and μ real scalar quantities.

Under the above assumptions we have

$$\frac{d}{dt}\frac{\partial L_D}{\partial \dot{q}_k} - \frac{\partial L_D}{\partial q_k} = \lambda\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} + \mu\frac{\partial L}{\partial q_k}. \tag{2.4}$$

Hence, instead of considering the Lagrange equations corresponding to L_D we can equivalently use the following *pseudo-Lagrangian equations for discrete systems*

$$\lambda \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} + \mu \frac{\partial L}{\partial q_k} = f_k \tag{2.5}$$

using the same expression of L for both regions I and II since the variations of the kinetic and potential energies are represented by the parameters involved in formulation. Then L is the true Lagrangian density for the conservative region and is assumed to be the *pseudo-Lagrangian density* for the dissipative region.

Similarly we can introduce the Hamiltonian densities

$$\begin{aligned} H &= T(p) + V(q) \\ H_D &= T'(p) + V'(q), \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} T'(p) &= \lambda T(p), \\ V'(q) &= -\mu V(q), \end{aligned} \tag{2.7}$$

by which

$$\begin{aligned} p_k &= \frac{\partial L_D}{\partial q_k} = \lambda \frac{\partial L}{\partial q_k}, \\ \frac{\partial H_D}{\partial p_k} &= \lambda \frac{\partial H}{\partial p_k}, \\ \frac{\partial H_D}{\partial q_k} &= -\mu \frac{\partial H}{\partial q_k}. \end{aligned} \tag{2.8}$$

Thus for the dissipative region II, instead of using the Hamilton equations corresponding to H_D, we can introduce the equations

$$\begin{aligned} \dot{q}_k &= \lambda \frac{\partial H}{\partial p_k}, \\ \dot{p}_k &= \mu \frac{\partial H}{\partial q_k} + f_k, \end{aligned} \qquad k = 1, 2, \ldots, n, \tag{2.9}$$

which we assume as the *pseudo-Hamiltonian equations* for discrete systems. Thus H has the meaning of true Hamiltonian density for the conservative region and of *pseudo-Hamiltonian density* for the dissipative region.

We note that the total time derivative of H is now

$$\frac{dH}{dt} = \frac{\partial H}{\partial q_k} \dot{q}_k + \frac{\partial H}{\partial p_k} \dot{p}_k = (\lambda + \mu) \frac{\partial H}{\partial q_k} \frac{\partial H}{\partial p_k}. \tag{2.10}$$

Thus the energy is conserved only for $\lambda = -\mu$, i. e. for the conservative formulation. Similarly the total time derivative of any function $F(q, p, t)$ (for $f_k = 0$) is given by

$$\frac{dF}{dt} = \left(\lambda \frac{\partial F}{\partial q_k} \frac{\partial H}{\partial p_k} + \mu \frac{\partial H}{\partial q_k} \frac{\partial F}{\partial p_k} \right) + \frac{\partial F}{\partial t} \tag{2.11}$$

and the generalization of the Poisson bracket for any two functions $A(q, p)$ and $B(q, p)$ is

$$(A, B) = \lambda \frac{\partial A}{\partial q_k} \frac{\partial B}{\partial p_k} + \mu \frac{\partial B}{\partial q_k} \frac{\partial A}{\partial p_k}. \tag{2.12}$$

Let us note that the above expression can be written

$$(A, B) = \varrho \left(\frac{\partial A}{\partial q_k} \frac{\partial B}{\partial p_k} - \frac{\partial B}{\partial q_k} \frac{\partial A}{\partial p_k} \right) + \sigma \left(\frac{\partial A}{\partial q_k} \frac{\partial B}{\partial p_k} + \frac{\partial B}{\partial q_k} \frac{\partial A}{\partial p_k} \right), \quad \begin{cases} \lambda = \sigma + \varrho, \\ \mu = \sigma - \varrho. \end{cases} \tag{2.13}$$

Thus, if the following supplementary condition on the parameters holds

$$\lambda^2 + \mu^2 = 2 \qquad \text{or} \qquad \varrho^2 + \sigma^2 = 1, \tag{2.14}$$

then the bracket (2.12) can be written

$$(A, B) = \cos \alpha \, [A, B] + \sin \alpha \, \{A, B\} \tag{2.15}$$

where $[A, B]$ is the usual Poisson bracket and $\{A, B\}$ is the totally symmetric expression on the r.h.s of Eq. (2.13).

Under the above assumption Eqs. (2.5), (2.9) and (2.11) can be written

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = \frac{f_k}{\sin \alpha + \cos \alpha} - 2 \frac{\tan \alpha}{1 + \tan \alpha} \frac{\partial L}{\partial q_k} \tag{2.16}$$

$$\begin{aligned} \dot{q}_k &= (\sin \alpha + \cos \alpha) \frac{\partial H}{\partial p_k}, \\ \dot{p}_k &= (\sin \alpha - \cos \alpha) \frac{\partial H}{\partial q_k} + f_k, \end{aligned} \tag{2.17}$$

$$\frac{dF}{dt} = \cos \alpha [F, H] + \sin \alpha \{F, H\} + \frac{\partial F}{\partial t}. \tag{2.18}$$

In this way the transition from the conservative region to a dissipative one is characterized by an "angle" α, while for $\alpha = 0^\circ$ all the formulation coincides with the usual Hamiltonian mechanics.

For supplementary properties of the bracket (2.12) see Appendix A.

Let us note that from an algebraic viewpoint the one-parameter and two-parameter formulations of the pseudo-Hamiltonian mechanics are equivalent. Indeed the algebra characterized by the product (2.12) is isomorphic to the isotopic algebra [6], [7] $U^*(\varepsilon)$ induced by the product

$$(A, B)^* = \frac{\partial A}{\partial q_k} * \frac{\partial B}{\partial p_k} + \frac{1}{\varepsilon} \frac{\partial B}{\partial q_k} * \frac{\partial A}{\partial p_k} \qquad (2.19)$$

for $\varepsilon = \lambda/\mu$, where the associative product AB is substituted by $A*B = \lambda AB$.

In connection with the problems involved in a possible quantization of the formulation, we note that the bracket (2.12) looks, in a stimulating way, like the classical correspondence of the product (1.1) defining the mutation algebras $A(\lambda, \mu)$. However the algebra $U(\lambda, \mu)$ characterized by (2.12) and the algebra $A(\lambda, \mu)$ characterized by (1.1) generally differ. Indeed the flexibility condition $[A, B, A] = ((A, B), A) - (A, (B, A)) = 0$ is not verified by $U(\lambda, \mu)$ since

$$[A, B, A] = (\lambda^2 - \mu^2)\left(\frac{\partial A}{\partial q_k} \frac{\partial B}{\partial q_i} \frac{\partial^2 A}{\partial p_k \partial p_i} - \frac{\partial^2 A}{\partial q_k \partial p_i} \frac{\partial B}{\partial p_i} \frac{\partial A}{\partial q_k}\right). \qquad (2.20)$$

Furthermore the algebra $[U(\lambda, \mu)]^+$ induced by the product

$$\frac{1}{2}((A, B) + (B, A)) = \frac{\lambda + \mu}{2}\left(\frac{\partial A}{\partial q_k} \frac{\partial B}{\partial p_k} + \frac{\partial B}{\partial q_k} \frac{\partial A}{\partial p_k}\right) \qquad (2.21)$$

is not a (commutative) Jordan algebra since the Jordan identity $[A^2, B, A] = 0$ is not verified. Consequently $U(\lambda, \mu)$ is not a realization of the noncommutative Jordan algebras as for the $A(\lambda, \mu)$ algebras [6]. However, if we consider a set of elements $A, B, C, \ldots$ satisfying the identities

$$\frac{\partial^2 A}{\partial q_k \partial q_i} = \frac{\partial^2 A}{\partial p_k \partial p_i} = 0\,, \qquad \frac{\partial^2 B}{\partial q_k \partial q_i} = \frac{\partial^2 B}{\partial p_k \partial p_i} = 0\,, \ \ldots \qquad (2.22)$$

then flexibility and Jordan-admissibility conditions are verified and in the transition $U(\lambda, \mu) \to A(\lambda, \mu)$ the algebraic structure remains unchanged.

As an example of application we consider the Liouville equation for the particle density $\varrho(q, p)$ corresponding to the conservative regions I and III of Fig. 1

$$\frac{\partial \varrho}{\partial t} + \nabla \cdot (V\varrho) = \frac{\partial \varrho}{\partial t} + V \cdot \nabla \varrho = 0\,, \qquad (2.23)$$

where V is the velocity in phase space such that $\nabla \cdot V = 0$. For the dissipative region II $\nabla \cdot V \neq 0$ and Eq. (2.23), by using (2.9) and (2.12) becomes

$$\frac{\partial \varrho}{\partial t} + (\varrho, H) + \varrho(\lambda + \mu) \frac{\partial^2 H}{\partial q_k \partial p_k} = 0\,. \qquad (2.24)$$

Let the Hamiltonian be of linear velocity force type, i. e.

$$\frac{\partial^2 H}{\partial q_k \partial p_k} = K \qquad (2.25)$$

where K is a constant. Then by (2.24) and by the total time derivative of ϱ, using condition (2.14), we get

$$\varrho = \varrho_0 \exp\left[-2Kt \sin \alpha\right]. \qquad (2.26)$$

Thus the particle density in phase space is not longer invariant, and more exactly we have a decrease or an increase of density in time corresponding respectively to the cases $K \sin\alpha \gtrless 0$.

We consider now a collision term of the Fokker-Planck form $(\delta\varrho/\delta t)_c = -(\varrho - \varrho_0)/t_c$ where t_c is the relaxation time and ϱ_0 the equilibrium distribution. The Boltzman equation can be then written

$$\frac{\partial \varrho}{\partial t} + v \cdot \triangle^q \varrho + F \cdot \triangle^p \varrho = -\frac{\chi\varrho - \varrho_0}{t_c} \qquad (2.27)$$

where $v = \dot{q}$, $F = \dot{p}$ and $\chi = 1 + Kt_c(\lambda + \mu)$. If the above equation refer to an electron gas under the action of an electric field E then, for small deviations from the equilibrium configuration, we can write

$$\varrho = \frac{1}{\chi}\varrho_0 - \frac{t_c}{\chi}(v \cdot \nabla^q \varrho + eE \cdot \nabla^p \varrho) \qquad (2.28)$$

by which to the first order, by assuming ϱ_0 independent of q,

$$\varrho = \frac{1}{\chi}\varrho_0 - \frac{t_c}{\chi} eE \cdot \nabla^p \varrho_0\,. \qquad (2.29)$$

Under the above assumptions the electric current and the conductivity tensor defined respectively by [11]

$$J_k = -\frac{e^2 t_c}{m\chi} E_i \int v_k \frac{\partial \varrho_0}{\partial v_i}\, dv \qquad (2.30)$$

$$T_{ij} = -\frac{e^2 t_c}{m\chi} \int v_i \frac{\partial \varrho_0}{\partial v_j}\, dv \qquad (2.31)$$

corresponding to a Maxwell distribution become

$$J = \frac{e^2 t_c n_0}{m\chi} E \qquad (2.32)$$

$$T_{ij} = \frac{e^2 t_c n_0}{m\chi}\,\delta_{ij} \tag{2.33}$$

which shows as an example the possibility of connecting the λ and μ parameters of the formulation to physical quantities.

3. Extension to continuous systems.

We consider now the case when regions I and III of Fig. 1 represent classical conservative complex fields characterized by the functions $\psi_k(x)$, $\psi_k^*(x)$ and the Lagrangian density $L = L(\psi_k, \psi_k^*, \partial_\mu\psi_k, \partial_\mu\psi_k^*)$ where $x \equiv (x^0, x^1, x^2, x^3)$, $x^0 = ict$, $k = 1, ..., n$, $\mu = 0, 1, 2, 3$, and $\partial_\mu = \partial/\partial x_\mu$. Corresponding to region II we suppose that the fields become dissipative under the action of external systems characterized by the functions $\Gamma_k(x)$ and $\Gamma_k^*(x)$. For an analytical dynamics formulation of the dissipative region satisfying requirements A, B and C of Sec. 1 we introduce the following equations as covariant extension of Eqs. (2.5) to continuous systems

$$\lambda\partial_\mu \frac{\partial L}{\partial\partial_\mu\psi_k} + \mu\frac{\partial L}{\partial\psi_k} = \Gamma_k\,, \qquad k = 1, 2, ..., n, \tag{3.1}$$

$$\lambda\partial_\mu \frac{\partial L}{\partial\partial_\mu\psi_k^*} + \mu\frac{\partial L}{\partial\psi_k^*} = \Gamma_k^*\,,$$

which we assume as the *pseudo-Lagrangian equations for continuous systems*. We note that Eqs. (3.1) are obtained by performing the extension of Eqs. (2.1)-(2.4) to continuous systems, that is by assuming the original expression of the Lagrangian density of the conservative region as valid for the dissipative region too since the variations of the kinetic and potential energies are represented by the parameters involved in the formulations.

The transition to equations corresponding to (2.9) can be performed by defining the conjugate momenta, as for (2.7), according to

$$\pi_k = \lambda\frac{\partial L}{\partial\psi_k}\,, \qquad \pi_k^* = \mu\frac{\partial L}{\partial\psi_k^*} \tag{3.2}$$

and by introducing the expression

$$H = \psi_k\frac{\partial L}{\partial\psi_k} + \psi_k^*\frac{\partial L}{\partial\psi_k^*} - L' \tag{3.3}$$

where

$$L' = L + D \tag{3.4}$$

and D is a real scalar function of the spatial derivative $\partial_i\psi_k$ and $\partial_i\psi_k^*$ only, satisfying the properties

$$\frac{\partial D}{\partial\partial_i\psi_k} = -\frac{\lambda+\mu}{\mu}\frac{\partial L}{\partial\partial_i\psi_k}\,, \qquad \frac{\partial D}{\partial\partial_i\psi_k^*} = -\frac{\lambda+\mu}{\mu}\frac{\partial L}{\partial\partial_i\psi_k^*}\,. \tag{3.5}$$

Thus $D = 0$ for $\lambda = -\mu$. Then Eqs. (3.1) are equivalent to [2]

$$\dot\psi_k = \lambda\frac{\delta H}{\delta\pi_k}\,, \qquad \dot\psi_k^* = \lambda\frac{\delta H}{\delta\pi_k^*}$$
$$\dot\pi_k = \mu\frac{\delta H}{\delta\psi_k} + \Gamma_k\,, \qquad \dot\pi_k^* = \mu\frac{\delta H}{\delta\psi_k^*} + \Gamma_k^* \tag{3.6}$$

which we assume as the *pseudo-Hamiltonian equations for continuous systems*. Indeed, by introducing the global quanties

$$\mathscr{L} = \int L d^3x\,, \qquad \mathscr{H} = \int H d^3x\,, \tag{3.7}$$

and by differentiating both sydes of Eqs. (3.3)

$$d\mathscr{H} = \int\left(\frac{\delta H}{\delta\psi_k}d\psi_k + \frac{\delta H}{\delta\pi_k}d\pi_k + \frac{\delta H}{\delta\psi_k^*}d\psi_k^* + \frac{\delta H}{\delta\pi_k^*}d\pi_k^*\right)d^3x \tag{3.8}$$

$$d\mathscr{H} = \int\left(-\frac{\delta L'}{\delta\psi_k}d\psi_k + \frac{1}{\lambda}\psi_k d\pi_k - \frac{\delta L'}{\delta\psi_k^*}d\psi_k^* + \frac{1}{\lambda}\psi_k^* d\pi_k^*\right)d^3x\,, \tag{3.9}$$

Eqs. (3.6) follow by means of (3.1), (3.2) and (3.4).

We note that the total time derivative of $\mathscr{H}$ is not longer equal to zero, being

$$\frac{d\mathscr{H}}{dt} = \int\left[(\lambda+\mu)\left(\frac{\delta H}{\delta\psi_k}\frac{\delta H}{\delta\pi_k} + \frac{\delta H}{\delta\psi_k^*}\frac{\delta H}{\delta\pi_k^*}\right)\right]d^3x. \tag{3.10}$$

Indeed, by definition, pseudo-Hamiltonian mechanics applies to systems which dissipate energy. Hence the quantities which are usually conserved in the Hamiltonian formulation for free fields, such as energy, momentum and angular momentum, are not longer conserved in the above

[2] We use the functional derivative

$$\frac{\delta A}{\delta\psi_k} = \frac{\partial A}{\partial\psi_k} - \partial_i\frac{\partial A}{\partial\partial_i\psi_k}\,.$$

formulation because of the exchange between the fields and the external systems.
Similarly the total derivative of any function $\mathscr{F} = \int F d^3x$ can be written

$$\frac{d\mathscr{F}}{dt} = \int \left[\left(\lambda \frac{\delta F}{\delta \psi_k} \frac{\delta H}{\delta \pi_k} + \mu \frac{\delta H}{\delta \psi_k} \frac{\delta F}{\delta \pi_k} \right) + \left(\lambda \frac{\delta F}{\delta \psi_k^*} \frac{\delta H}{\delta \pi_k^*} + \right. \right.$$

$$\left. \left. + \mu \frac{\delta H}{\delta \psi_k^*} \frac{\delta F}{\delta \pi_k^*} \right) + \frac{\partial F}{\partial t} \right] d^3x \,. \qquad (3.11)$$

We can thus introduce the following expressions for the generalization of the bracket (2.12) to continuous systems

$$(A, B)_c = \lambda \frac{\delta A}{\delta \psi_k} \frac{\delta B}{\delta \pi_k} + \mu \frac{\delta B}{\delta \psi_k} \frac{\delta A}{\delta \pi_k} \qquad (3.12)$$

$$(A, B)_c^* = \lambda \frac{\delta A}{\delta \psi_k^*} \frac{\delta B}{\delta \pi_k^*} + \mu \frac{\delta B}{\delta \psi_k^*} \frac{\delta A}{\delta \pi_k^*} \qquad (3.13)$$

by which Eqs. (3.6) can be written

$$\dot{\psi}_k = (\psi_k, H)_c \,, \qquad \dot{\psi}_k^* = (\psi_k^*, H)_c^* \,,$$

$$\dot{\pi}_k = (\pi_k, H)_c + \Gamma_k \,, \qquad \dot{\pi}_k^* = (\pi_k^*, H)_c^* + \Gamma_k^* \,. \qquad (3.14)$$

By introducing the supplementary condition (2.14) we can write

$$(A, B)_c = \cos\alpha [A, B]_c + \sin\alpha \{A, B\}_c \,, \qquad (3.15)$$

$$\partial_\mu \frac{\partial L}{\partial \partial_\mu \psi_k} - \frac{\partial L}{\partial \psi_k} = \frac{\Gamma_k}{\sin\alpha + \cos\alpha} - 2 \frac{\tan\alpha}{1 + \tan\alpha} \frac{\partial L}{\partial \psi_k} \qquad (3.16)$$

$$\psi_k = \cos\alpha [\psi_k, H]_c + \sin\alpha \{\psi_k, H\}_c \,,$$

$$\pi_k = \cos\alpha [\pi_k, H]_c + \sin\alpha \{\pi_k, H\}_c + \Gamma_k \,, \qquad (3.17)$$

and similarly for the relations corresponding to ψ_k^* and π_k^*, while

$$\frac{d\mathscr{H}}{dt} = 2 \sin\alpha \int \left(\frac{\delta H}{\delta \psi_k} \frac{\delta H}{\delta \pi_k} + \frac{\delta H}{\delta \psi_k^*} \frac{\delta H}{\delta \pi_k^*} \right) d^3x \qquad (3.18)$$

by which we see that the formulation reduces to the usual one when the value of the "dissipativity angle" α is equal to zero.

We now briefly discuss some example of classical fields represented in terms of the given dissipative formulation. We consider a free field characterized by the Lagrangian L_0 corresponding to regions I and III of Fig. 1. We suppose that the field becomes dissipative under the action of external systems characterized by the interaction Lagrangian L_i. Then the dissipative region II can be characterized by means of Eqs. (3.1) or (3.6) in the following two equivalent ways:

i) By assumimg $\Gamma_k = 0$ and $\Gamma_k^* = 0$. Then the expression for L can be given by the usual form

$$L = L_0 + f L_i \,, \qquad (3.19)$$

where f is the coupling constant and L_0 is not comprehensive of the external term, while H can be constructed by means of expressions (3.3) and (3.5).

ii) By assuming Γ_k and $\Gamma_k^* \neq 0$. Then

$$L = L_0 \,, \qquad (3.20)$$

H can be constructed in terms of (3.3) and (3.5), while Γ_k and Γ_k^* are generally given in terms of L_i by

$$\Gamma_k = -f \frac{\partial L_i}{\partial \psi_k} \,, \qquad \Gamma_k^* = -f \frac{\partial L_i}{\partial \psi_k^*} \,. \qquad (3.21)$$

We note that the usual methods consider always (3.19) in terms of Lagrange equations, but without a deep characterization of the dissipativity induced by the interaction. If we construct field equations by means of the given dissipative formulation we find that the equations formally remain unchanged with respect to the usual ones, since the parameters of the procedure affect the numerical constants of the system, i. e. masses and coupling constants. Thus at a classical level, or formally at a first quantization, the procedure presents a possibility of direct renormalization only. However at a level of second quantization there are deeper consequences concerning, for instance, the algebraic invariance or the statistical characterization of the interacting dissipative region.

Let us consider for example a classical complex scalar (or pseudo-scalar) field characterized by the following Lagrangian for the conservative regions I and III of Fig. 1

$$L_0 = -\partial_\mu \psi_0^* \partial_\mu \psi_0 - m_0^2 \psi_0^* \psi_0 \qquad (3.22)$$

and the field equations

$$(\Box - m_0^2) \psi_0(x) = 0, \qquad (\Box - m_0^2) \psi_0^*(x) = 0 \,. \qquad (3.23)$$

We suppose that the field becomes dissipative under the action of an external source characterized by $\Gamma_k = f \varrho^*(x)$ and $\Gamma_k^* = f \varrho(x)$. Then by introducing the expression formally equal to (3.22) as in (ii)

$$L = -\partial_\mu \psi^* \partial_\mu \psi - m^2 \psi^* \psi \qquad (3.24)$$

Eqs. (3.1) gives rise to the new inhomogeneous equations

$$(\Box - m_i^2)\psi(x) = f'\varrho(x)\,, \qquad (\Box - m_i^2)\psi^*(x) = f'\varrho^*(x) \tag{3.25}$$

where

$$m_i^2 = -\frac{\mu}{\lambda}m^2, \qquad f' = \frac{1}{\lambda}f \tag{3.26}$$

or in terms of the "dissipativity angle" α

$$m_i^2 = \frac{1-\tan\alpha}{1+\tan\alpha}m^2, \qquad f' = \frac{1}{\sin\alpha+\cos\alpha}f \tag{3.27}$$

where clearly $-45^\circ < \alpha < +45^\circ$.

In the same way, if we consider an interaction of the state ψ corresponding to (3.24) with an external field ϕ characterized by $\Gamma_k = f\phi^*(x)\psi^*(x)$ and $\Gamma_k^* = f\phi(x)\psi(x)$, then Eqs. (3.1) give rise to the following expressions

$$(\Box - m_i^2)\psi(x) = f'\phi(x)\psi(x)\,, \qquad (\Box - m_i^2)\psi^*(x) = f'\phi^*(x)\psi^*(x)\,. \tag{3.28}$$

If ϕ is, for instance, a spinor, then by introducing the expression

$$L = -\frac{1}{2}\bar{\phi}\gamma_\mu\partial_\mu\phi - \frac{1}{2}\partial_\mu\bar{\phi}\gamma_\mu\phi - m'\bar{\phi}\phi \tag{3.29}$$

we get the inhomogeneous equations by assuming ψ as external

$$(\gamma_\mu\partial_\mu + m_i')\phi(x) = f'\psi(x)\phi(x)\,, \qquad (\gamma_\mu\partial_\mu + m_i')\bar{\phi}(x) = f'\psi^*(x)\bar{\phi}(x) \tag{3.30}$$

where

$$m_i' = -\frac{\mu}{\lambda}m', \quad \text{or} \quad m_i' = \frac{1-\tan\alpha}{1+\tan\alpha}m' \tag{3.31}$$

And similarly, if we consider a dissipative coupling of ψ and ϕ induced by an interaction of Yukawa type $L_i = -\bar{\phi}\phi\psi$, then we get for ψ and ψ^*

$$(\Box - m_i^2)\psi(x) = f'\bar{\phi}(x)\phi(x)\,, \qquad (\Box - m_i^2)\psi^*(x) = f'\bar{\phi}(x)\phi(x)\,, \tag{3.32}$$

while for $\bar{\phi}$ and ϕ the equations are always given by (3.30).

Eqs. (3.25), (3.28), (3.30) and (3.32) formally coincide with the usual ones but they no longer admit Lie algebras as invariance algebras. Indeed they are constructed with an analytical dynamics procedure which induces Lie-admissible algebras by means of the generalizations (3.12) and (3.13) of the Poisson bracket for dissipative conditions. The new invariance algebras are able to satisfy the asymptotic conditions in the meaning that they reduce directly to Lie algebras when the fields become conservative.

As a concluding remark, let us note that a second quantization of the given formulation, instead of a statistics (or parastatistics) characterization of the interpolating fields would give rise to the recently proposed unified statistics [8], which are parametrized structures, intermediate between those of Fermi and Bose, expressed in terms of Lie-admissible mutation algebras and able to regenerate Fermi or Bose statistics (i. e. a physical asymptotic state) when the Lie-admissible algebras reduce to Lie algebras. Indeed the D function defined by (3.5) generally breaks the symmetric (Bose) or antisymmetric (Fermi) character of the states, while quantization of equations (3.14) directly expresses the Lie-admissible structure of the formulation, which are the essential requirements for unified statistics. Conceivably, the construction of a physical model of interaction or decay in terms of the proposed formulations would be interesting, for instance, for investigating a dynamical interpretation of problems like the P-violation in weak interactions.

Appendix A.

Let us briefly list some properties of the bracket (2.12). (A, B) is neither totally antisymmetric nor totally symmetric. Thus the "square value" of a function $A = A(q, p)$, i. e.

$$(A, A) = (\lambda + \mu)\frac{\partial A}{\partial q_k}\frac{\partial A}{\partial p_k}\,, \tag{A.1}$$

vanishes only for $\lambda = -\mu$, that is for the conservative region.

(A, B) is a linear function on the arguments A and B since

$$(A+B, C) = (A, C) + (B, C), \qquad (A, B+C) = (A, B) + (A, C)\,. \tag{A.2}$$

The "differentiation property" follows as for the Poisson bracket

$$(AB, C) = (A, C)B + A(B, C)\,, \qquad (A, \delta) = 0\,, \tag{A.3}$$

$$(A, BC) = (A, B)C + B(A, C),$$