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ORIGIN, PROBLEMATIC ASPECTS AND INVARIANT FORMULATION OF CLASSICAL AND OPERATOR DEFORMATIONS

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In this paper we study three aspects of generalized classical and operator theories, herein generically called *deformations*, which do not appear to have propagated in the rather vast literature in the field: (1) the first known studies on classical and operator deformations; (2) their rather serious physical and mathematical shortcomings due to lack of invariance when conventionally formulated; and (3) the ongoing efforts for the achievement of invariant formulations preserving the axiomatic consistency of the original theories. We begin by recalling the mathematical beauty, axiomatic consistency and experimental verifications of the special relativity at both classical and quantum levels, and its main axiomatic properties: universal invariance of the fundamental units of space and time; preservation of hermiticity-observability at all times; uniqueness and invariance of numerical predictions; and other known properties. We then review the first known, generally ignored, classical and operator deformations. We then study the generally ignored problematic aspects of classical and operator deformations in their current formulation which include: lack of invariance of the fundamental units of space and times with consequential inapplicability to real measurements; loss of observability in time; lack of uniqueness and invariance of numerical predictions; violation of causality and probability laws; and, above all, violation of Einstein's special relativity. We finally outline the generally ignored ongoing efforts for the resolutions of the above shortcomings, and show that they require the necessary use of *new mathematics* specifically constructed for the task. We finally present a systematic study for the identical reformulation of existing classical and operator deformations in an invariant form.

Keywords: Classical and quantum deformations, Lie-isotopic and Lie-admissible theories, isomathematics and genomathematics.

1. Statement of the Problem

The first part of this century will undoubtedly be considered in the history of physics as signaling the triumph of the special relativity[1] in both its classical and quantum versions because of its mathematical beauty, axiomatic consistency and experimental verifications.

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Quite likely, the second part of this century will also pass to the history of physics as being characterized by numerous attempts at broadening the mathematical, physical and experimental, as well as nonrelativistic and relativistic foundations of the special relativity. The latter theories are generically known under the name of (classical and operator) *deformations*, and we shall preserve such a terminology in this study.

This paper is devoted to the differences in axiomatic structure between the special relativity and the various generalized theories. In particular, we shall show that, as currently formulated on conventional spaces over fields, deformations have an undeniable mathematical beauty but, in their current formulation, are afflicted by a number of rather serious problematic aspects of *physical* consistency, because they lose the main properties of the special relativity, such as: invariance of the fundamental units of space and time; preservation in time of hermiticity-observability; invariant probabilities; causality; and other known features.

The literature of this century contains numerous generalizations-deformations, first, of the classical setting underlying the special relativity, Hamiltonian mechanics, and then of its operator foundations, (nonrelativistic or relativistic) quantum mechanics, which require an individual inspection.

The first objective of this paper is to recall the origin of what are today called classical deformations which are generally ignored in the vast literature in the field, and which can be traced back to the founders of analytic mechanics, such as Lagrange,[2] Hamilton,[3] Jacobi,[4] and others. In fact, Lagrange and Hamilton presented their celebrated equations *with external terms* which were then removed in this century, resulting in what are often called the "truncated Lagrange and Hamilton equations." Similarly, Jacobi [*loc. cit.*] did not prove his celebrated theorem for the analytic equations of the contemporary literature, but rather for the original ones with external terms.

Despite the successes of special relativity (which mostly motivated the removal of the external terms in the analytic equations), the legacy of Lagrange and Hamilton has persisted and actually increased with the passing of time. In essence, the special relativity is exactly valid under the conditions of its original conception, which were historically referred to as those of the *exterior dynamical problem*, here denoting particles which can be well approximated as being pointlike when moving in the homogeneous and isotropic vacuum under action-a-distance/potential interactions. Typical examples of exterior dynamical problems are a spaceship in a stationary orbit around earth or an electron in an atomic cloud.

By contrast, in Lagrange's and Hamilton's view there exist conditions under which one sole quantity, today called Lagrangian or Hamiltonian, cannot represent the entire physical reality and, for this reason, they added the external terms to their celebrated equations. The latter conditions were historically referred as those of the *interior dynamical problem*, here denoting particles which cannot be approximated as being pointlike when moving within a generally inhomogeneous and anisotropic physical medium. Typical examples of interior dynamical problems are a spaceship during re-entry in our atmosphere, or a neutron in the core of a neutron star, which experience both action-a-distance, potential-Hamiltonian as well as contact nonpotential–non-Hamiltonian interactions due to the motion of an extended object (whether a spaceship or a neutron) within a physical medium.

Lagrange's and Hamilton's historical view is therefore amply sufficient to provide physical motivations for the study of generalized theories.

However, the addition of the external terms in the dynamical equations has profound mathematical and physical implications because it implies the loss of the entire Lie theory in favor of a covering formulation known as *Lie-admissible theory*, whose algebraic axioms were identified by Albert[5] in 1948 and whose applicability to Hamilton's equation with external term was identified by Santilli[6(a)] in 1967 (see Refs. 7(a) and 7(b) for a general presentation up to 1983,[7(e),7(f)] for a recent study, and Ref. 8 for independent accounts).

An important feature of Lie-admissible formulations is their *direct universality*,[7(b)] i.e. their capability to represent all possible Newtonian systems (universality) in the frame of the experimenter (direct universality). In fact, the Hamiltonian represents all potential interactions, while the external terms represent all non-Hamiltonian forces and effects. By comparison, conventional Hamiltonian mechanics can represent only a rather limited class of Newtonian systems in the frame of the observer.[7(c)]

We shall exclude throughout our analysis the use of Darboux's transforms and only use *direct representations*, i.e. representations in the fixed coordinates of the observers. The main reason is that, since they map non-Hamiltonian into Hamiltonian systems, Darboux's transformations are highly *nonlinear* in all variables. As such, *Darboux's transforms imply the loss of the inertial character of the reference frames with consequential loss of Galilei's and Einstein's relativities.* Only *after* non-Hamiltonian systems have been represented in the *fixed* inertial frame of the observer, may the transformation theory acquire a physical value, precisely as it is the case for conventional conservative systems.

It is evident that the transition from Lie's theory to its Lie-admissible covering implies structural departures from the physical foundations of the special relativity. Even though mathematically appealing, this creates the rather serious problems of identifying a new covering relativity, proving its axiomatic consistency and then establishing it experimentally.

Another objective of this paper is to recall the complementary line of generalized classical formulations initiated by (G. D.) Birkhoff[9] in 1927. His main point was also the expected insufficiency of one single quantity, the Hamiltonian, to represent the entire physical reality. Rather than adding the external terms, Birkhoff's considered the most general possible, first-order, Pfaffian, action functional.

This implied a broadening of the brackets of the Hamiltonian time evolution into the so-called *generalized Lie brackets*, as studied by numerous authors (see, e.g. Ref. 10). Birkhoff's approach was studied in details by Santilli[6(d),7(c),7(d)] who showed that the brackets characterized by Birkhoff's equations imply a step-by-step, axiom-preserving generalization of Lie's theory in all its branches (enveloping algebras, Lie algebras, Lie groups, representation theory, etc.).

The latter theory was submitted under the name of *Lie-isotopic theory*[6(d),7(d)] where the term "isotopic" is used in its Greek meaning of being "axiom-preserving." The emerging new mechanics was then submitted under the name of *Birkhoffian mechanics*.[7(d)] The more general mechanics of Lie-admissible type was submitted under the name of *Birkhoff-admissible mechanics*[6(d),7(b)] to denote the fact that the Lie-isotopic theory is a particular case of the Lie-admissible version.

In particular, Lie-isotopic formulations also resulted to be directly universal although for the more restricted class of well behaved, local-differential and analytic systems in a star-shaped regular point of their variables.[7(d)]

It is evident that, despite the preservation of the Lie character, a departure from the *canonical* realization of space–time symmetries implies an inevitable departure from the special relativity, thus creating again the problems of identifying a covering relativity, proving its axiomatic consistency and then establishing it experimentally.

Note since these introductory lines that, in view of the totally antisymmetric character of the product, Lie-isotopic theories admit conventional total conservation laws under generalized internal forces represented precisely by the generalized brackets (see systematic studies[7]). On the contrary, since their product is neither totally antisymmetric nor totally symmetric, the covering Lie-admissible theories are particularly set to represent open nonconservative systems under unrestricted external forces.[7]

In summary, as a result of their direct universality, all possible, well behaved, unconstrained, classical deformations can be classified into:

(I) Deformations preserving the Lie character of Hamiltonian mechanics, in which case they can studied via one of the various realizations of the Lie-isotopic theory; or

(II) Deformations abandoning the Lie character of Hamiltonian mechanics in favor of a covering of the Lie and Lie-isotopic theories, in which case they can be studied via the Lie-admissible theory.

(III) Deformations of the still broader multivalued type currently under study by a restricted class of experts, which will not be studied in this paper for brevity (see later on Refs. 28 for details).

The above classification is important because it permits the study of axiomatic profiles in a unified way, rather than for individually for a seemingly disparate variety of deformations.

Additional types of generalized theories, such as the antiautomorphic images of Classes (I), (II), (III) currently under study for a classical treatment of antimatter, will not be considered at this time for brevity (see Refs. 7(e) and 7(f)). The extension of the results of this papers to Classes (I), (II), (III) and their antiautomorphic images to the case with subsidiary constraints is left to the interested reader.

Another objective of this paper is the identification of the origin of what are today called operator deformations which can be traced back to the inception of quantum mechanics itself, such as the theories relaxing the linearity of quantum mechanics (see the historical accounts in Refs. 12), or relaxing the potential character,[13] or relaxing the local structure,[7(d)] or relaxing the algebraic structure via external collision terms.[14]

The first known deformations of the Lie product $[A, B] = A \times B - B \times A$ of quantum mechanics was identified by Santilli[6(a)] back in 1967 via the expression $(A, B) = p \times A \times B - q \times B \times A$ where p and q are non-null parameters. The q-deformations with product $(A, B) = A \times B - q \times B \times A$ introduced by Biedenharn[15(a)] and Macfarlane[15(b)] in 1989, are an evident particular case Santilli's (p, q)-deformations. The latter q-deformations were then studies by a large number of authors (see the representative list[16]) although without the quotation of the origination of the deformations in Ref. 6(a).

The above studies were then follows by a large variety of operator deformations. Without any claim of completeness due to their shear number, we here mention: the deformations under the somewhat misleading name of "quantum groups;"[17] the k-deformations (which are a particular relativistic version of quantum groups);[18] the so-called "star theories"[19] whose product is, as we shall see, the basic isoassociative product of the Lie-isotopic theories; theories with nonassociative envelopes;[20] the so-called "squeezed states theories;"[21] a nonunitary statistical mechanics by Prigogine and his associates;[22] the Ellis–Mavromatos–Nanopoulos model of black hole dynamics with Santilli's Lie-admissible structure;[23] noncanonical time theories;[24] supersymmetric theories;[25] Kac–Moody superalgebras;[26] and others.

As we shall see, all the above generalized operator deformations can also be classified depending on their algebraic character. In fact, all (p, q)- and q-deformations evidently abandon the Lie character of quantum mechanics in favor of a generalized algebra which, since it is not totally antisymmetric or symmetric, it also results to be of Lie-admissible type as originally proposed by Santilli.[6(a)] However, quantum groups,[17] generalized statistical formulations[22] and other theories preserve the Lie character of the underlying algebra, although expressed in a generalized form, in which case they can be considered as a particular class of Lie-isotopic theories.

We should note that other theories, such as the nonlinear models[12] *appear* to have a conventional Lie algebra structure in their brackets, while at a deeper inspection such a structure results to be generalized, as evidently expected from the strictly *linear* character of Lie's theory when compared to the *nonlinear* character of the models here considered.

In regards to the generalized operator formulations it is therefore sufficient for us to consider only the classes of Lie-isotopic and Lie-admissible generalizations, because the latter have also resulted to be directly universal in operator settings.[7(f)] The understanding is that, again, the former are a particular case of the latter.

To avoid trivial cases, we shall solely consider classical (operator) deformations *outside the class of equivalence of Hamiltonian (quantum) mechanics.* Also, we shall solely consider deformations as currently treated, that is, on conventional spaces over fields.

To avoid possible misrepresentations, we shall use the generic term "operator" rather than "quantum" deformations, because we are dealing with theories outside the class of equivalence of quantum mechanics.

The above definition of deformations evidently includes current gravitational theories. No study of the problematic aspects of available deformations can therefore be considered as sufficiently exhaustive without a consideration of classical and operator theories of gravity.

In summary, the first objective of this paper is to review the origin of classical and operator deformations and their unified treatments via Santilli's Lie-isotopic and Lie-admissible formulations[6,7] which are significant in the study of deformations because they unify seemingly disparate approaches, yet are generally ignored in the vast literature in the field.

The second objective of this paper is to point out that, even though with an unquestionable mathematical beauty, all possible classical and quantum deformations as currently treated are afflicted by rather serious problems of physical consistency. These problematic aspects have been studied in Refs. 27 and 28(b) mainly for the operator version. To our best knowledge, this is the first systematic study of the problematic aspects of deformations beginning at the classical level and then passing to their operator counterpart.

It is at this point where the mathematical beauty, axiomatic consistency and experimental validity of the special relativity emerge in their full light. A fundamental quantity of the special relativity is the four-dimensional unit

$$I = \mathrm{Diag}(\{1,1,1\},1)\,, \tag{1.1}$$

which represents in a dimensionless form the basic units of space $\{1,1,1\}$ (e.g. 1 cm, 1 cm, 1 cm for the three Euclidean axes), as well as the basic unit of time (e.g. 1 sec).

A pillar of the axiomatic consistency of the special relativity at both classical and quantum levels is *the universal invariance of the basic space and time units* (1.1), where the term "universal" stands to indicate invariance under all possible space–time symmetries as well as dynamical equations.

In fact, quantity (1.1) is the fundamental unit of the Minkowski space and of its basic Poincaré symmetry. As such, unit (1.1) is the unit of the universal enveloping associative algebra of the acting space–time symmetry, which is the *definition of unit* tacitly implied hereon.

The invariance of the basic unit is not a mere mathematical curiosity because it carries fundamental physical implications. In fact, it first implies lack of ambiguities in the physical applications and experimental verifications of the theory, evidently because the basic units used in measurements say, (1 cm, 1 cm, 1 cm, 1 sec) are universal invariants. The same invariance has then implications at all axiomatic and physical levels.

A primary objective of this paper is to study the problematic aspect of deformations implied by the following theorem whose classical proof will be presented in Sec. 3 and its operator counterpart in Sec. 4.

Theorem 1.1. *All possible classical and operator deformations, here defined as being outside the class of equivalence of conventional theories yet defined on conventional spaces over conventional fields, do not possess invariant units of space, time, energy, etc.*

As a result, the numerical applications and experimental verifications of deformed theories, whether classical or operator, are in question because of the lack of invariance of the basic units used for the measurements themselves.

A subsequent objective of this paper is to show that the lack of invariance of the basic units implies additional rather serious problematic aspects, again, of physical character.

As an example, one may attempt to bypass the problematic aspects of Theorem 1.1 by assuming that the rest of the universe is deformed jointly with that of the basic units, thus implying valid measurements.

Such a position is evidently questionable for the measurements of far away objects which, as such, are independent from local dynamics.

Independently from that, the above position is insufficient to resolve the shortcomings, because the lack of invariance of the unit has additional, rather serious implications. For example, it implies: the loss of the base field with evidently disastrous axiomatic consequences; the lack of preservation in time of the hermiticity with consequential lack of physically acceptable observables; the lack of uniqueness as well as invariance of the numerical predictions; the loss of invariant probabilities; the violation of causality; and, above all, the violation of the axioms of the special relativity.

These problematic aspects are sufficiently serious, first, to warrant their collegial awareness, and then to require systematic studies for their resolutions.

Note that we have used the terms "problematic aspects," rather than "inconsistencies," because we do not claim at all that theories[12–26] are physically inconsistent. We only insist that their problematic aspect should be addressed in the only possible scientific way, via publications.

As a matter of fact, our third objective is to indicate the ongoing efforts on the invariant formulation of Lie-admissible and Lie-isotopic theories which also appear to be largely ignored in the vast literature on deformations.

In fact, in the memoirs[28] we present mathematical and physical studies for an apparent first solution of the above problematic aspects which consists of generalized classical and operator theories constructed under the fundamental requirement of preserving the axiomatic structure of the special relativity, thus including universally invariant basic units, the preservation of hermiticity-observability at all times, uniqueness and invariance of the numerical predictions, etc.

Other resolutions of the problematic aspects are also possible, and their study is here encouraged, but under the physically uncompromisable condition of possessing invariant basic units.

A summary of the content of this paper is presented in note.[27(g)]

2. The Notions of Lie-Admissibility and Lie-Isotopy

2.1. *The first notion of Lie-admissibility*

In 1948 Albert[5] introduced the *first notion of Jordan admissible and Lie-admissible algebras* as generally nonassociative algebras U with elements a, b, c, and abstract product ab which are such that the attached algebras U^+ and U^-, which are the same vector spaces as U equipped with the products $\{a, b\}_U = ab + ba$ and $[a, b]_U = ab - ba$, are Jordan and Lie algebras, respectively. Albert then studied the algebra with product

$$(A, B) = p \times A \times B + (1 - p) \times B \times A\,, \tag{2.1}$$

where p is a parameter, A, B are matrices or operators (hereon assumed to be Hermitian), and $A \times B$ is the associative product. It is easy to see that the above product is indeed jointly Jordan- and Lie-admissible because $\{A, B\}_U = A \times B + B \times A$ and $[A, B]_U = (2p - 1) \times (A \times B - B \times A)$.

Note that for $p = 0$ product (2.1) becomes that of a *commutative Jordan algebra,* but there exist no (finite) value of p under which product (2.1) recovers the Lie product. As a result, product (2.1) cannot be used for possible coverings of current physical theories. In fact, Albert [*loc. cit.*] was primarily interested in the *Jordan,* rather than in the Lie content of nonassociative algebras (see Ref. 5 for more details).

2.2. *The second and third notions of Lie-admissibility*

In view of the above occurrence, in 1967 Santilli[6(a)] introduced the *second notion of Lie-admissibility* which is Albert's first notion [*loc. cit.*], plus the condition that the algebras U admit Lie algebras in their classification or, equivalently, that the generalized Lie product admits the conventional one as a particular case.

Santilli[6(a)–6(c)] therefore introduced the algebra with product

$$(A, B) = p \times A \times B - q \times B \times A\,, \tag{2.2}$$

with related time evolution in the infinitesimal and finite forms ($\hbar = 1$)

$$i \times \frac{dA}{dt} = p \times A \times H - q \times H \times A\,, \tag{2.3a}$$

$$A(t) = e^{i \times q \times t \times H} \times A(0) \times e^{-i \times p \times t \times H}\,, \tag{2.3b}$$

where: p and q are non-null parameters with non-null values $p \pm q$; A, B are Hermitian operator (or matrices), and $A \times B$ is also the associative product. It is easy to see that product (2.2) is Lie- and Jordan-admissible and admits the Lie and Jordan products as particular (nondegenerate) cases.

The second notion of Lie-admissibility[6(a)] also resulted to be insufficient for physical applications because, as we shall see shortly, the parameters p and q become *operators* under the time evolution of the theory. Santilli[6(b),6(e)] therefore introduced the *third notion of Lie-admissibility* (also called *general Lie-admissibility*) which is the second notion plus the condition that the algebras U admit *Lie-isotopic* (rather then Lie) algebras in their classification (see below).

The latter notion was realized via the *general Lie-admissible product* (first introduced in Ref. 6(e), p. 719)

$$(A, B) = A \times P \times B - B \times Q \times A, \tag{2.4}$$

and time evolution in infinitesimal and finite forms (Ref. 6(e), pp. 741, 742)

$$i \times \frac{dA}{dt} = A \times P \times H - H \times Q \times A, \tag{2.5a}$$

$$A(t) = e^{i \times H \times Q \times t} \times A(0) \times e^{-i \times t \times P \times H}, \tag{2.5b}$$

where P, Q and $P \pm Q$ are nonsingular, generally non-Hermitian operators with nonsingular values $P \pm Q$ admitting of the parametric values p and q as particular cases. The conventional Heisenberg's equations are evidently recovered for $P = Q = 1$.

Note that the P and Q operators must be sandwiched in between the elements A and B to characterize an algebra as commonly understood in mathematics. In fact, the script $P \times A \times B - Q \times B \times A$ with P, Q fixed, is acceptable when P and Q are parameters, but it would *not* characterize an algebra for P and Q operators because of the violation of the right distributive and scalar laws (see Ref. 7(d) for details).

2.3. *The notion of Lie-isotopy*

A fundamental property of the general Lie-admissible algebras U identified in Refs. 6(d) and 6(e) is that their attached antisymmetric algebras U^- *are not* characterized by the traditional Lie product $[A, B] = A \times B - B \times A$, but rather by the product (first introduced in Ref. 6(e), p. 725)

$$\begin{gathered} [A \hat{;} B]_U = A \hat{\times} B = B \hat{\times} A = A \times T \times B - B \times T \times A\,, \\ T = P + Q = T^\dagger\,, \end{gathered} \tag{2.6}$$

called *Lie-isotopic*, because verifying the Lie axioms although in a more general way. The product $A \hat{\times} B = A \times T \times B$ is called *isoassociative* because more general then the conventional associative product $A \times B$, yet preserving associativity, $A \hat{\times} (B \hat{\times} C) \equiv (A \hat{\times} B) \hat{\times} C$.[6(e)]

According to the above results, the *nonassociative* algebra U with product (A, B), Eq. (2.4), can be replaced with an algebra $\hat{\xi}$ with *isoassociative* product

$A \hat{\times} B = A \times T \times B$, in the characterization of the attached antisymmetric algebra[6(e),7(d)]

$$[A \hat{;} B]_U = (A, B) - (B, A) \equiv [A \hat{;} B]_{\hat{\xi}} = A \hat{\times} B - B \hat{\times} A. \tag{2.7}$$

The latter property permitted the construction of the *Lie-isotopic theory*,[6(d),6(e),7(d)] i.e. a step-by-step axiom-preserving lifting of the conventional formulation of Lie theory in terms of the isoassociative product $A \hat{\times} B$, including the lifting of numbers, spaces, enveloping algebras, Lie algebras, Lie groups, Lie symmetries, transformation and representation theory, etc. The emerging new theory is today called *Lie–Santilli isotheory*.[8,30,31]

As a particular case of the broader Lie-admissible formulations, Santilli[6(e),7(d),7(f)] therefore studied the Lie-isotopic time evolution in infinitesimal and finite forms for $T = T^\dagger$ (first introduced in Ref. 6(e), p. 752)

$$i \times \frac{dA}{dt} = [A \hat{;} H]_{\xi} = A \hat{\times} H - H \hat{\times} A = A \times T \times H - H \times T \times A\,, \tag{2.8a}$$

$$A(t) = e^{i \times H \times T \times t} \times A(0) \times e^{-i \times t \times T \times H}\,, \tag{2.8b}$$

which admit conventional quantum equations for $T = 1$.

Note for future needs that the Lie-admissible product (2.4) the (Lie-isotopic product (2.7)) are the most general possible nonantisymmetric (antisymmetric) product, respectively. This property is at the foundation of the unified treatment of deformations as we shall see. For additional details we refer the interested reader to monographs.[7(e),7(f)]

3. Origin and Problematic Aspects of Classical Deformations

3.1. *Birkhoffian mechanics*

No operator theory has sufficient depth without well-defined classical foundations. For this reasons, Santilli studied the classical counterparts of the preceding theories, as reported in monographs.[7] In essence, the classical action underlying Lie-isotopic theories resulted to be the most general possible, first-order, Pfaffian action in phase space[7(d)]

$$\begin{gathered} A = \int_{t_1}^{t_2} dt \left[R_\mu(b) \frac{db^\mu}{dt} + H(t, b) \right], \\ b = \{b^\mu\} = \{r^k, p_k\}\,, \qquad R = \{R_\mu\} = \{A_k(r,p), B^k(r,p)\}\,, \\ \mu = 1, 2, \ldots, 6\,, \qquad k = 1, 2, 3\,, \end{gathered} \tag{3.1}$$

whose variations yield *Birkhoff's equations*[9] in the covariant and contravariant forms (see Ref. 7(d) for all historical notes and references)

$$\Omega_{\mu\nu}(b) \frac{db^\nu}{dt} = \frac{\partial H(t, b)}{\partial b^\mu}\,, \tag{3.2a}$$

$$\frac{db^\mu}{dt} = \Omega^{\mu\nu}(b) \frac{\partial H(t, b)}{\partial b^\nu}\,, \tag{3.2b}$$

with (nowhere degenerate) covariant and contravariant tensors

$$\Omega_{\mu\nu} = \frac{\partial R_\nu}{\partial b^\mu} - \frac{\partial R_\mu}{\partial b^\nu}, \tag{3.3a}$$

$$\Omega^{\mu\nu}(b) = (|\Omega_{\alpha\beta}|^{-1})^{\mu\nu}. \tag{3.3b}$$

The ensuing mechanics, called *Birkhoffian mechanics* in Ref. 7(d), was said to be isotopic because it preserves the main axioms of conventional Hamiltonian mechanics although realized in their most general possible form. We are here referring to: (1) derivability from the most general possible first order action (analytic isotopy); (2) characterization by the most general possible, regular symplectic structure in local coordinates (geometric isotopy),

$$\Omega = \Omega_{\mu\nu}(b) db^\mu \wedge db^\nu ; \tag{3.4}$$

and (3) characterization by the most general possible regular (unconstrained) brackets verifying the Lie axioms (algebraic isotopy)

$$[A, B]^* = \frac{\partial A}{\partial b^\mu} \Omega^{\mu\nu}(b) \frac{\partial B}{\partial b^\nu}. \tag{3.5}$$

Conventional classical Hamiltonian mechanics is admitted as a particular case at all levels for $R = R^0 = (p, 0)$, as one can easily verify.

One may consult Ref. 7(d) for additional aspects, including: the unified treatment via *the conditions of variational selfadjointness*; the isotopies of Lie's theory; the proof of the "direct universality" of the mechanics for local-differential and analytic systems; and other aspects.

Since Eqs. (2.8) and (3.5) have the same generalized (unconstrained and regular) Lie structures, the latter were introduced in Refs. 6(e) and 7(d) as the classical counterpart of the former, an assumption subsequently confirmed by specific studies.[7(f)]

3.2. *Hamilton-admissible and Birkhoff-admissible mechanics*

References 6(b), 6(d), 7(a) and 7(b) were devoted to the study of the classical counterpart of Lie-admissible equations (2.5). Conventional Newtonian forces are divided into variationally *self-adjoint* (SA) and *non-self-adjoint forces* (NSA),[7(c)] $F_k(t, b) = F_k^{\mathrm{SA}} + F_k^{\mathrm{NSA}}$. The SA forces are represented in terms of a conventional potential $U(t, b)$ via the techniques of the inverse problem [*loc. cit.*]. The NSA forces are represented via the algebraic tensor of the theory, according to the equations first introduced in Ref. 6(d)

$$\frac{db^\nu}{dt} - S^{\mu\nu}(t, b)\frac{\partial H(t, b)}{\partial b^\mu} \equiv \frac{m\, dv_k}{dt} - F_k^{\mathrm{SA}}(t, b) - F_k^{\mathrm{NSA}}(t, b), \tag{3.6a}$$

$$(S^{\mu\nu}) = (\omega^{\mu\nu}) + (s^{\mu\nu}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & (F^{\mathrm{NSA}}/(\partial H/\partial p)) \end{pmatrix}, \tag{3.6b}$$

where $\omega^{\mu\nu}$ is the familiar canonical Lie tensor and $S^{\mu\nu}$ is a Lie-admissible tensor because

$$S^{\mu\nu}(t,b) - S^{\nu\mu}(t,b) = 2\omega^{\mu\nu} . \tag{3.7}$$

Consequently, the brackets of the time evolution

$$\frac{dA}{dt} = (A,H) = \frac{\partial A}{\partial b^\mu} S^{\mu\nu}(t,b) \frac{\partial H}{\partial b^\nu} , \tag{3.8}$$

are of Lie-admissible type,

$$(A,B) - (B,A) = 2[A,B] , \tag{3.9}$$

(with a trivial character of this type because the factor 2 is constant) with a compatible lifting of the symplectic two-form (3.4) called *symplectic-admissible.*[7(b)]

The emerging mechanics was called in Ref. 7(b) *Hamilton-admissible mechanics* when the attached antisymmetric tensor is Lie (as in Eqs. (3.6)) or *Birkhoff-admissible mechanics* when the attached antisymmetric tensor is the Birkhoffian one.

Note the simple direct universality of the Hamilton-admissible mechanics (without any need to go to the broader Birkhoff-admissible case) for all possible Newtonian systems, owing to general algebraic solution (3.6b). This simple direct universality should be compared with the rather complex direct universality of Birkhoff's equations (3.4).[7(d)]

It is important to know that Lie-admissible equations (3.7) were constructed along the original Hamilton's equations, those with *external terms* here denoted F_k^{NSA}. In fact, the number of independent functions in the external terms F_k^{NSA} and that in the Lie-admissible tensor $S^{\mu\nu}$ coincide.

Reformulation (3.6) is requested by the fact that the brackets of Hamilton's equations with external terms violate the conditions to form any algebra, let alone Lie algebras, thus preventing the construction of a covering of conventional Hamiltonian mechanics. On the contrary, brackets (3.8), first of all, verify all conditions to characterize an algebra, and, second, that algebra results to be Lie-admissible, i.e. a covering of the algebraic structure of conventional Hamiltonian mechanics.

Note also that *the* (*autonomous*) *Lie-isotopic equations* (3.4) *are structurally reversible*, that is, they are reversible for reversible Hamiltonians. On the contrary, *Lie-admissible equations* (3.6) *are structurally irreversible*, that is, they are irreversible even for reversible Hamiltonians. These main characteristics will persist at the operator level of the next section.

Therefore, the Lie-admissible equations are particularly suited for an *axiomatization of irreversibility*, that is, its representation via the *structure* of the theory, rather than the addition of symmetry breaking terms in a time-symmetric Lagrangian or Hamiltonian.

Since Eqs. (2.5) and (3.6) have the maximal possible (unconstrained and regular) Lie-admissible structures, the latter were assumed in Refs. 6(e) to be the classical image of the former, as confirmed by subsequent studies.[7(f)]

A most dominant characteristics of Birkhoff's equations is that of being derivable via *noncanonical* transformations $b \to b'(b)$ of Hamilton's equations.[7(d)] This can be seen from the fact that the most general possible, nowhere degenerate symplectic tensor $\Omega_{\mu\nu}(b)$ is a noncanonical image of the fundamental symplectic tensor $\omega_{\mu\nu}$ according to the rules (see Ref. 7(d), Subsec. 5.3 for details)

$$\Omega_{\mu\nu}(b) = \frac{\partial b^\alpha}{\partial b'^\mu}\omega_{\alpha\beta}\frac{\partial b^\beta}{\partial b'^\nu} . \tag{3.10}$$

The transition from the fundamental symplectic tensor $\omega_{\alpha\beta}$ to the symplectic-admissible tensor $S_{\mu\nu}$ is more general, and requires *two* noncanonical transformations $b \to b'(b)$ and $b \to b''(b)$, one acting to the right and one from the left, according to the rules

$$S_{\mu\nu}(b) = \frac{\partial b^\alpha}{\partial b'^\mu}\omega_{\alpha\beta}\frac{\partial b^\beta}{\partial b''^\nu} . \tag{3.11}$$

3.3. *Problematic aspects of the Birkhoffian and Birkhoffian-admissible mechanics*

We are now equipped to present the following:

Proof of Theorem 1.1 (classical profile). The fundamental units of space $I = \mathrm{Diag}(1,1,1)$ are embedded in the canonical symplectic structure according to the familiar form

$$(\omega_{\mu\nu}) = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} , \tag{3.12}$$

with an evident extension to four-dimensional space–time units for relativistic formulations.

An important axiomatic property of conventional Hamiltonian mechanics is that of admitting the basic unit $I = \mathrm{Diag}(1,1,1)$ as the fundamental invariant. In fact, the transformation theory, including all possible symmetries, are given by canonical transformations, that is, transformations leaving invariant structure (3.12) by assumption. The emerging mechanics is then axiomatically and physically consistent, as well known.

It is then easy to see that all deformations of Hamiltonian mechanics of Lie-isotopic type, by their very conception, do not leave invariant the basic units because they must deform the fundamental canonical structure as a necessary condition to exit from the class of equivalence of the conventional theory.

Liftings of type (3.10) are reducible to the generalization of the unit $I = \mathrm{Diag}(1,1,1)$ into 3×3 *matrices* whose elements have an arbitrary functional dependence on local quantities (see Ref. 7(d) for details),

$$(\omega_{\mu\nu}) = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \quad \rightarrow (\Omega_{\mu\nu}) = \begin{pmatrix} 0 & -\hat{I}(b') \\ \hat{I}(b') & 0 \end{pmatrix}, \tag{3.13}$$

which evidently implies the loss of the units of the original theory.

All deformations of Hamiltonian mechanics of Lie-admissible type also imply a greater loss of the basic units of the original theory, because the ensuing matrix $(S_{\mu\nu})$ is no longer totally antisymmetric, i.e. of the type

$$(\omega_{\mu\nu}) = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \quad \rightarrow (S_{\mu\nu}) = \begin{pmatrix} \hat{A}(b') & -\hat{I}(b') \\ \hat{I}(b') & 0 \end{pmatrix}, \tag{3.14}$$

thus implying the loss of the very image of the units of the original theory, and this proves Theorem 1.1 for the classical profile. q.e.d.

Thus, all classical deformations of the conventional Hamiltonian mechanics do not appear to be physically viable as conventionally treated, because they do not possess invariant units of measurements and, as such, their physical applications and experimental verifications are afflicted by evident problematic aspects.

The classical implications of Theorem 1.1 are rather serious. The lack of invariance of the basic unit evidently implies the following (see also Refs. 27 and 28(b) for more details):

Corollary 3.1. *Classical deformations do not possess an invariant field of numbers.*

But conventional Euclidean and Minkowskian spaces are centrally dependent on the field on which they are defined. We therefore have the following additional problematic aspect (see also Ref. 23(a) for more details)

Corollary 3.2. *Classical deformations do not have invariant carrier spaces.*

The following additional problematic aspects, considered the most important one by this author, can then be easily proved:

Corollary 3.3. *Classical deformations violates the axioms of the special relativity.*

The above problematic aspects illustrates the reasons why, after the laborious study of the classical Lie-admissible and Lie-isotopic theories reported in monograph,[7] this author had to restart his classical studies from the beginning in order to reach generalizations of classical Hamiltonian mechanics with universally invariant basic units, as reported in memoirs.[28]

We close this section by nothing that *Theorem 1.1 also applied to the Riemannian geometry as well as to all gravitational theories with non-null curvature.* In fact, all such theories in $3+1$ dimensions are *noncanonical* images of Minkowskian theories. As a result, *all gravitational theories with non-null curvature do not possess basic units of space, time, etc., which are invariant under the time evolution and/or under the symmetries of the line element.* For a detailed study of this occurrence we refer the interested reader to the memoir.[29(h)]

4. Origin and Problematic Aspects of Operator Deformations

4.1. *The majestic axiomatic and physical consistency of quantum mechanics*

The axiomatic beauty and physical consistency of quantum mechanics are the result of an articulated body of individually consistent and mutually compatible formulations, including (see, e.g. Ref. 11 and historical accounts quoted therein): the underlying *linear, local and potential structure*; the *universal enveloping associative algebra* with fundamental unit I (say, $I = \mathrm{Diag}(1,1,1)$ representing the three-dimensional Euclidean units of length), elements A, B, H, etc. given by operators that are Hermitian on a Hilbert space $\mathcal{H}$ over the field of complex numbers C, associative product $A \times B$, etc.

$$\begin{aligned} \xi: \quad & I,\ A \times B\,, \quad A \times (B \times C) = (A \times B) \times C\,, \\ & I \times A = A \times I = A\,, \quad \forall A \in \xi\,; \end{aligned} \tag{4.1}$$

the fundamental *Heisenberg finite and infinitesimal time evolution* for *Hermitian–Hamiltonians*

$$\begin{aligned} A(t) &= U \times A(0) \times U^{\dagger} = e^{iHt} \times A(0) \times e^{-itH} \\ &\approx -i(A \times H - H \times A) = -i[A, H]\,; \end{aligned} \tag{4.2}$$

the equally fundamental *Schrödinger equation*,

$$H(t, r, p) \times \phi(t, r) = E \times \phi(t, r)\,, \tag{4.3}$$

the underlying *unitary structure* of the theory,

$$U \times U^{\dagger} = U^{\dagger} \times U = I\,; \tag{4.4}$$

the equivalence of the Heisenberg and Schrödinger representations; the invariance of the basic units of time, length, energy, etc. with consequential consistent application to measurements; the preservation of hermiticity under the time evolution of the theory with consequential physically acceptable observables; the verification of causality and probability laws; the rigorous validity of the axioms of Galileo's and Einstein's special relativities (e.g. via *Mackay imprimitivity theorem*[11]); and other features.

4.2. *Initial operator deformations*

Attempts at generalizing the structure of quantum mechanics via the relaxation of one or the other of its axioms can be traced back to the inception of the theory itself. In this section we show that *all* such broader theories, with no exception known to this author, have rather serious physical shortcomings when they are outside the class of equivalence of quantum mechanics, are regular-unconstrained and are formulated with the conventional mathematics of quantum mechanics.

In the next section we shall then show that the problems herein studied emerge in the treatment of *deviations* from conventional quantum axioms when represented via *conventional* mathematics, such as conventional numbers and fields, conventional vector and Hilbert spaces, etc. On the contrary, when deviations from quantum axioms are treated with *new mathematics* specifically conceived for that purpose, realistic possibilities of regaining the original axiomatic beauty and physical consistency exist.

One of the oldest attempts that comes to mind is the *relaxation of the axiom of linearity of quantum mechanics,* i.e. the formulation of theories nonlinear in the wave functions (see, e.g. the recent Refs. 12 and historical accounts therein). In essence, doubts on the capability of a strictly linear theory to describe the entire universe date back to the very birth of quantum mechanics. They have been investigated throughout this century with nonlinear equations of the type

$$H(t,r,p,\phi) \times \phi(t,r) = E \times \phi(t,r)\,. \tag{4.5}$$

Even though mathematically impeccable and generally unitary, when formulated on conventional spaces over conventional fields, *nonlinear theories of type* (4.5) *violate the superposition principle*, as one can verify. As a result, *nonlinear theories cannot be used for physically consistent representations of composite systems*, such as hadrons, nuclei, atoms, stars, etc. The same theories also have additional deeper shortcomings, e.g. in the topology, with consequential loss of the imprimitivity theorem[11] and evident violation of established relativities (owing to their notorious linearity).

The above problematic aspects of nonlinear theories have been studies by Santilli and others (see Ref. 7(f) and literature quoted therein). Additional comprehensive studies on the limitations of nonlinear theories have been studied by Schuch (see Ref. 27(e) and papers quoted therein).

A second type of attempts which can also be traced to the early part of this century consists in *the relaxation of the axiom of locality of quantum mechanics.* In fact, doubts have always existed as to whether the operator description of a finite set of isolated points can represent the entire universe.[7(f)] As experimentally established, all massive particles have extended wavepackets and their deep overlapping is evidently not reducible to a finite number of isolated points. Since the sole known operator theory was Hamiltonian, it became rather nature to attempt the representation of nonlocal interactions via the addition in the Hamiltonian of "integral potentials," i.e. potentials characterized by surface or volume integrals.

Regrettably, these broader theories have no mathematical or physical value of any known type for various reasons. Mathematically, they imply the violation of the underlying nonrelativistic and relativistic topologies (because they are well known to be strictly local-differentials), with consequential invalidation of the mathematical foundations of the theory.

Physically, nonlocal effects due to wave-overlappings are known to be of "contact" type for which the notion of potential has no known meaning (i.e. it would be like representing the resistive forces experienced by the space-shuttle during re-entry in atmosphere with a potential, whether local-differential or nonlocal-integral). As a result, nonlocal interactions due to wave-overlapping, to have physical sense, should be represented with anything *except* the Hamiltonian.

For comprehensive studies on nonlocal interactions and related literature one may consult Santilli.[7]

A third, rather old, generalized formulation that comes to mind was attempted via *the relaxation of the the axiom of potentiality.* As indicated in Sec. 1, doubts as to whether all interactions in the universe are derivable from a potential can be traced back to Lagrange, Hamilton and Jacobi, and they persisted in the transition to operator theories in the early part of this century. Since, again, the only known operator theory is Hamiltonian, it appeared logical to attempt the representation of nonpotential effects via "imaginary potentials" $iV(t, r)$. This leads to the well-known *dissipative nuclear models* (see, e.g. Ref. 13) with non-Hermitian Hamiltonians, $H = H_0 + iV \neq H^\dagger$, and consequential finite and infinitesimal time evolution,

$$\begin{aligned} A(t) &= U \times A(0) \times U^\dagger = e^{iHt} \times A(0) \times e^{-itH^\dagger} \\ &\approx -i(A \times H^\dagger - H \times A) = -i(A, H, H^\dagger) \,. \end{aligned} \tag{4.6}$$

Unfortunately, the above theories are seriously flawed on physical grounds because *dissipative nuclear models* (4.6) *lose "all" algebras in the brackets of their time evolution, let alone all Lie algebras*, evidently because of the loss of the *bilinear* Lie product $[A, H]$ in favor of the *triple product* $(A, H, H^\dagger)$. As a consequence, *the notion of nucleons with spin 1/2 has no known mathematical or physical meaning for dissipative nuclear model* (4.6) and the same fate occurs for all other notions characterized by a Lie algebra.

This is due to a "hidden," yet fundamental requirement of dynamics according to which *all applicable symmetries must be characterized by the brackets of the time evolution*, because the use of different brackets implies the referral to different systems. This fundamental requirement is indeed verified for quantum mechanics, while it is not verified for models (4.6). As a consequence, the SU(2)-spin symmetry cannot be even conceived, let alone treated for the models considered, thus implying the complete loss of meaning of the terms "spin 1/2." Moreover, dissipative nuclear models (4.6) have a *nonunitary* finite time evolution, thus suffering of all the additional shortcomings of Theorem 1.1 studied below.

Studies on the problematic aspects in representing dissipation via triple systems have been conducted by Santilli and others (see Refs. 7(a), 7(b) and 7(f) and papers quoted therein).

A fourth representative class of rather old generalizations of quantum mechanics can be today characterized via *the relaxation of the universal enveloping associative algebra.* This third attempt occurred in statistical mechanics at large, and in the behavior of the density matrix in particular. In essence, the insufficiency of the

Hamiltonian for the representation of collisions was identified since the inception of quantum mechanics, resulting in the addition of an *external collision term* to the Liouville equation for the density, with equations used throughout this century (see, e.g. Ref. 14)

$$\frac{i\,d\rho}{dt} = (\rho, H) = \rho \times H - H \times \rho + C = [\rho, H] + C\,. \tag{4.7}$$

The latter models suffer of physical shortcoming greater than the preceding ones. In fact, as it is the case for models (4.6), models (4.7) also lose *all* algebras in the brackets of the time evolution (ρ, H), let alone *all* Lie algebras (this time because of the violation of the scalar and distributive laws). In addition, the latter models lose the exponentiation of the time evolution to a finite form, thus losing any consistent application of basic topics such as rotations or Lorentz transforms, let alone violating the premises for the implementation of any known relativity.

At a deeper inspection, the technical reason for the above occurrences is that *the addition of an external term to the Lie brackets implies the loss of any consistent enveloping algebra, whether associative or not.* By recalling that the enveloping algebra is at the foundation of quantum mechanics, e.g. for the representation theory,[11] its loss essentially implies the collapse, whether in a direct or indirect way, of the totality of the original axiomatic structure and physical consistency. In particular, *models of type* (4.7) *have no unit at all*, thus losing physically consistent applications to real measures, contrary to popular beliefs in the field throughout this century.

Studies on the physical inconsistencies of theories with external collision terms have been conducted by Santilli,[6(e),7(a),7(b)] Jannussis and Skaltsas[27(c)] and others.

Numerous additional attempts exist in the literature of the first part of this century at relaxing other axioms of quantum mechanics, and they all suffer of physical shortcomings similar to the preceding ones when treated with the mathematics of quantum theories.

4.3. *Lie-admissible and Lie-isotopic deformations*

In 1967 Santilli[6(a)] (see Refs. 7(a) and 7(b) of 1978 and 1982 for detailed initial studies, and Refs. 28 for the latest advances) introduced the first known *parameter-deformations* of the Lie product according to the second notion of Lie-admissibility (and Jordan-admissibility of Subsec. 2.2), Eq. (2.2), which we now write

$$\begin{aligned}(A, B) &= p \times A \times B - q \times B \times A \\ &= m \times (A \times B - B \times A) + n \times (A \times B + B \times A)\,,\end{aligned} \tag{4.8}$$

where $p = n + m$, $q = n - m$ and $p \pm q$ are non-null real or complex parameters (or functions).

Subsequently, in 1978 Santilli[6(e)] (see Refs. 7(c) and 7(d) of 1978 and 1983 for detailed initial studies, and Refs. 28 for recent advances) introduced the first known *operator-deformations* of the Lie product according to the third (or general) notion of Lie-admissibility (and Jordan admissibility, see Subsec. 2.2), Eq. (2.4), which we now write

$$\begin{aligned}(A,B) &= A\times P\times B - B\times Q\times A\\ &= (A\times M\times B - B\times M\times A) + (A\times N\times B + B\times N\times A)\,, \end{aligned}\tag{4.9}$$

where $P = N+M$, $Q = N-M$ and $P\pm Q$ are nonsingular, generally non-Hermitian operator (or real-valued, nonsingular and nonsymmetric matrices).

The motivation of the latter deformations is due to the fact that the time evolution of the parameter-deformations, Eqs. (2.3), is *nonunitary* when formulated on a conventional Hilbert space over the conventional field of complex numbers. The operator-deformations then follow from the parameter ones via a simple nonunitary transform with

$$U\times U^\dagger \neq I\,,\qquad P = p\times (U\times U^\dagger)^{-1}\,,\qquad Q = q\times (U\times U^\dagger)^{-1}\,,\tag{4.10}$$

as one can verify (see also the next section).

Another property that does not appear to have propagated in the rather vast literature on deformations is that *the operator broadening of parameter-deformations emerges as inevitable, even when not desired, under the mere time evolution of the theory.* Equivalently, *all parameter-deformations are solely valid at one, single, fixed value of time.*

Note that, when reached in this way, operator deformations (4.9) remain a realization of the third notion of Lie-admissibility under any additional nonunitary transform, as the reader can also verify (see Sec. 5 for the problem of invariance). In particular, deformations (4.9) are characterized by the most general possible (regular, unconstrained, bilinear, single-valued) product defining an algebra (over a field of characteristic zero). As such, they admit as particular cases all other infinitely possible quantum deformations with a well-defined algebra in their product, as we shall see in more details below.

Yet another aspect that does not appear to have propagated in the literature is that *deformations* (4.8) *or* (4.9) *and all their non-antisymmetric particular cases, can only represent open nonconservative systems.* This is evidently due to the fact that, from Eqs. (2.5) we have the *time-rate-of-variation of the energy* $i\,dH/dt = (H,H) = H\times(P-Q)\times H \neq 0$. By contract, numerous particular cases, such as the q-deformations reviewed below, as often applied in the literature for *conservative* cases, with evident inconsistency.

In 1978 Santilli[6(d)] (see Refs. 7(c) and 7(d) of 1978 and 1983 for detailed studies, and Refs. 28 for recent accounts) introduced the simpler class of axiom-preserving operator deformations of the Lie product, Eq. (2.7),

$$\begin{aligned}[A\hat{,}B] &= (A,B)-(B,A) = A\,\hat{\times}\,B - B\,\hat{\times}\,A\\ &= A\times T\times B - B\times T\times A\,,\\ T &= P-Q\,,\end{aligned}\tag{4.11}$$

which identifies the "Lie-content" of the general Lie-admissible deformations (see Refs. 6 and 7 for the complementary Jordan content which we cannot possibly consider here for brevity).

Note that deformations (4.11) emerge quite naturally via a nonunitary transform of the conventional Lie product with

$$U \times U^{\dagger} \neq I\,, \qquad T = (U \times U^{\dagger})^{-1}\,, \tag{4.12}$$

and they preserve their Lie-isotopic structure under additional arbitrary nonunitary transforms, as one can also verify (see again Sec. 5 for invariance). As such, brackets (4.11) are the most general possible (regular, unconstrained, bilinear, single-valued) operator realization of the Lie product (on a field of characteristic zero).

From time evolution (2.8) it is evident that *antisymmetric deformations* (4.11) *can only represent closed-isolated systems.* In fact, in this case we have the *conservation law of the energy* $i\,dH/dt = H \times (T - T) \times H \equiv 0$, and the same occurs for other total conservation laws.[7(f)]

More generally, Lie-admissible deformations (4.9) are used to characterize the brackets of the time evolution of *open, nonconservative and irreversible systems with unrestricted external interactions*, admitting as classical counterpart the historical Hamilton's equations with external terms, only re-written in the identical, algebraically consistent form (3.6).

By comparison, Lie-isotopic deformations (4.11) where instead used for the characterization of the brackets of the are evolution of *closed, isolated and irreversible systems with linear and nonlinear, local and nonlocal and potential as well as nonpotential internal interactions*, possessing Birkhoff's equations (3.2) as their classical counterpart (see Refs. 6–8 and 27–31 for details).

It should be stressed that, as better indicated in the next section, the above Lie-admissible and Lie-isotopic operator-deformations were introduced under the specific requirement that they are treated via *new mathematics*, called *genomathematics and isomathematics,* respectively.[6(e),28(a)] In this section we study said deformations with the *conventional* mathematics of quantum mechanics.

4.4. *Additional deformations*

Several years following the origination of the above deformations, numerous other deformations appeared in the literature, although generally without the quotation of the former, such as:

(1) the q-parameter deformations introduced in 1989 by Biedenharn[15(a)] and Macfarlane,[15(b)] which were then followed by a vast literature (see, e.g. Ref. 16), with product

$$(A, B) = A \times B - q \times B \times A\,; \tag{4.13}$$

(2) the so-called quantum groups[17] generally consisting of parameter-dependent deformations of the structure constants of a given Lie-algebra, while keeping the conventional Lie product unchanged,

$$X_i \times X_j - X_j \times X_i = C_{ij}^k \times X_k \to X_i \times X_j - X_j \times X_i = D_{ij}^k(q) \times X_k \,; \tag{4.14}$$

(3) the so-called k-deformations (see, e.g. Refs. 18), which essentially are a particular relativistic form of quantum groups;

(4) the so-called *star models*[19] with lifting of the associative product

$$A * B = A \times T \times B = \text{isoassociative}\,; \tag{4.15}$$

(5) a particular form of nonlinear deformations introduced by Weinberg[20(a)] via a nonassociative deformation of the associative enveloping algebra of quantum mechanics,

$$A * B = \text{nonassociative}\,; \tag{4.16}$$

(6) the so-called squeezed states theories (see, e.g. Refs. 21), which are characterized by a nonunitary image of quantum mechanical theories;

(7) the statistics by Prigogine *et al.*,[22] which also has a nonunitary structure;

(8) noncanonical time theories;[23]

(9) the Ellis–Mavromatos–Nanopoulos model of black hole dynamics with Santilli's Lie-admissible structure;[24]

(10) supersymmetric theories;[25]

(11) Kac–Moody superalgebras;[26]

and other models.

Another class of operator theories which are deformations of quantum mechanics according to our definition is known under the name of *grantum gravity.* In fact, it is easy to prove that, as a necessary condition to acquire curvature, all quantum gravity theories are *nonunitary* image of conventional quantum theories when formulated on a conventional Hilbert space over a conventional field, as studied in detail in Refs. 28(b) and 29(h).

4.5. *Lie-admissible and Lie-isotopic unification of operator deformations*

The first property which is recommendable for a systematic study of the axiomatic consistency of the above disparate deformations is the following:

Proposition 4.1. *All operator deformations of quantum mechanics which are outside the class of equivalence of the original theory and admit an algebra which is (is not) totally antisymmetric, are particular cases of Santilli's Lie-isotopic (Lie-admissible) formulations.*

The proof is so simple that can be merely illustrated. In fact: q-deformations as per Eqs. (4.13) are an evident particular case of the (p,q)-deformations (4.8); quantum groups are a particular case of Lie-isotopic theories because the deformation of the structure constants (e.g. of the SU(2) algebra) can only be achieved via nonunitary transforms; k-deformations can therefore be more adequately studied within the context of Lie-isotopic theory (as a full nonunitary image of a Lie algebra, rather than deforming the structure constants via nonunitary maps, while preserving the original vector space unchanged); star-models as per Eq. (4.15) *coincide* with Santilli's isoassociative envelope of the Lie-isotopic theory; Weinberg nonlinear theory as per envelope (4.16) also *coincides* with the very basic axiom of general Lie-admissibility, Eqs. (4.11), as done in reformulation[20(b)] where rule (4.11) is formulated via indices; squeezed states, Prigogine's statistics and noncanonical time theories have a clear nonunitary Lie structure, thus being a clear realization of the Lie-isotopic theory; The Ellis–Nanopoulos–Mavromatos model is one of the few with a quoted Lie-admissible structure; in view of their mixing commutators and anticommutators, supersymmetric theories are a clear particular case of the Lie-admissible deformations (4.9) with P and Q constants; the Kac–Moody superalgebras, which also mix commutators and anticommutators, are also a clear particular case of the Lie-admissible theories (4.9), this time, for $M = 1$ and N a phase factor (see the original definition of general Lie-admissibility of Ref. 6(d) in which the quantities M and N depend on the generators); and similar occurrences hold for other deformations, as the reader is encouraged to verify.

It is also important to see that all historical generalizations of quantum mechanics recalled in Subsec. 4.2 can also be identically reformulated in terms of the Lie-isotopic or Lie-admissible theories. Besides providing a unification of evident value for axiomatic studies, the reformulation produces a consistent algebra when it does not exist in the conventional formulation.

For instance, all nonlinear theories (4.5) can be *identically* rewritten in terms of the Lie-isotopic theory via the rules

$$H(t,r,p,\phi) \times |\phi\rangle = H_0(t,r,p) \times T(\phi) \times |\phi\rangle = E \times |\phi\rangle\,. \tag{4.17}$$

Similarly, nonpotential models treated via "imaginary potentials," Eqs. (4.6), admit the following *identical* reformulation in terms of Lie-admissible equations (4.9),

$$\begin{gathered}(A, H, H^\dagger) = A \times H^\dagger - H \times A = A \times P \times H_0 - H_0 \times Q \times A\,,\\ H = H_0 \times Q\,, \qquad H_0 = H_0^\dagger\,, \qquad P = Q^\dagger\,.\end{gathered} \tag{4.18}$$

Along similar lines, models with external collision terms (4.7) can also be *identically* rewritten via Lie-admissible theories (4.9) according to the rules

$$\begin{gathered}[\rho, H] + C = \rho \times H - H \times \rho + C = \rho \times P \times H - H \times Q \times \rho\,,\\ P = 1\,, \qquad Q = 1 + H^{-1} \times C \times \rho^{-1}\,.\end{gathered} \tag{4.19}$$

Similar identical reformulations exist for other generalized models, such as for string theories, theories with discrete space and time structures, and others.[7(f)] As we shall see in the next section, the above reformulations are not mere mathematical curiosities because they permit a resolution of the problematic aspects of the original versions.

The unified treatment of all possible deformations via Lie-isotopic or the broader Lie-admissible formulations permits the unified study of the axiomatic constituency of operator deformations at large, which should otherwise be conducted on an individual basis.

4.6. *Nonunitary character of operator deformations*

Ironically, by the time of the appearance of papers[15] by Biedenharn and Macfarlane, Santilli[6,7] had already abandoned this line of inquiry because of insurmountable problematic aspects of *physical* character reported in Refs. 27 and 28(b), whose study can be initiated with the following:

Lemma 4.1. *The general Lie-admissible time evolution* (2.5) *and its Lie-isotopic particularization* (2.8) *are nonunitary when formulated on a conventional Hilbert space $\mathcal{H}$ over the conventional field of complex numbers C for all nontrivial* (*i.e. operator*) *realizations of P, Q, T.*

Proof. Heisenberg's time evolution in finite form has a *bimodular Lie structure*, in the sense of being characterized by an action to the right, here denoted $U^> = \exp\{iH \times t)\}$ and an action to the left, here denoted $^<U = \exp\{-it \times H\}$,

$$A(t) = U^> \times A(0) \times {}^<U = e^{iH\times t} \times A(0) \times e^{-it\times H} . \tag{4.20}$$

The unitarity of the time evolution follows from the familiar conjugation

$${}^<U = (U^>)^\dagger . \tag{4.21}$$

The familiar condition of unitarity, Eq. (4.4), then acquires the more detailed forms

$$\begin{aligned} U \times U^\dagger &= U^\dagger \times U = U^> \times {}^<U = {}^<U \times U^> = U^> \times (U^>)^\dagger \\ &= (U^>)^\dagger \times U^> = {}^<U \times ({}^<U)^\dagger = ({}^<U)^\dagger \times {}^<U = I . \end{aligned} \tag{4.22}$$

The general Lie-admissible law (2.5) violates, first, condition (4.21) and then each condition (4.22) because of the lack of commutativity of P and Q with H. The Lie-isotopic time evolution (2.8) verifies condition (4.21), but violates conditions (4.22), again, because of the lack of general commutativity of T and H. Therefore, time evolutions (2.5) and (2.8) are nonunitarity. The same occurs for all particular cases, such as q- or k-deformations, supersymmetric models, Kac-Moody algebras, etc. q.e.d.

The above occurrence was evidently expected from its classical counterpart, namely, the noncanonical character of the transformation theory of the Birkhoffian and Birkhoffian-admissible mechanics.[7]

4.7. *Problematic aspects of operator deformations*

The nonunitary structure of operator deformations (as defined in this section) has devastating implications for the axiomatic and physical consistencies of the theories. In fact, we have the following

Proof of Theorem 1.1 (*operator profile*). Consider the case of the regular representation of the rotational symmetry on Euclidean space over the reals. The space unit $I = \mathrm{Diag}(1,1,1)$ of quantum mechanics is the unit of the enveloping associative operator algebra ξ with generic elements A, $B, \ldots$ and conventional associative product $A \times B$, Eqs. (4.1). It is well known that, by definition, the above unit is invariant under unitary transformations, $I \to I' = U \times I \times U^\dagger \equiv I$, thus recovering the known axiomatic consistency of the theory.

By comparison, from their very definition, nonunitary transforms do not preserve the unit,

$$I \to I' = U \times I \times U^\dagger \neq I\,. \tag{4.23}$$

The same unit is not preserved under the time evolution of all nonunitary deformations, e.g.

$$\frac{id}{dt} = (I, H) = I \times P \times H - H \times Q \times I \neq 0\,, \tag{4.24a}$$

$$\frac{i\,dI}{dt} = [I\hat{,}H] = I \times T \times H - H \times T \times I \neq 0\,. \tag{4.24b}$$

Thus, all nonunitary deformations lose the invariance of the unit with respect to both the transformation theory as well as the dynamical equations, and this proves Theorem 1.1 for all possible the operator profiles. q.e.d.

As a result, q-, k-, quantum-, supersymmetric, Kac–Moody, Lie-isotopic, Lie-admissible and all other operator deformations with nonunitary structure cannot be unambiguously applied to measurements, because, e.g. it is not possible to measure distances with a (stationary) meter of length varying in time.

Corollary 4.1. *Nonunitary deformations do not preserve hermiticity when defined on conventional Hilbert spaces over conventional fields.*

Proof. Under a nonunitary transform, the familiar associative modular action of the Schrödinger's representation $H \times |\psi\rangle$, where H is an operator Hermitian at the initial time $t = 0$, becomes

$$\begin{aligned} U \times H \times |\psi\rangle &= U \times H \times U^\dagger \times (U \times U^\dagger)^{-1} \times U \times |\psi\rangle \\ &= \hat{H} \times \hat{T} \times |\hat{\psi}\rangle\,, \end{aligned} \tag{4.25a}$$

$$U \times U^\dagger \neq I\,, \quad T = (U \times U^\dagger)^{-1}\,, \quad |\hat{\psi}\rangle = U \times |\psi\rangle\,, \tag{4.25b}$$

$$\hat{H} = U \times H \times U^\dagger\,. \tag{4.25c}$$

By noting that T is Hermitian, the initial condition of hermiticity of H on $\mathcal{H}$, $\langle\psi| \times \{H \times |\psi\rangle\} \equiv \{\langle\psi| \times H^\dagger\} \times |\psi\rangle$, when applied to the Hilbert space $\mathcal{H}$ with states $|\hat{\psi}\rangle$, $|\hat{\phi}\rangle$, etc. requires the action of the transformed operator on a *conventional* inner product, resulting in the loss of hermiticity at subsequent times, a property first studied by Lopez[27(b)] and known as *Lopez's Lemma,*

$$\langle\hat{\psi}| \times \{\hat{H} \times \hat{T} \times |\hat{\psi}\rangle\} \equiv \{\langle\hat{\psi}| \times \hat{T} \times \hat{H}^\dagger \times |\hat{\psi}\rangle\,,$$

i.e.

$$\hat{H}^\dagger = T^{-1} \times \hat{H} \times T \neq \hat{H}\,. \tag{4.26}$$

In fact, hermiticity is not generally preserved because of the lack of general commutativity of T and $\hat{H}$. q.e.d.

The problematic aspect of Corollary 4.1 is rather serious because it implies that all nonunitary deformations, as conventionally treated, do not possess physically acceptable observables.

Corollary 4.2. *Nonunitary deformations on conventional Hilbert spaces over conventional fields do not possess unique and invariant numerical predictions.*

Proof. The lack of uniqueness is a well-known problematic aspect in the literature,[16] and can be seen from the lack of uniqueness of q- or k-special functions such as the exponentiation. The lack of invariance can be seen from the fact that the Lie-isotopic and Lie-admissible equations are not form-invariant under their own time evolutions when formulated on $\mathcal{H}$ over C.

Specifically, for the case of Lie-isotopic equations (2.8) we have

$$\begin{aligned} \frac{i\,dA}{dt} &= A \times T \times H - H \times T \times A \to \frac{i\,dA'}{dt} \\ &= A' \times T' \times H' - H' \times T' \times A'\,. \end{aligned} \tag{4.27}$$

The lack of conservation of the numerical value of T then implies the evident lack of invariance of the numerical predictions. An even more general lack of invariance occurs for Lie-admissible equations (2.5). q.e.d.

As is well known, the numerical predictions of quantum mechanics are the result of data elaboration via special functions and transforms, e.g. partial wave analysis based on Legendre polynomials. These numerical predictions are then unique and invariant because of the uniqueness and invariance of the special functions and transforms.

The problematic aspect of Corollary 4.2 is also serious because it implies the lack of physical meaning of the numerical predictions of nonunitary deformations. In fact, the numerical predictions of operator deformations are also elaborated via special functions and transforms, the so-called q-, k-, and other special functions and transforms. But these special functions are not invariant under nonunitary time evolutions (because, e.g. the q-parameter becomes the Q operator as per rule (4.10). The lack of invariance of the numerical predictions then follows.

The above occurrence also implies the following problematic aspect (where the definition on $\mathcal{H}$ over C is hereon ignored):

Corollary 4.3. *Nonunitary deformations do not possess unique and invariant physical laws.*

It is then easy to prove the following:

Corollary 4.4. *Nonunitary deformations do not possess invariant probabilities.*

Recall that the causality of quantum mechanics follows from the unitarity of its time evolution. We therefore have the additional:

Corollary 4.5. *Nonunitary deformations violate causality.*

Besides all the preceding problematic aspects, that considered by this author most serious, is the following one of evident derivation from Theorem 1.1:

Corollary 4.6. *Nonunitary deformations violate the axioms of Galileo's and Einstein's special relativities.*

The above occurrence can be easily illustrated by noting that, e.g. the k-deformed Minkowski spaces of Refs. 18 are not compatible with the Lorentz transforms, or that the k-deformed Poincaré symmetry is not isomorphic to the conventional symmetry.

Corollary 4.6 then implies the rather difficult problems of identifying new relativities, proving their axiomatic and physical consistencies, and establishing them experimentally.

Theorem 1.1 and all its Corollaries also apply to the contemporary theories on *quantum gravity* which, when formulated on a conventional Hilbert space over a conventional field, have a nonunitary structure, thus lacking invariant units of space, time, etc. losing the original hermiticity-observability under the time evolution, and having other serious physical problematic aspects studied in details in Refs. 28(b) and 29(h).

We complete this section with a review of *Okubo's No Quantization Theorem*[27(a)] which *prohibits the use of nonassociative envelopes* because they imply *the lack of equivalence between the Heisenberg-type and Schrödinger-type representations.*

More specifically, the Schrödinger representation is based on the *right, modular, associative action* of an operator H on a state $|\phi\rangle$, $H \times |\phi\rangle$, where $\times$ is such that $A \times (B \times |\phi\rangle) = (A \times B) \times |\phi\rangle$. When the enveloping operator algebra is based on a product, say, $A \circ B$ for which $A \circ (B \circ C) \neq (A \circ B) \circ C$, the Schrödinger-type and Heisenberg-type representations cannot possibly be equivalent, as first identified by Okubo, while a "nonassociative extension of Schrödinger representation" would have problems of physical consistency because in this case $A \circ (B \circ |\phi\rangle) \neq (A \circ B) \circ |\phi\rangle$.[27(a)]

In turn, the above nonequivalence has disastrous consequences for the physical consistency of the theory studied in details Jannussis, Mignani and Santilli,[27(d)] such as:

(1) General loss of all units, let alone of their invariance, because nonassociative algebras do not generally admit a quantity I such that $I \circ A = A \circ I = A$ for all possible A, thus suffering from a drawback more severe than that of Theorem 1.1.
(2) No quantity whose definition depends on the Schrödinger representation (such as the notions of hermiticity and observability) can be even defined at any time, thus suffering again from a drawback more serious than that of Theorem 1.1.
(3) For the same reasons there is the lack of consistent definition of exponentiation, thus lacking finite transforms such as the Galilei or Lorentz transforms, with consequential loss of causality, probability laws, etc.

The above shortcomings prevent the achievement of axiomatic consistency for any deformation based on a nonassociative envelope, such as Weinberg's nonlinear theory.[20(a)] Jordan[20(b)] attempted a reformulation of Weinberg's theory via the use of our fundamental rule (2.7) although in a disguised expression in terms of indices $A_{ij}T_{jk}C_{kl}$. Despite that, the latter formulation of Ref. 20(b) suffers of all the shortcomings of Theorem 1.1 because it is possesses a *nonunitary* structure when defined on *conventional* spaces and fields.

This completes our study of the problematic aspects of classical and operator deformations with the understanding that, by no means, the above study exhaust all deformations existing in the literature. Interested readers can identify the problematic aspects of other deformations via the techniques presented in this paper. The author would be grateful to colleagues who care to bring to his attention additional important deformations deserving a specific inspection.

5. Invariant Formulation of Classical and Operator Deformations

5.1. *Preliminaries*

The third and final objective of this paper is to outline as well as to upgrade the efforts conducted so far for the achievement of an invariant formulation of classical and operator, Lie-isotopic and Lie-admissible deformations, and then present, apparently for the first time, the invariant reformulation of known deformations.

These studies have been conducted under the name of *hadronic mechanics*[6–8,27–31] which includes the *Lie-isotopic and Lie-admissible branches* considered in this paper, plus additional branches not considered in this study for brevity, such as the *multivalued hyperstructural branch* (used for multi-dimensional cosmologies or biological structures), as well as antiautomorphic images of the preceding branches called *isoduals* (used for the classical and operator treatment of antimatter).

Therefore, by conception and construction, hadronic mechanics contains as particular cases all possible or otherwise known operator deformations. Studies for invariance conducted within the context of hadronic mechanics are then directly applicable to all known deformations.

The main structural difference is that hadronic mechanics achieves invariance via *new mathematics* specifically conceived for *each* branch, while deformations are treated via the *conventional* mathematics of quantum mechanics, thus suffering the shortcomings outlined in the preceding section.

Note that all existing deformations, including q-, k-, star-deformations, quantum groups, nonlinear nonlocal and nonunitary theories, supersymmetric and Kac–Moody algebras, etc. can be *identically* treated via the new mathematics of hadronic mechanics, by achieving in this way the invariance needed for physical applications.

However, the reader should be aware from the outset that, even though the dynamical equations are the same, the invariant numerical predictions are different than the noninvariant ones, as evidently expected from the fact that their elaborations are different.

To avoid excessive initial complexities, in this section we shall review first the invariant formulation of classical and operator Lie-isotopic theories, and then pass to the more complex invariant formulation of Lie-admissible theories. We shall then indicate the application of the results to the invariant reformulation of deformations.

It should be indicated that the achievement of invariant generalized theories resulted to be rather difficult. In fact, for reasons reviewed below, it took about two decades to achieve an invariant formulation of Lie-isotopic theories following their original formulation,[6(a),6(e)] while it took three decades to reach an invariant formulation of the broader Lie-admissible theories following their original proposal in 1967.[6(a)]

The reader should therefore be aware that all studies directly or indirectly related to hadronic mechanics prior to 1996, including all studied conducted by this author until that time, *are not* invariant.

Sufficient maturity in the new mathematics was only reached in the recent memoir[28(a)] and papers.[30] Sufficient maturity in the physical formulation of the Lie-isotopic and Lie-admissible branches of hadronic mechanics was only reached in the recent memoirs,[28(b),28(c)] respectively. Sufficient maturity on symmetry profiles was reached in Refs. 29 and memoir.[30(c)]

In this paper we cannot possibly review all these studies and, to avoid a prohibitive length, we can only outline and upgrade the main aspects and suggest the consultation of Refs. 27–30 for technical details.

5.2. *Hermitian isounit and isomathematics*

A sound foundation of contemporary classical or operator dynamics is the assumption that conventional Hamiltonians can represent the totality of interactions which are linear (in the wave functions), local-differential and derivable from a potential.

As a consequence, all deformations aim at the representation of *effects, characteristics or interactions which are non-Hamiltonian by conception.*

The main issue here addressed is therefore *how to represent non-Hamiltonian terms in an invariant way*, that is, in a way capable of recovering the same axiomatic consistency of Hamiltonian formulations.

The solution submitted by Santilli in 1978[6,7] is *to represent all non-Hamiltonian terms via a generalization of the basic unit of the theory.* The main motivation for this solution is the well known property that the unit is the fundamental invariant of all theories. The embedding of non-Hamiltonian terms in generalized units can therefore be safely assumed as a sound basis for the achievement of the desired invariance.

Other solutions are evidently possible, and their study is here encouraged, *provided* that they achieve an invariance at least equivalent to that offered by the unit.

The fundamental assumption for the invariant representation of Lie-isotopic deformations is then the lifting of the conventional n-dimensional unit $I = \mathrm{Diag}(1,1,\ldots,1)$ of classical or quantum theories into an $(n\times n)$-dimensional matrix $\hat{I}$ which is nonsingular, Hermitian and positive-definite, but otherwise possesses an unrestricted functional dependence on time t, coordinates r, velocities dr/dt, wave functions ϕ and their derivatives $\partial\phi/\partial r$ (for the operator case), and any other needed quantity[6,28]

$$I = \mathrm{Diag}(1,1,\ldots,1) \to \hat{I} = (\hat{I}^i_j) = \hat{I}\left(t, r, \frac{dr}{dt}, \phi, \frac{\partial\phi}{\partial r}, \ldots\right) = \frac{1}{\hat{T}}\,. \tag{5.1}$$

An invariant formulation of the deformations then follows when the *totality* of the original mathematics is reconstructed in such a way to admit $\hat{I}$, rather than I, as the correct left and right new unit.

This requires the lifting of *all* conventional associative products $A \times B$ among generic quantities A, B (e.g. numbers, vector fields, operators, etc.) into the form [*loc. cit.*]

$$A \times B \to A \,\hat{\times}\, B = A \times \hat{T} \times B\,, \quad \hat{T} = \text{fixed}\,, \tag{5.2}$$

under which $\hat{I}$ is indeed the correct left and right unit

$$\hat{I} \,\hat{\times}\, A = A \,\hat{\times}\, \hat{I} = A\,, \tag{5.3}$$

for all possible quantities A.

The emerging new theories were called *isotopic*[6(d)] in the Greek meaning of being "axiom-preserving." In fact, by assumption, $\hat{I}$, $A \,\hat{\times}\, B$, etc. preserve all original axiomatic properties. Under these conditions (only), $\hat{I}$ is called the *isounit*, T is called the *isotopic element*, $A \,\hat{\times}\, B$ is called an *isoassociative product* (or *isoproduct* for short), and the prefix "iso" is used in similar cases.

For consistency, liftings (5.1)–(5.3) must be applied to the *totality* of the original mathematics, with no exception unwon to this author. In fact, it is now well known to experts in isotopies that any exception in this basic rule (essentially implying a mixture of conventional and generalized settings), does not imply invariance.

The implementation of the above basic rule mandates the construction of a new mathematics, specifically (and solely) applicable to the Lie-isotopic branch of hadronic mechanics, constructed under the name of *isomathematics*,[6(d),7(e),8,27–31] which includes: new *isonumbers* $\hat{n} = n \times \hat{I}$; new *isogeometries* such as the *iso-Euclidean geometry* with *isometric* $\hat{\delta} = T \times \delta$ and *isointerval* $r^{\hat{2}} = (r^i \times \hat{\delta}_{ij} \times r^j) \times \hat{I}$; *iso-Hilbert spaces*; *isoalgebras*, *isotopologies*, *isospecial functions and isotransforms*, etc.

Regrettably, we cannot possibly review this new mathematics for brevity. Its technical knowledge is however essential for a technical understanding of this section. For instance, all operations depending on multiplications must be lifted for consistency. This evidently includes all quotients which have to be lifted into the *isoquotient*. But the latter assume the simple form $a\hat{/}b = (a/b) \times \hat{I}$. As a result, the isoproduct of an isoquotient and a generic quantity coincides with the conventional product $(a\hat{/}b)\,\hat{\times}\, A \equiv (a/b) \times A$. The latter scripture will be used mainly for the reader's convenience with the understanding that invariance occurs only for the former.

Despite the use of the new isomathematics, the dynamical equations of all Lie-isotopic theories formulated prior to 1996 were still *noninvariant* for reasons that escaped identifications for about two decades. It was finally discovered in the memoir[28(a)] that the origin of the noninvariance was where one would expect it the least: in the *ordinary differential calculus.*

In essence, all treatises on differential calculus are silent on its dependence on the unit because the (tacitly assumed) conventional unit, the number $I = +1$, has a trivially null differential, $dI = 0$. This is no longer the case when the assumed unit $\hat{I} = \hat{I}(t, r, \ldots)$ depends on the local variables, for which $d\hat{I} \neq 0$. The memoir[28(a)] identified this occurrence and constructed, apparently for the first time, the *isotopies of the differential calculus* (or *isodifferential calculus* for short) via the basic expressions

$$\hat{d}t = \hat{I}_t \times dt\,, \qquad \hat{d}r^k = \hat{I}^k_{s\,i} \times dr^i\,, \tag{5.4a}$$

$$\frac{\hat{\partial}}{\hat{\partial} t} = \hat{T}_t \times \frac{\partial}{\partial t}\,, \qquad \frac{\hat{\partial}}{\hat{\partial}} r^k = \hat{T}^i_k \times \frac{\partial}{\partial r^i}\,, \tag{5.4b}$$

where $\hat{I}_t$ is evidently a scalar.

Note that the new calculus is an isotopic image of the old one and as such, it preserves all original axioms *on isospace over isofields* (although not necessarily in its projection on conventional spaces over conventional fields), including the commutativity of the derivatives. Note in particular that $\hat{\partial} r^i / \hat{\partial} r^j = \delta^i_j$.

The lack of invariance of all isotheories prior to Ref. 28(a) is now clear. In fact, it occurred in the most vital part of any physical theory, its fundamental dynamical equations, because all dynamical equations of isotheories prior to Ref. 28(a) were expressed in terms of the conventional differential calculus, thus being non-invariant.

The isodifferential calculus completed all isotopies that could possibly be constructed and permitted the achievement of an invariant formulation of Lie-isotopic theories which was first reached in the memoir[28(b)] of 1997, and whose results are evidently extendable to all possible deformations with an antisymmetric product in the time evolution.

In the following we shall assume the notation generally used in isotopies according to which all quantities with a "hat" are computed on isospaces over isofields (i.e. with respect to generalized units), while all quantities without the "hat" are the *projections* of the former on conventional spaces over conventional fields (i.e. are computed with respect to conventional units).

We shall also assume that all repeated indices in isospaces imply sum with respect to the isometric $\hat{\delta} = \hat{T} \times \delta$, and all repeated indices between isounits $\hat{I}$ or isotopic elements $\hat{T}$ and other indices are computed with respect to the conventional metric δ.

While conventional theories and all their deformations have a *unique* formulation, the usual one, all isotopic theories have instead *two* formulations, one on isospaces over isofield and the other on conventional spaces over conventional fields.

Invariance is achieved by *identically* reformulating the latter in the former context.

5.3. *Classical iso-Hamiltonian mechanics*

The achievement of invariant formulation of classical deformations was not a mere mathematical detail, inasmuch as it implied the birth of a basically novel mechanics presented for the first time in the memoir[28(a)] under the name of the *iso-Hamiltonian mechanics.* In particular, the novelty occurred in the physical most important aspects, the equations of motion.

In fact, the memoir[28(a)] first introduced the *iso-Newton equations*

$$\hat{m}\,\hat{\times}_t\,\frac{\hat{d}\hat{v}_k}{\hat{d}t} = -\frac{\hat{\partial}\hat{V}}{\hat{\partial}r^k}\,, \tag{5.5}$$

where $\hat{m}_t = m \times \hat{I}_t$, $\hat{\times}_t = \times\hat{T}_t\times$, $\hat{v}_k = \hat{d}\hat{r}_k/dt = \hat{T}_t \times d(\hat{T}^i_k(t,r,v,\ldots) \times r_i/dt)$, and $\hat{\partial}\hat{V}/\hat{\partial}r^k = \hat{T}^i_k \times \partial V/\partial r^i$.

The memoir[28(a)] then proved that the above isoequations are "directly universal" (Sec. 1) for all infinitely possible, unconstrained, well-behaved systems of particles which are extended, nonspherical and deformable under linear and nonlinear, local and nonlocal and potential as well as nonpotential forces.

As one can verify, these representations are permitted by realizations of the space isotopic element of the class

$$\hat{T} = \mathrm{Diag}\left(\frac{1}{n_1^2}, \frac{1}{n_2^2}, \frac{1}{n_3^2}\right) \times \Gamma(t,r,v,\ldots)\,, \tag{5.6}$$

where the factor $\mathrm{Diag}(1/n_1^2, 1/n_2^2, 1/n_3^2)$ represents the extended, nonspherical and deformable shapes of the particles (here assumed to be a spheroidal ellipsoid), while the factor $\Gamma(t, r, v, \ldots)$ represents all nonlinear, nonlocal and nonpotential forces.

The time isotopic element $\hat{T}_t$ is generally redundant in iso-Newtonian mechanics and can be assumed to be the value $+1$ (not so for relativistic mechanics where the value $\hat{T}_t = c_0^2/n_4^2$ permits a representation of the locally varying speed of light $c = c_0/n_4$ — see the memoir[29(h)] for brevity).

Note that Eqs. (5.5) constitute a *generalization of Newton's equations in Newtonian mechanics* (thus prior to any relativistic gravitational or constrained extension). In particular, Eqs. (5.5) signal the transition from the Galilean–Newtonian representation of "massive points" to extended, nonspherical and deformable bodies.

The latter extension was made possible by the underlying novel *iso-Euclidean geometry* (see the monograph[7(e)] and the latest presentation[29(h)] for the relativistic case) and related *isotopology* specifically constructed for the task (see Refs. 28(a), 30(b) and 30(d) for brevity).

Note the mechanism of invariance under non-Newtonian characteristics. It consists in embedding all non-Newtonian terms in the isotopic elements in such a way that *the iso-Newton equations on iso-Euclidean space coincide with the conventional Newton equation for conservative systems.*

The memoir[28(a)] then proved that all possible Eqs. (5.5) are derivable from the new *isoaction principle*[28(a)]

$$
\begin{aligned}
\hat{A} &= \int_{t_1}^{t_2} dt \left[p_k \times \frac{\hat{d} r^k}{\hat{d} t} + H(t, r, p) \right] \\
&= \int_{t_1}^{t_2} dt \left[p_k \times \hat{I}_i^k(t, r, v, a, \ldots) \times \frac{dr^i}{dt} + H(t, r, p) \right] \\
&= \int_{t_1}^{t_2} dt \left[\hat{R}_{0\mu}(b) \hat{\times} \frac{\hat{d} b^\mu}{\hat{d} t} + H(t, b) \right] \\
&= \int_{t_1}^{t_2} dt \left[R_{0\mu}(b) \times \hat{I}_\nu^\mu \times \frac{db^\nu}{dt} + H(t, b) \right],
\end{aligned}
\tag{5.7a}
$$

$$
\begin{gathered}
(\hat{I}_\nu^\mu) = \mathrm{Diag}(\hat{I}, \hat{T}), \quad \hat{I} = (\hat{I}_j^i) = \frac{1}{\hat{T}}, \\
R_0 = (p, 0), \quad b = (r^k, p_k), \quad \mu = 1, 2, \ldots, 6, \ k = 1, 2, 3.
\end{gathered}
\tag{5.7b}
$$

The "direct universality" of the above variational principle is evident due to the arbitrariness of the functional dependence of the isounit $\hat{I}$. Note also that, as part of this direct universality, action functionals of *arbitrary order* (i.e. dependent on accelerations or higher derivatives) can always be rewritten in the identical *first-order* isotopic form (i.e. dependent at most on the velocities).

Note again the achievement of invariance via the reconstruction on isospace over isofields of the *conventional canonical action* $A = \int_{t_1}^{t_2} dt[p_k \times dr^k/dt + H(t,r,p)]$ which, to our best knowledge at this writing, is *solely* permitted by the isodifferential calculus.

The invariant form of the *iso-Hamilton equations*[28(a),28(b)] (also called *Hamilton–Santilli isoequations*[8,30,31]) as characterized by the above isoaction principle, is given in the infinitesimal and finite forms by

$$\frac{\hat{d}b^\mu}{\hat{d}t} = \hat{\omega}^{\mu\nu} \hat{\times} \frac{\hat{\partial}H(t,b)}{\hat{\partial}b^\nu} = \omega^{\mu\nu} \times \frac{\hat{\partial}H(t,b)}{\hat{\partial}b^\nu}$$

$$= \omega^{\mu\nu} \times \hat{T}^\rho_\nu \times \frac{\partial H(t,b)}{\partial b^\rho}, \tag{5.8a}$$

$$b^\mu(t) = \left(e^{\hat{\omega}^{\alpha\beta} \hat{\times} \frac{\hat{\partial}H}{\hat{\partial}b^\beta} \hat{\times} \frac{\hat{\partial}}{\hat{\partial}b^\alpha}}\right) b^\mu(0), \tag{5.8b}$$

where one should keep in mind that the summation in the symplectic structure $\omega = dr^k \wedge dp_k$ is now generalized (because on iso-Euclidean spaces).

The fundamental algebraic brackets of the theory can then be written

$$[X \hat{,} Y] = \frac{\hat{\partial}X}{\hat{\partial}b^\mu} \hat{\times} \hat{\omega}^{\mu\nu} \hat{\times} \frac{\hat{\partial}Y}{\hat{\partial}b^\nu} = \frac{\hat{\partial}X}{\hat{\partial}b^\mu} \times \omega^{\mu\nu} \times \frac{\hat{\partial}Y}{\hat{\partial}b^\nu}, \tag{5.9}$$

which, as such, formally *coincide* with the conventional canonical Lie brackets, thus assuring the preservation of the Lie axioms in isospace (but generally *not* in their projection on conventional spaces).

Therefore, iso-Hamiltonian mechanics constitutes the classical realization of the Lie–Santilli isotheory.

We finally mention the invariant form of the *isotopic Hamilton–Jacobi equations*[28(a),28(b)] (also called *Hamilton–Jacobi–Santilli isoequations*[8,29])

$$\frac{\hat{\partial}\hat{A}}{\hat{\partial}t} + H(t,r,p) = 0, \qquad \frac{\hat{\partial}\hat{A}}{\hat{\partial}r^k} - p_k = 0, \qquad \frac{\hat{\partial}\hat{A}}{\hat{\partial}p_k} = 0. \tag{5.10}$$

Note that, while the conventional Hamiltonian mechanics in general, and Hamilton's equations in particular, can only represent in the fixed frame of the experimenter a rather small class of Newtonian systems,[7(a)] the Hamilton–Santilli isomechanics is directly universal for all possible iso-Newtonian systems possessing a conserved Hamiltonian. In particular, the above universality includes all possible (well-behaved) *nonlocal and nonpotential forces*, as desired.

Moreover, the construction of Hamiltonian representations (when they exist) of given local-differential Newtonian systems is quite complex, because it requires the solution of *nonlinear partial differential equations* (these are the *conditions of variational selfadjointness of the inverse Newtonian problem*[7(c)]). On the contrary, the construction of the iso-Hamilton representations from given nonlocal systems is quite simple because it requires the solution of generally overdetermined *algebraic* equations (see the memoir[28(a)] for brevity).

The reader should be aware that the construction of the iso-Hamiltonian mechanics was possible because of the prior construction, not only of the iso-Euclidean geometry indicated earlier, but also of the *isosymplectic geometry* we cannot possibly review here.[28(a)]

5.4. *Lie-isotopic branch of hadronic mechanics*

Following simple, yet unique and unambiguous isotopies of the naive or symplectic quantization,[28(a),28(b)] the operator image of the Hamilton–Jacobi–Santilli isoequations (5.10) characterize the following *iso-Schrödinger equations* formulated via the isodifferential calculus for the first time in Ref. 28(a) (see Ref. 7(f) for literature on the earlier formulations)

$$\begin{aligned} i\hat{\partial}_t|\hat{\phi}\rangle &= i\hat{T}_t \times \partial_t|\hat{\phi} = \hat{H} \hat{\times} |\hat{\phi}\rangle \\ &= \hat{H}(t,r,p) \times \hat{T}(t,r,p,\phi,\ldots) \times |\hat{\phi}\rangle \\ &= \hat{E} \hat{\times} |\hat{\phi}\rangle = E \times |\hat{\phi}\rangle\,, \end{aligned} \tag{5.11a}$$

$$\hat{p}_k \hat{\times} |\hat{\phi}\rangle = p_k \times \hat{T} \times |\hat{\phi}\rangle = -i\hat{\partial}_k|\hat{\phi}\rangle \doteq -i\hat{T}^i_k \times \partial_i|\hat{\phi}\rangle\,, \tag{5.11b}$$

$$\begin{aligned} [b^\mu \hat{,} b^\nu] \hat{\times} |\hat{\phi}\rangle &= (b^\mu \times \hat{T} \times b^\nu - b^\nu \times \hat{T} \times b^\mu) \times \hat{T} \times |\hat{\phi}\rangle \\ &= i\hat{\omega}^{\mu\nu} \hat{\times} |\hat{\phi}\rangle = i\omega^{\mu\nu} \times |\hat{\phi}\rangle\,, \end{aligned} \tag{5.11c}$$

which are now formulated on *iso-Hilbert spaces* $\hat{\mathcal{H}}$ with *isoinner product* $\langle\hat{\phi}| \times \hat{T} \times |\hat{\psi}\rangle \times \hat{I}$ and *isonormalization* $\langle\hat{\phi}| \times \hat{T} \times |\hat{\phi}\rangle \times \hat{I} = \hat{I}$ defined over the field of *isocomplex numbers* $\hat{C}(\hat{c}, +, \hat{\times})$ with elements $\hat{c} = c \times \hat{I}$, conventional sum $+$ and isoproduct $\hat{\times} = \times\hat{T}\times$.

The corresponding invariant form of the *iso-Heisenberg equations*, first proposed by Santilli[6(e)] in 1978 and first formulated in terms of the isodifferential calculus in Ref. 28(a), is given in the infinitesimal and finite forms by

$$\begin{aligned} i \times \frac{\hat{d}\hat{A}}{\hat{d}t} &= [\hat{A} \hat{,} \hat{H}] = \hat{A} \hat{\times} \hat{H} - \hat{H} \hat{\times} \hat{A} \\ &= \hat{A} \times \hat{T}\left(t, r, p, \phi, \frac{\partial \phi}{\partial r}, \ldots\right) \times \hat{H}(t,r,p) \\ &\quad - \hat{H}(t,r,p) \times \hat{T}\left(t, r, p, \phi, \frac{\partial \phi}{\partial r}, \ldots\right) \times \hat{A}\,, \end{aligned} \tag{5.12a}$$

$$\begin{aligned} \hat{A}(t) &= \hat{U} \hat{\times} \hat{A}(0) \hat{\times} U^\dagger = \hat{e}^{i\hat{H}\times t} \hat{\times} \hat{A}(0) \hat{\times} \hat{e}^{-it\times\hat{H}} \\ &= e^{i\hat{H}\times\hat{T}\times t} \times \hat{A}(0) \times e^{-it\times\hat{T}\times\hat{H}}\,, \end{aligned} \tag{5.12b}$$

and they can be easily proved to be equivalent to the iso-Schrödinger form.

Note that we have used in Eqs. (5.12b) the *isoexponentiation*, which is the lifting of the conventional exponentiation on the isoenvelope (thanks to the isotopies of the Poincaré–Birkhoff–Witt theorem first formulated in Ref. 6(d)) $\hat{e}^X = \hat{I} + X/1! + X \hat{\times} X/2! + \cdots = (e^{X\times\hat{T}}) \times \hat{I} = \hat{I} \times (e^{\hat{T}\times X})$.

It is evident that time evolution (5.12) characterizes the operator realization of the *Lie–Santilli isotheory*,[6–8,27–31] with the finite form characterizing a one-parameter isogroup of *isounitary transforms* on $\hat{\mathcal{H}}$ over $\hat{C}$,

$$\hat{U} \hat{\times} \hat{U}^{\dagger} = \hat{U}^{\dagger} \hat{\times} \hat{U} = \hat{I}\,. \tag{5.13}$$

The effects of isomathematics in the operator formulation now begin to be visible. In fact, the above operator theory is *directly universal* as its classical origin, although this time for all infinitely possible, well-behaved, nonlinear, nonlocal and nonpotential interactions with a conserved Hamiltonian (owing to the unrestricted functional dependence of the isotopic element $\hat{T}$). Yet, *the eigenvalues of the fundamental isocommutation rules* (5.11c) *coincides with the conventional quantum values* $i\omega^{\mu\nu}$.

The latter property signals the *abstract identity between hadronic and quantum mechanics*, including the preservation of all conventional quantum axioms and physical laws under said unrestricted non-Hamiltonian interactions. As an example, the preservation of Heisenberg's uncertainties can be easily proved from the preservation of the conventional eigenvalue of the isotopic commutator $[r^i \hat{,} p_j] = \delta^i_j$, and similar occurrences hold for the preservation of all other quantum principles, such as Pauli's exclusion principle.[28(b)]

Besides preserving *all* characteristics of quantum mechanics, as assured by the very nature of the isotopies, *isohermiticity coincides with conventional hermiticity* as the reader is encouraged to verify (see Ref. 28(b) for details). This implies that *all quantities that are observable for quantum mechanics remain observables for the isotopic branch of hadronic mechanics*, thus including the real-valuedness of the eigenvalues.

In actuality, *all observables of quantum mechanics are preserved identically in hadronic mechanics*, and are merely rewritten on isospaces over isofields. This occurrence can be first seen from the preservation completely unchanged of the conventional Hamiltonian as representing the total energy. Non-Hamiltonian effects are represented via *different operations*, because physical quantities such as linear momentum, angular momentum, energy, etc. cannot possibly be changed by the interactions.

The above occurrence is technically established by the Lie–Santilli isotheory in which conventional space–time and internal symmetries are subjected to isotopies while preserving identically the original generators (see, for brevity, Refs. 29 and 30(c)).

The abstract identity of hadronic and quantum mechanics can be best illustrated by the property that *the isoexpectation values of the isounit recover the conventional unit*[28(b)]

$$\frac{\langle\hat{\phi}| \hat{\times} \hat{I} \hat{\times} |\hat{\phi}\rangle}{\langle\hat{\phi}| \hat{\times} |\hat{\phi}\rangle} = I\,. \tag{5.14}$$

Despite the above property, the reader should keep in mind that the same operator H has different eigenvalues in quantum and hadronic mechanics, evidently because of the different eigenvalue equations $H \times |\phi\rangle = E_0 \times |\phi\rangle$ and $H \times T \times |\hat{\phi}\rangle = E \times |\hat{\phi}\rangle$, $E \neq E_0$.

As a result, *hadronic mechanics is a "completion" of quantum mechanics according to the historical argument by Einstein, Podolsky and Rosen*[32(a)] (see the title of the memoir[28(b)] and paper[29(g)]). In fact, the isoeigenvalue equations $H \hat{\times} |\hat{\phi}\rangle = E(\hat{T}) \times |\hat{\phi}\rangle$ constitute *an explicit and concrete realization of the theory of hidden variables*, evidently with $\lambda = \hat{T}$. In reality, hadronic mechanics provided an explicit and concrete *operator* realization of the theory of hidden variables.

Note that von Neumann's theorem[32(b)] and Bell's inequality[32(c)] *are not* applicable to hadronic mechanics, evidently because of its nonunitary structure. In fact, the isotopies of Bell's inequality do indeed admit classical counterparts,[29(g)] thus confirming the achievement of a "completion" of quantum mechanics according to the historical EPR argument.

We finally note that *hadronic mechanics is permitted by the following novel degree of freedom of Hilbert spaces* first identified in the memoir[28(b)] (here expressed for an isotopic element independent from the integration variables)

$$\langle\phi| \times |\psi\rangle \times I \equiv \langle\phi| \times T \times |\psi\rangle \times T^{-1} = \langle\phi| \hat{\times} |\psi\rangle \times \hat{I} . \tag{5.15}$$

The reader should not be surprised that the above degree of freedom of Hilbert spaces has remained unknown throughout this century, because its identification required the prior discovery of *new numbers*, those with arbitrary units.

5.5. *Simple construction of Lie-isotopic theories*

To complete our outline and upgrade of the antisymmetric branch of hadronic mechanics, the reader may be interested in knowing the existence of simple, yet unique and unambiguous means for the explicit construction of the above classical and operator isotheories.

As recalled earlier, the fundamental requirement of all classical (operator) deformations is that of being noncanonical (nonunitary) images of the original Hamiltonian theory. Another fundamental assumption of the isotheories is the *identification of the isounit precisely with said noncanonical (nonunitary) transforms*, according to the rule

$$U \times U^{\dagger} = \hat{I} \neq I , \tag{5.16}$$

where $\dagger$ represents transpose for the classical case and Hermitian conjugation for the operator counterpart.

The entire classical (operator) Lie-isotopic theory can then be constructed in its invariant form via the systematic application of the above noncanonical (nonunitary) transform, provided that it is applied to the *totality* of the original formalism. Again, any exception implies a mixture of conventional and generalized mathematics, with consequential inconsistencies.

In this way we have: the lifting of conventional real or complex numbers into the isonumbers, $n \rightarrow \hat{n} = U \times n \times U^{\dagger} = n \times (U \times U^{\dagger}) = n \times \hat{I}$; the lifting of the conventional associative product into the isoproduct $n \times m \rightarrow \hat{n} \hat{\times} \hat{m} = U \times (n \times m) \times U^{\dagger} = \hat{n} \times \hat{T} \times \hat{m}$, $\hat{T} = (U \times U^{\dagger})^{-1} = \hat{I}^{-1}$, under which $\hat{I}$ is the correct left and right unit of the theory; the lifting of the Hilbert product into the isoinner product $\langle\phi| \times |\psi\rangle \rightarrow U \times \langle\phi| \times |\psi\rangle \times U^{\dagger} = \langle\hat{\psi}| \times \hat{T} \times |\hat{\phi}\rangle \times \hat{I}$, $|\hat{\phi}\rangle = U \times |\phi\rangle$; and the same occurs for all other aspects, as the reader is encouraged to verify.

The above simple construction has resulted to have particular value for applications. As one illustrations among the several applications currently available,[28(b)] we mention that the first achievement of an explicitly *attractive* interactions between the *identical* electrons of the Cooper pair in superconductivity was reached precisely via a judicious yet simple selection of a nonunitary transform of the conventional Coulomb theory.

Note that, once achieved via a noncanonical (nonunitary) transform, isotheories *are not* invariant under additional noncanonical (nonunitary) transforms.

5.6. *Resolution of the problematic aspects of Theorem 1.1 for antisymmetric products*

The above classical and operator isotopies resolve all known problematic aspects of conventional deformations with antisymmetric brackets in the time evolution, as studied in details in Refs. 7(e), 7(f) and 28(b). Evidently these studies can only be outlined here for brevity.

First, *Lie-isotopic theories reconstruct linearity, locality and canonicity on isospaces over isofields.* This resolves all problematic aspects of conventional nonlinear, nonlocal and nonpotential theories with an antisymmetric algebra (Subsec. 4.2).

Secondly, isotopic theories are invariant under noncanonical (or nonunitary) transforms provided that they are treated via the isomathematics. This requires their reformulation in the isocanonical (or isounitary) form

$$U = \hat{U} \times \hat{T}^{1/2}, \qquad U \times U^{\dagger} = \hat{U} \hat{\times} \hat{U}^{\dagger} = U^{\dagger} \hat{\times} \hat{U} = \hat{I}\,. \tag{5.17}$$

In fact, it is easy to see that, at the classical level, *isocanonical transforms preserve the isocanonical tensor*, i.e. $\hat{U} \hat{\times} \hat{\omega} \hat{\times} \hat{U}^{t} = \hat{\omega}$ (see the memoir[28(a)] for a proof). Equivalently, the algebraic tensor of Hamilton–Santilli isomechanics is the *canonical* Lie tensor, just multiplied by the isounit, $\hat{\omega} = \omega \times \hat{I}$. The invariance of the isotheory is then ensured.

The invariance of the operator theory under isounitary transforms is then consequential. In fact we have: the *numerical invariance of the isounit* $\hat{I} \rightarrow \hat{I}' = \hat{U} \hat{\times} \hat{I} \hat{\times} \hat{U}^{\dagger} \equiv \hat{I}$; the invariance of the isoassociative product $\hat{A} \hat{\times} \hat{B} \rightarrow \hat{U} \hat{\times} (\hat{A} \hat{\times} \hat{B}) \hat{\times} \hat{U}^{\dagger} = A' \times \hat{T} \times B' = \hat{A}' \hat{\times} \hat{B}'$, with consequential invariance of the Lie–Santilli isoproduct; and similar invariances hold for all other aspects. The invariance of the fundamental dynamical equations (5.11) and (5.12) then follows, as *guaranteed* by the correct application of the isotopies.

The above features assure: the invariance of the basic units of time, space, energy, etc. thus permitting unambiguous applications to measurements; the preservation of the original hermiticity at all times, thus permitting physically acceptable observables; the uniqueness and invariance of special functions and transforms, with consequential uniqueness and invariance of the numerical predictions; the validity of causality and probability laws; and the validity of all conventional axiomatic properties.

Finally, the exact reconstruction of the axioms of the special relativity holds because (unlike the case of k and other deformations) *the isotopic images of the Minkowski space and of the Poincaré symmetry are isomorphic to the conventional forms.*[29] This property is not a mere mathematical curiosity, because it establishes that, contrary to a popular belief throughout this century, *the special relativity remains exactly valid under nonlinear, nonlocal and nonpotential interactions*, as well as *for arbitrary local speeds of light*, of course, when treated with the adequate mathematics (see Refs. 7(e) and 7(f) and the latest memoir[29(h)]).

The isotopies also permit the axiomatically consistent inclusion of gravitation in unified gauge theories of electroweak interactions[29(i)] and other applications in various fields too numerous to mention here.

A list of experimental verifications and predictions as of early 1997 is available in the memoir.[28(b)]

5.7. *Lie-isotopic reformulation of antisymmetric deformations*

The identical invariant reformulation of all possible, generalized classical theories with antisymmetric brackets in the time evolution can be readily done via the rules here expressed for Birkhoff's equations (3.2)

$$\frac{db^\mu}{dt} = \Omega(b)^{\mu\nu} \times \frac{\partial H(t,b)}{\partial b^\nu} = \omega^{\mu\nu} \times \frac{\hat{\partial} H(t,b)}{\hat{\partial} b^\nu}\,, \tag{5.18}$$

namely, by: (1) factorizing the conventional canonical Lie tensor in Birkhoff's (contravariant) tensor, $\Omega^{\mu\nu} = \omega^{\mu\rho} \times \hat{T}_\rho^\nu$; (2) by assuming the factor $\hat{T}$ as the inverse of the isounit; and (3) by embedding said factor in the differential calculus.

In this way, when formulated on isospaces over isofields (that is, when referred to the isounit $\hat{I}$), the theory becomes purely Hamiltonian, thus assuring the recovering of the original invariance. Non-Hamiltonian terms only emerge in the *projection* of the theory on conventional spaces over conventional fields (that is, when referred to the unit I).

The identical invariant isotopic reformulation of operator deformations with antisymmetric brackets is equally straightforward.

First, we recall Okubo's No-Quantization Theorem[27(a)] which prevents the equivalence of the Heisenberg-type and Schrödinger-type representations, as reviewed in Subsec. 4.7. This mandates the preservation of the associative character of the envelope, thus leaving its isoassociative realization as the only viable alternative.

In turn, the only known invariant formulation of isoassociative envelopes is that on isospaces over isofields, thus mandating the use of the isotopic branch of hadronic mechanics as the only invariant operator theory known to this author at this time.

Once these foundations are understood, the reformulation of operator deformations with antisymmetric products is straightforward. In fact, we have:

(A) Star deformations[19] remain completely unchanged in hadronic mechanics, and are merely reformulated on isospaces over isofields which then assure their invariance.

(B) Nonlinear theories[12] are also identically reformulated in hadronic mechanics via the mere embedding of all nonlinear terms in the isotopic element as per Eqs. (4.17). In this case the regaining of linearity on isospace over isofield guarantees the regaining of the superposition principle with consequential applicability of nonlinear theories to composite systems, as well as the regaining of the necessary topology to apply the imprimitivity theorem.[7(f)]

(C) Prigogine's nonunitary statistics[22] is also identically reformulated in hadronic mechanics, and merely expressed on isospaces over isofields which also guarantee its invariance.

(D) Squeezed states theories[21] are also identically reformulated as the preceding ones.

(E) Quantum groups require a redefinition for their formulation as invariant realizations of the Lie–Santilli isotheory consisting of: first, the identification of the generally *nonunitary* map from the quantum mechanical structure constants C_{ij}^k to the new, q-parameter dependent structure constants $D_{ij}^k(q)$, and the application of the same nonunitary map to the totality of the original theory, beginning with the generators and then including numbers, associative product, functional analysis, etc. As recalled in Sec. 4, in their current formulations, quantum groups are generally unitary, thus escaping the drawbacks of Theorem 1.1. However, their physical relevance is obscure due to the fact that they are a mixture of structure constants outside the class of equivalence of quantum mechanical symmetries with purely quantum mechanical generators and product.

Similar reformulation are possible for all other deformations with antisymmetric brackets in the time evolution, such as the axiomatically consistent treatment of *string theories* or *nonlocal interactions*, which is achieved via their embedding in the isounit of all nonlinear and nonlocal terms.

The reader should however be aware that, *even though the above isotopic reformulations are identical, the numerical predictions are generally altered.* The best illustration is given by squeezed states theories which have been constructed to reach deviations from Heisenberg's uncertainties via nonunitary transforms. However, once identically reformulated in terms of the isotopies, squeezed states theories verify Heisenberg's uncertainties in their entirely, as established by hadronic mechanics.[28(b)]

Particularly simple is the invariant Lie-isotopic reformulation of Riemannian theories of gravitation. In fact, the Riemannian metric in $3+1$ dimensions $g(x)$ is precisely a *noncanonical* image of the Minkowski metric m. The invariant iso-Minkowskian reformulation of the Riemannian geometry is then simply given by factoring the Minkowski metric in any given Riemannian metric, $g(x) = \hat{T}(x) \times m$, and by assuming $\hat{T}$ as the inverse of the new unit. The identical, invariant reformulation of gravity is then given by the reconstruction of the conventional Minkowskian geometry via the new isounit $\hat{I} = 1/\hat{T}$ (see Refs. 28(b) and 29(h) for details).

5.8. *Non-Hermitian genounits and genomathematics*

The fundamental assumption of the Lie-admissible branch of hadronic mechanics is the relaxation of the hermiticity of the basic unit, $\hat{I}^\dagger \neq \hat{I}$, while preserving its nonsingularity and the other conditions. This implies the existence of *two different generalized units* denoted and interconnected as follows:

$$^<\hat{I} = \frac{1}{\hat{P}}, \qquad \hat{I}^> = \frac{1}{\hat{Q}}, \tag{5.19a}$$

$$\hat{I}^> = (^<\hat{I})^\dagger . \tag{5.19b}$$

In turn, a theory with two different units necessarily requires the following, corresponding *ordering of the products*,

$$A < B = A \times \hat{P} \times B, \qquad A > B = A \times \hat{Q} \times B, \tag{5.20}$$

for which $\hat{I}^>$ and $^<\hat{I}$ are indeed left and right units

$$\hat{I}^> > A = A > \hat{I}^> = A, \qquad ^<\hat{I} < A = A < {^<\hat{I}} = A. \tag{5.21}$$

Under these conditions (only), the new units are called *genounits* while the related products are called *genoproducts*, where the prefix "geno" indicates the *genotopies* first introduced by Santilli[6(e)] in 1978 in the Greek meaning of "inducing covering theories." In fact, the isotopic theories are recovered identically for Hermitian generalized units, while conventional quantum theories are recovered identically when the Hermitian unit is the trivial value I.

The assumption of an ordering in the product then necessarily requires the construction, this time, of *two* isomathematics interconnected by Hermitian conjugation. The emerging dual isomathematics has the structure of an *isobimodule* and it is called *genomathematics* (see the latest account in the memoir[28(a)]).

In this way, we have: *genonumbers* $n^> = n \times \hat{I}^>$, $^<n = {^<\hat{I}} \times n$ with corresponding genoproducts (5.20), *genofields* $\hat{F}^>(\hat{n}^>, +, >)$ and $^<\hat{F}(^<\hat{n}, +, <)$ and *bigenofields* $\{^<\hat{F}, \hat{F}^>\}$; *geno-Hilbert spaces* $\hat{\mathcal{H}}^>$, $^<\hat{\mathcal{H}}$ on corresponding genocomplex numbers with *genoinner product* $^<\hat{I} \times \langle\hat{\phi}| \times \hat{P} \times |\psi\rangle$ and $\langle\hat{\phi}| \times \hat{Q} \times |\psi\rangle \times \hat{I}^>$, and their combination into the *bigeno-Hilbert spaces* $\{^<\hat{H}, \hat{\mathcal{H}}^>\}$; the *genodifferential calculus* with main rules $\hat{d}^> r^k = \hat{I}_i^{>k} \times dr^i$, $\hat{\partial}^> / \hat{\partial}^> r^k = \hat{Q}_k^i \times \partial / \partial r^i$, etc. (see Ref. 28(a) for details).

5.9. *Classical geno-Hamiltonian mechanics*

Genotopic theories were originally proposed for the *axiomatic treatment of the origin of irreversibility*. In fact, they are *structurally irreversible*, i.e. irreversible also for reversible Lagrangians or Hamiltonians.

The ordered product $>$ ($<$) with corresponding unit $\hat{I}^>$ ($^<\hat{I}$) and related formalism is generally assumed to characterize the *forward (backward) motion in time* with interconnection characterized by transpose. Irreversibility is then reduced to the most primitive possible setting, the difference of the two units and related products.

The physical foundations are the historical teaching by Lagrange and Hamilton. In fact, all known action-at-a-distance interactions are time-reversal invariant. The teaching by Lagrange and Hamilton is to *represent irreversibility via the external terms* in their equations. Genotopic theories merely represent this historical teaching in the only invariant way known at this writing, that via real-valued nonsymmetric generalized units.

As it was the case for the iso-Hamiltonian mechanics, the invariant reformulation of the Hamilton- and Birkhoff-admissibles mechanics of Sec. 3 implied a structurally novel mechanics first identified in the memoir.[28(a)] Evidently, we can here review only the main lines. Assume an ordering of time, say, the forward one. We then have [*loc. cit.*]:

(1) the *geno-Newton equations*

$$\hat{m}^> >_t \times_t^> \frac{\hat{d}^> \hat{v}_k^>}{\hat{d}^> t^>} = -\frac{\hat{\partial}^> \hat{V}^>}{\hat{\partial}^> r^> k}, \tag{5.22}$$

where $\hat{m}^> = m \times \hat{I}_t^>$, $>_t = \times \hat{Q}_t \times$, $\hat{v}_k^> = \hat{d}^> \hat{r}_k^> / \hat{d}^> t^> = \hat{Q}_t \times d(\hat{Q}^>{}_k{}^i(t, r, v, \ldots) \times r_i^> / dt^>)$, and $\hat{\partial}^> \hat{V}^> / \hat{\partial}^> r^{>k} = \hat{Q}_k^i \times \partial V / \partial r^{>i}$);

(2) the *genoaction principle*

$$\begin{aligned}\hat{A}^> &= \int_{t_1^>}^{t_2^>} dt^> \left[p_k^> \times \frac{\hat{d}^> r^{>k}}{\hat{d}^> t^>} + H^>(t, r, p) \right] \\ &= \int_{t_1^>}^{t_2^>} dt^> \left[p_k \times \hat{I}_i^{>k} \times \frac{dr^{>i}}{dt^>} + H^>(t, r, p) \right] \\ &= \int_{t_1^>}^{t_2^>} dt > \left[\hat{R}_{0\mu}^>(b^>) > \frac{\hat{d}^> b^{>\mu}}{\hat{d}^> t^>} + H^>(t, b) \right] \\ &= \int_{t_1^>}^{t_2^>} dt \left[R_{0\mu}^>(b^>) \times \hat{I}_\nu^{>\mu} \times \frac{db^{>\nu}}{dt^>} + H^>(t, b) \right], \end{aligned} \tag{5.23a}$$

$$\begin{gathered}(\hat{I}_\nu^{>\mu}) = \mathrm{Diag}(\hat{I}^>, \hat{Q}), \quad \hat{I}^> = \hat{I}_j^{>i} = \frac{1}{\hat{Q}}, \quad R_0^> = (p^>, 0), \\ b^> = (r^{>k}, p_k^>), \quad \mu = 1, 2, \ldots, 6, \quad k = 1, 2, 3; \end{gathered} \tag{5.23b}$$

(3) the *geno-Hamilton equations* (also called the *Hamilton–Santilli genoequations*[8,30]) here written in their infinitesimal and finite forms

$$\frac{\hat{d}^{>}b^{>\mu}}{\hat{d}^{>}t^{>}} = \hat{\omega}^{>\mu\nu} > \frac{\hat{\partial}^{>}H^{>}(t,b)}{\hat{\partial}^{>}b^{>\nu}} = \omega^{\mu\nu} \times \frac{\hat{\partial}^{>}H^{>}(t,b)}{\hat{\partial}^{>}b^{>\nu}} = \omega^{\mu\nu} \times \hat{Q}^{\rho}_{\nu} \times \frac{\partial H^{>}(t,b)}{\partial b^{>\rho}}, \tag{5.24a}$$

$$b^{>\mu}(t^{>}) = \left(e^{\hat{\omega}^{>\alpha\beta}> \frac{\hat{\partial}^{>}H^{>}}{\hat{\partial}^{>}b^{>\beta}} > \frac{\hat{\partial}^{>}}{\hat{\partial}^{>}b^{>\alpha}}}\right) b^{>\mu}(0)\,; \tag{5.24b}$$

(4) the *genobrackets* characterized by the preceding equations

$$(X\hat{,}Y) = \frac{\hat{\partial}^{>}X}{\hat{\partial}^{>}b^{>\mu}} > \hat{\omega}^{>\mu\nu} > \frac{\hat{\partial}^{>}Y}{\hat{\partial}^{>}b^{>\nu}} = \frac{\hat{\partial}^{>}X}{\hat{\partial}^{>}b^{>\mu}} \times \omega^{\mu\nu} \times \frac{\hat{\partial}^{>}Y}{\hat{\partial}^{>}b^{>\nu}}\,; \tag{5.25}$$

(5) the *geno-Hamilton–Jacobi equations*

$$\frac{\hat{\partial}^{>}\hat{A}^{>}}{\hat{\partial}^{>}t^{>}} + H^{>}(t,r,p) = 0\,, \qquad \frac{\hat{\partial}^{>}\hat{A}^{>}}{\hat{\partial}^{>}r^{>k}} - p^{>}_{k} = 0\,, \frac{\hat{\partial}^{>}\hat{A}^{>}}{\hat{\partial}^{>}p^{>}_{k}} = 0\,; \tag{5.26}$$

with corresponding backward genoequations.

The above equations are also directly universal as it is the case for the iso-Hamiltonian particularization. The main difference is that the former constitute an open, nonconservative and irreversible system, while the latter are still irreversible, but are closed-isolated, i.e. they possess total conserved energy.

The reader should be aware that the construction of geno-Hamiltonian mechanics required the prior construction of the *geno-Euclidean and genosymplectic geometries* we cannot possibly review here for brevity. For numerous examples and additional aspects, the interested reader may consult Refs. 7(e) and 28(a).

5.10. *Lie-admissible branch of hadronic mechanics*

A simple genotopy of the conventional quantization, or just the relaxation of hermiticity of the isounit in the isoquantization, yields in a unique and unambiguous way the *forward geno-Schrödinger equations* (see Refs. 7(f), 28(a) and 28(b) for original contributions)

$$\begin{aligned} i\hat{\partial}^{>}_{t}|\hat{\phi}^{>}\rangle &= i\hat{Q}_t \times \partial_t|\hat{\phi}^{>}\rangle = H^{>} > |\hat{\phi}^{>}\rangle \\ &= H(t,r,p) \times \hat{Q}\left(t, r, p, \phi, \frac{\partial \phi}{\partial r}, \dots\right) \times |\hat{\phi}^{>}\rangle \\ &= E^{>} > |\hat{\phi}^{>}\rangle = E \times |\hat{\phi}^{>}\rangle\,, \end{aligned} \tag{5.27a}$$

$$p^{>}_{k} > |\hat{\phi}^{>}\rangle = p_k \times \hat{Q} \times |\hat{\phi}^{>}\rangle = -i\hat{\partial}^{>}_{k}|\hat{\phi}^{>}\rangle = -i\hat{Q}^{i}_{k} \times \partial_i|\hat{\phi}^{>}\rangle\,, \tag{5.27b}$$

which are now defined on the *forward geno-Hilbert space* $\hat{\mathcal{H}}^{>}$ with *genostates* $|\hat{\phi}^{>}\rangle$ and *forward genoinner product* $\langle^{<}\hat{\phi}| \times Q \times |\hat{\psi}^{>}\rangle \times \hat{I}^{>}$ defined over the *genofield* $\hat{C}^{>}(\hat{c}^{>}, +, >$ with *genocomplex numbers* $\hat{c}^{>} = c \times \hat{I}^{>}$.

The equivalent *geno-Heisenberg equations*, first proposed in Ref. 6(e), can be written following the genomathematics of Ref. 28(a) in their finite and infinitesimal form

$$\begin{aligned} A(t) &= e_{>}^{iH\times t} > A(0) < e_{<}^{-it\times H} \\ &= e^{iH\times\hat{Q}\times t} \times A(0) \times e^{-it\hat{P}\times H}\,, \end{aligned} \tag{5.28a}$$

$$\begin{aligned} \frac{i\,\hat{d}A}{\hat{d}t} &= i\hat{W}_t \times \frac{dA}{dt} = (A, H) = A\langle H - H\rangle A \\ &= A \times \hat{P}\left(t, r, p, \phi, \frac{\partial\phi}{\partial r}, \dots\right) \times H(t, r, p) \\ &\quad - H(t, r, p) \times \hat{Q}\left(t, r, p, \phi, \frac{\partial\phi}{\partial r}, \dots\right) \times A\,, \end{aligned} \tag{5.28b}$$

$$(b^{\mu}, b^{\nu}) = \omega^{\mu\nu} \times \hat{W}\,, \qquad b = (r^{k}, p_{k})\,, \tag{5.28c}$$

where $e_{>}$ and $e_{<}$ are exponentiations in the corresponding genoenvelopes, $\hat{W}$ and $\hat{W}_t$ can be either the forward or the backward genotopic elements of space and time, respectively.

Note that, while the forward (backward) geno-Schrödinger equations are definite on spaces $\hat{\mathcal{H}}^{>}$ $(^{<}\hat{\mathcal{H}})$, the geno-Heisenberg equations are defined instead on the *bigenomodular spaces* $\{^{<}\hat{\mathcal{H}}, \hat{\mathcal{H}}^{>}\}$. This is due to their inclusion of both forward and backward actions.

Note that, as pointed out earlier, the same bimodular character exists in the conventional and isotopic Heisenberg equations. In the latter cases the bimodular structure can be correctly ignored because the forward and backward products coincide.

It is an instructive exercise for the interested reader to verify that all axiomatic properties of the Lie-isotopic branch of hadronic mechanics persist under the above genotopies.

We should note in particular the preservation on genospaces over genofields of conventional quantum laws, such as Heisenberg's uncertainty, Pauli's exclusion principle, etc. as evident from the preservation of the conventional eigenvalues of the genocommutator rules (5.28c). In particular, the Lie-admissible branch of hadronic mechanics is also a "completion" of quantum mechanics along the celebrated argument by Einstein, Podolsky and Rosen.[29(g),32]

Most intriguingly, the formalism provides the first occurrence known to this author according to which *the nonconserved Hamiltonian H is Hermitian, thus observable.* In fact, the preservation of the original hermiticity of H under genotopies can be easily proved, and so does its conservation from the geno-Heisenberg equations for which $idH/dt = H \times (\hat{P} \times -\hat{Q}) \times H \neq 0$. By comparison, in all existing nonconservative models the Hamiltonian is not Hermitian. But then, it should not be considered as observable.

5.11. *Simple construction of Lie-admissible theories*

The reader should be aware that, despite their seemingly complex mathematical structure, classical (operator) Lie-admissible formulations can be constructed in all their aspects via the simple application of *two* noncanonical (nonunitary) transforms

$$V \times V^\dagger \neq I, \qquad W \times W^\dagger \neq I, \tag{5.29}$$

to the totality of the original canonical theory, according to the rules

$$V \times W^\dagger = \hat{I}^>, \quad {}^<\hat{I} = W \times V^\dagger, \quad \hat{I}^> = ({}^<\hat{I})^\dagger, \tag{5.30a}$$

$$V \times (A \times B) \times W^\dagger = A' > B', \quad W \times (A \times B) \times V^\dagger = A' < B', \tag{5.30b}$$

and the same happens for all other aspects, as the reader is encouraged to verify.

Note that, after having reached a Lie-admissible structure via the above simple method, the same structure *is not* invariant under the same transforms, e.g. because $\hat{I}^> \to \hat{I}'^> = V \times \hat{I}^> \times W^\dagger \neq \hat{I}^>$. The same noninvariance exists for other cases.

5.12. *Resolution of the problematic aspects of Theorem 1.1 for nonantisymmetric products*

As it is the case for the Lie-isotopic theories, it is easy to see that classical (operator) Lie-admissible formulations are invariant under noncanonical (nonunitary) transforms, provided that they are formulated for one fixed direction of time and expressed in the related genomathematics.

This requires the reformulation of any noncanonical (nonunitary) transform in the following *forward genocanonical (genounitary) transform*

$$U \times U^\dagger = \hat{I}^> \neq I, \quad U = \hat{U}^> \times \hat{Q}^{1/2}, \tag{5.31a}$$

$$U \times U^\dagger = \hat{U}^> > \hat{U}^{>\dagger} = U^{>\dagger} > \hat{U}^> = \hat{I}^>, \tag{5.31b}$$

with similar expressions for the backward case, where again $\dagger$ presents transpose for the classical case and Hermitian conjugation for the operator case.

The classical or operator Lie-admissible theories are then invariant. In fact, each genounit is *numerically preserved* by the corresponding genounitary transforms, $\hat{I}^> \to \hat{I}'^> = \hat{U}^> > \hat{I}^> > \hat{U}^{>\dagger} \equiv \hat{I}^>$; genoassociative products are invariant; and the same occurs for genonumbers, geno-Hilbert spaces, etc. as the reader is encouraged to verify. The invariance of the geno-Heisenberg equations follows from their bi-modular character, thus requiring the use of both forward and backward genotransforms for the corresponding products.

The above invariance properties, as well as the preservation of the main axiomatic structure and physical laws of quantum mechanics, permit the resolution of all the shortcomings of Theorem 1.1 studied in Secs. 3 and 4. The exact reconstruction of the special relativity is currently under study on bimodular genospaces over bimodular genofields.[28(c)]

5.13. *Invariant Lie-admissible reformulation of nonantisymmetric deformations*

It is easy to see that the classical noninvariant Hamilton- or Birkhoff-admissible equations (3.6a) can be *identically* reformulated in terms of our geno-Hamilton equations (5.24), according to the rules:

$$\frac{db^\mu}{dt} = S^{\mu\nu} \times \frac{\partial H(t,b)}{\partial b^\nu} \equiv \omega^{\mu\nu} \times \frac{\hat{\partial}^> H(t,b)}{\hat{\partial}^> b^\nu} \tag{5.32}$$

which, in a way similar to the reformulation of Birkhoff's equations, Eqs. (5.18), requires: the factorization of the conventional canonical Lie tensor in the Lie-admissible tensor, $S^{\mu\nu} = \omega^{\mu\rho} \times \hat{T}_\rho^{>\nu}$; the assumption of the remaining factor $T^>$ as the inverse of the forward genounit; and (3) the embedding of the factor $\hat{T}^>$ in the forward genodifferential calculus.

In all earlier formulations of classical Lie-admissible theories up to 1996 the Lie-admissible tensor was decomposed into the *sum* $S^{\mu\nu} = \omega^{\mu\nu} + s^{\mu\nu}$. This permitted the regaining of a consistent algebra in Hamilton's historical equations, those with external terms, but the emerging theory remained noninvariant, thus not suitable for physical applications, because treated with the conventional mathematics of Hamiltonian mechanics.

In reformulation (5.18) the Lie-admissible tensor must be decomposed into the *product* $S = \omega \times Q$, where Q is nonsymmetric. Invariance then requires the construction of the new genomathematics for the nonsymmetric unit $\hat{I}^> = 1/Q$ without any other alternative known to this author.

The *identical* reformulation of operator deformations with a nonantisymmetric product is equally straightforward. First, it is easy to see why q- or (p,q)-deformations *cannot* be invariant when formulated on a conventional space over a conventional field. In fact, these deformations change the product of the enveloping algebra $A \times B \to q \times A \times B$ *while preserving the original unit.* The lack of invariance is then an unavoidable consequence because the old unit is no longer invariant for the theory based on the new product. This indicates the reason why, after proposing these deformations back in 1967, Santilli[6(a)] had abandoned their conventional study by the time of the appearance of the simpler versions by Biedenharn[15(a)] and Macfarlane.[15(b)]

The Lie-admissible reformulation of (p,q)-deformations requires that, jointly with the liftings of the products of the envelope, the units are lifted by an amount which is the *inverse* as that of the deformations,

$$A \times B \to p \times A \times B\,, \qquad I \to {}^<\hat{I} = \frac{1}{p}\,, \tag{5.33a}$$

$$A \times B \to q \times A \times B\,, \qquad I \to \hat{I}^> = \frac{1}{q}\,. \tag{5.33b}$$

The above reformulation assures the invariance of the basic units and of the related theory *ab initio*, provided that, again, the transformation theory is reconstructed with respect to the above new genounits.

It should be noted that the emerging Lie-admissible theory is mathematically invariant, yet still with physical shortcomings because of the general violation of the crucial conjugation $\hat{I}^{>} = ({}^{<}\hat{I})^{\dagger}$ which is necessary for a physically consistent conjugation under time reversal (see also the concluding remarks of the next subsection). The latter condition is manifestly violated by the q-deformations and restricts the (p,q)-deformations to only those with *complex* parameters p and q such that $p = \bar{q}$.

Moreover, the p and q parameters are lifted into matrices under the time evolution of the theory on conventional spaces over conventional fields, and this explains the reason for passing from the (p,q)-parameter deformations[6(a)] to the (P,Q)-operator deformations[6(e)] under the conjugation $P = Q^{\dagger}$.

All other deformations with nonantisymmetric brackets are reformulated along similar lines. For instance, supersymmetric theories[25] have brackets which are a particular case of the (p,q)-deformations, according to the rules

$$\begin{aligned} (A,B) &= a \times (A \times B - B \times A) \\ &\quad + b \times (A \times B + B \times A) \\ &= p \times A \times B - q \times B \times A\,, \\ p &= a + b\,, \qquad q = a - b\,. \end{aligned} \tag{5.34}$$

Exactly the same occurrence holds for Kac–Moody superalgebras,[27] although with different values of the parameters.

Note that the identical Lie-admissible reformulation of the supersymmetric, Kac–Moody and other parameter deformations permits the achievement of their invariant. Nevertheless, the same theories remain insufficient for physical applications for the reasons indicated above, namely, the insufficiency of the mixture of commutator and anticommutators via *parameters* in favor of their mixture via *operators*, as well as the need of a conjugation for physical consistency under time reversal.

The interested reader can easily work out additional reformulations.

5.14. *Origin of Lie-isotopic and Lie-admissible formulations*

To understand the Lie-isotopic and Lie-admissible formulations to a sufficient depth, it may be recommendable to recall that they originate from the very structure of Lie's theory (see the forthcoming English translation of Lie's celebrated thesis[33]).

In fact, as identified in the original proposal[6(d),6(e)] and recalled earlier in this paper, a unitary Lie group has precisely the structure of a bi-module with an action from the left $U^{>} = e^{iH \times t}$ and an action from the right ${}^{<}U = e^{-it \times H}$

interconnected by Hermitian conjugation, and such a structure exists in both the finite and infinitesimal forms

$$A(t) = U^{>} > A(0) < {}^{<}U = e^{iH\times t} > A(0) < e^{-it\times H}\,, \tag{5.35a}$$

$$\frac{idA}{dt} = A < H - H > A\,, \tag{5.35b}$$

$${}^{<}U = (U^{>})^{\dagger}\,. \tag{5.35c}$$

In this case both products $A < B$ and $A > B$ are evidently conventional associative products, $A < B = A > B = A \times B$. The point is that axiomatic structure (5.35) *does not* require that such products have necessarily to be conventionally associative, because they can also be isoassociative, thus yielding the Lie-isotopic formulations. Moreover, axioms (5.35) *do not* require that the forward and backward isoassociative products have to be necessarily the same, because they can also be different, provided that conjugation (5.35) is met, in which case they yield the Lie-admissible formulations.

It then follows that *the abstract axioms of Lie's theory and Santilli's Lie-isotopic and Lie-admissible theories coincide*, the latter merely being *broader realizations* of the former. The basic property is the preservation of the abstract unity which holds to such an extent that we could have presented both the Lie-isotopic and the Lie-admissible formulations with the same abstract symbols used for Lie's theory.

The axiomatic consistency and invariance of the Lie-isotopic and Lie-admissible theories can then be derived from that of the Lie's theory, of course, when treated with the appropriate mathematics, that is, the mathematics leaving invariant the applicable units. The only applicable mathematics are then the iso- and geno-mathematics.

The identity of the abstract axioms of the Lie and Lie-admissible theories is confirmed by the property that, contrary to a widespread impression, *the Lie-admissible brackets* $(A, B) = A < B - B > A$ *are indeed antisymmetric and they do indeed verify the Lie axioms on genospaces over genofields.* In fact, these brackets are neither antisymmetry nor symmetric when computed on conventional spaces over conventional fields. The same brackets become antisymmetric when the products $A < B = A \times \hat{P} \times B$ and $A > B = A \times \hat{Q} \times B$ are computed with respect to units which are the *inverse* of the underlying deformations, $\hat{I}^{>} = 1/\hat{Q}$ and ${}^{<}\hat{I} = 1/\hat{P}$, respectively.

Alternatively, one can see that *the classical counterparts of the Lie-admissible brackets, Eqs.* (5.25), *are indeed transparently antisymmetric and fully Lie on genospaces over genofields,* and acquire their non-Lie/Lie-admissible form only in their projection on conventional spaces over conventional fields (see Ref. 28(c) for details).

The conclusion reached by this author after some thirty years of investigations of the topics treated in this paper is that *the sole known invariant formulations of classical and operator deformations are those capable of preserving the abstract axioms* (5.35) *of Lie's theory.*[33]

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