

Representation Theory of a New Relativistic Dynamical Group (*).

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(ricevuto il 9 Dicembre 1970)

Summary. — Via the method of induced representations, all irreducible unitary projective representations of the recently introduced new relativistic dynamical group \mathcal{G}_5 are deduced and classified. An explicit form of the transformation law is given. The properties of the corresponding infinite-dimensional basis functions are studied. It is shown that in the limiting case of $l = \infty$ (corresponding to $\tilde{\mathcal{G}}_5 \rightarrow \mathcal{G}_5$) the infinite spin-tower representations become reducible and decompose into irreducible representations of the Poincaré group. The reduction of the direct product of two irreducible unitary ray representations of $\tilde{\mathcal{G}}_5$ is studied. The Clebsch-Gordan coefficients are computed. Finally, some comments on the physical interpretation of the results are given.

1. — Introduction.

In a previous publication ⁽¹⁾ we introduced a new symmetry group (denoted by \mathcal{G}_5) for relativistic dynamics. This group acts on the Cartesian product space $\mathcal{E}_{3,1} \times \mathcal{E}_1$, where $\mathcal{E}_{3,1}$ is the Minkowski space with points x^μ and \mathcal{E}_1 is a one-dimensional manifold with points denoted by u . As was indicated in ref. ⁽¹⁾ and discussed in greater detail in a subsequent publication ⁽²⁾, the new kinematical variable u must be interpreted as the proper time. The defining trans-

(*) Work supported by the U.S. Air Force under Grant No. AFOSR-67-0385B.

⁽¹⁾ J. J. AGHASSI, P. ROMAN and R. M. SANTILLI: *Phys. Rev. D*, **I**, 2753 (1970).

⁽²⁾ J. J. AGHASSI, P. ROMAN and R. M. SANTILLI: *Journ. Math. Phys.*, **II**, 2297 (1970).

formations ⁽³⁾ of \mathcal{G}_5 are

$$(1.1) \quad x'_\mu = A'_\mu{}^\nu x_\nu + b_\mu u + a_\mu, \quad u' = u + \sigma.$$

Here A'_μ is a restricted Lorentz matrix, a_μ a constant translation vector, σ a constant scalar. The transformations associated with the constant vector b_μ are analogous to the boost (velocity) transformations of the nonrelativistic Galilei group. We call these the «zest» transformations. In obvious notation, the structure of \mathcal{G}_5 is as follows:

$$(1.2) \quad \mathcal{G}_5 = \{T_4^a \times T_1^\sigma\} \rtimes \{T_4^b \rtimes SO_{0(3,1)}\},$$

where \times and \rtimes denote direct and semi-direct products, respectively. Thus, \mathcal{G}_5 contains as a subgroup both the restricted Poincaré group and the non-relativistic Galilei group. Moreover, \mathcal{G}_5 is a group extension ⁽⁴⁾ of the restricted Lorentz group $SO_{0(3,1)}$. From these comments it follows that, on the one hand, our \mathcal{G}_5 is a natural generalization of the Poincaré group and, on the other hand, it is also a natural generalization of the nonrelativistic Galilei group.

In ref. ⁽¹⁾ it was pointed out that for the use in relativistic *quantum* mechanics, the central extension ⁽⁴⁾ of the covering group of \mathcal{G}_5 by a one-dimensional Abelian phase group T_1^0 must be used. This new relativistic quantum-mechanical dynamical group will be denoted by $\tilde{\mathcal{G}}_5$ and its structure is

$$(1.3) \quad \tilde{\mathcal{G}}_5 = \{T_4^a \times T_1^\sigma\} \rtimes \{T_4^b \rtimes (SL_{2,\sigma} \times T_1^0)\},$$

where $SL_{2,\sigma}$ appears as the covering of $SO_{0(3,1)}$.

The generators of $\tilde{\mathcal{G}}_5$ are denoted by $J_{\mu\nu}$, P_μ , Q_μ , S and they generate the subgroup $SL_{2,\sigma}$, T_4^a , T_4^b , T_1^σ , respectively. Since we shall not need them in this paper, we do not write out here the Lie algebra ⁽⁵⁾ in full. But we recall the most important relation, *viz.* ⁽⁶⁾

$$(1.4) \quad [P_\mu, Q_\nu] = -ig_{\mu\nu} l^{-1}.$$

Here the real constant l has the dimension of length and its appearance is connected with the phase group T_1^0 .

⁽³⁾ In a recent private communication M. NOGA (Purdue University) gave an alternative derivation of our group, emphasizing that it is actually the dynamical group of the standard equation of motion in relativistic mechanics. See also ref. ⁽⁷⁶⁾.

⁽⁴⁾ See ref. ⁽¹⁾, Appendix A.

⁽⁵⁾ See ref. ⁽¹⁾, eqs. (3.7) through (3.12).

⁽⁶⁾ We use the Minkowski metric $g_{00} = -g_{kk} = 1$. Note, incidentally, that the full carrier space $E_{3,1} \times E_1$ is not a metric space.

The Casimir operators of $\tilde{\mathcal{G}}_5$ are

$$(1.5a) \quad \mathcal{D} = P_\mu P^\mu + 2l^{-1}S,$$

$$(1.5b) \quad \mathcal{J} = \frac{1}{2}T_{\mu\nu}T^{\mu\nu},$$

$$(1.5c) \quad \mathcal{K} = \frac{1}{4}\varepsilon_{\mu\nu\varrho\sigma}T^{\mu\nu}T^{\varrho\sigma},$$

where

$$(1.6) \quad T_{\mu\nu} \equiv J_{\mu\nu} - lM_{\mu\nu}$$

with

$$(1.6a) \quad M_{\mu\nu} \equiv P_\mu Q_\nu - P_\nu Q_\mu.$$

Of course, in addition to \mathcal{D} , \mathcal{J} , \mathcal{K} , the operator $l\mathbb{1}$ is also an invariant of our group. As is well known (see, for example, ref. (7)) this leads to a superselection rule.

In ref. (1) we showed that $X_\mu \equiv -lQ_\mu$ is a perfectly acceptable relativistic space-time position operator (8) and $\mathcal{M}^2 \equiv -2l^{-1}S$ is a nontrivial relativistic mass operator. S also plays the role of an evolution operator with respect to proper time. Some other physical consequences of $\tilde{\mathcal{G}}_5$ were also explored (1,2), and finally we showed (2) that $\tilde{\mathcal{G}}_5$ is the contracted limit of the covering of the connected component of the inhomogeneous de Sitter group $ISO_{3,2}$.

The main purpose of the present paper is to study in detail and with sufficient mathematical rigor the representations of $\tilde{\mathcal{G}}_5$. We find this study crucial, because all further applications of $\tilde{\mathcal{G}}_5$ depend critically on the thorough understanding of the representations (9). Apart from this, the representation theory of $\tilde{\mathcal{G}}_5$ merits study from the purely mathematical point of view. The group has a sufficiently interesting structure (cf. (1.3)) and the mathematics involved is far from being trivial. It is true that there are some similarities with the nonrelativistic Galilei group, but in the present case the little group (see Subsect. 2'4) is noncompact; this makes the theory quite involved.

In Sect. 2 and 3 we systematically derive all irreducible unitary projective representations of \mathcal{G}_5 , in an explicit form. In Sect. 4 we study separately the $l = \infty$ limiting case, which corresponds (10) to replacing $\tilde{\mathcal{G}}_5$ by \mathcal{G}_5 . In Sect. 5 we give a discussion of the products of representations and their reduction. This turns out to be a rather involved problem. In Sect. 6 we discuss additional features of our group, pointing out also some problematic aspects.

(7) V. BARGMANN: *Ann. Math.*, **59**, 1 (1954).

(8) In this respect, see also J. E. JOHNSON: *Phys. Rev.*, **181**, 1755 (1969); L. CASTELL: *Nuovo Cimento*, **49 A**, 285 (1967).

(9) Among other things, we have in mind the establishing of wave equations for arbitrary spin.

(10) See ref. (1), Appendix C.

The main mathematical tool used in this paper will be the method of induced representations, developed by MACKEY⁽¹¹⁾. Actually, some parts of our calculations parallel rather closely the work of VOISIN⁽¹²⁾, who used Mackey's method to study the ray representations of the nonrelativistic Galilei group⁽¹³⁾.

2. - Some algebraic preliminaries.

2.1. *Factor system.* - Let us represent a generic element g of $\tilde{\mathcal{G}}_5$ by

$$(2.1) \quad g = (\exp [i\theta]; \sigma, a, b, \mathcal{A}),$$

where $\sigma, a, b, \mathcal{A}$ stand for the parameters in (1.1) and θ is the phase associated with the T_1^0 subgroup. As we already stated in ref. (1), the composition law of $\tilde{\mathcal{G}}_5$ can be written as

$$(2.2) \quad \begin{aligned} g_2 g_1 &\equiv (\exp [i\theta_2]; \sigma_2, a_2, b_2, \mathcal{A}_2) (\exp [i\theta_1]; \sigma_1, a_1, b_1, \mathcal{A}_1) = \\ &= (\omega(g_2, g_1) \exp [i(\theta_2 + \theta_1)]; \sigma_2 + \sigma_1, a_2 + \mathcal{A}_2 a_1 + \sigma_1 b_2, b_2 + \mathcal{A}_2 b_1, \mathcal{A}_2 \mathcal{A}_1). \end{aligned}$$

Here

$$(2.3) \quad \omega(g_2, g_1) \equiv \exp [if(g_2, g_1)]$$

is a phase factor (f is real), called the factor system⁽¹⁴⁾, which arises from the scalar extension of \mathcal{G}_5 to $\tilde{\mathcal{G}}_5$. Its appearance in (2.2) has deep implications for the representation theory of $\tilde{\mathcal{G}}_5$. Let us consider a homomorphism

$$(2.4) \quad g \rightarrow \mathcal{U}_g$$

from $\tilde{\mathcal{G}}_5$ to a family of unitary operators. The multiplication law for these

⁽¹¹⁾ A very readable account of this powerful tool can be found in G. W. MACKEY: *Induced Representations of Groups and Quantum Mechanics* (New York, 1968). A shorter, but more rigorous summary is given in G. W. MACKEY: *Group representations in Hilbert space*, which is the Appendix in I. E. SEGAL: *Mathematical Problems in Relativistic Physics* (New York, 1963). The latter contains also a bibliography of original publications.

⁽¹²⁾ J. VOISIN: *Journ. Math. Phys.*, **6**, 1519, 1822 (1965).

⁽¹³⁾ An alternative, somewhat more intuitive treatment of the ray representations of the nonrelativistic Galilei group was given by J.-M. LÉVY-LEBLOND: *Journ. Math. Phys.*, **4**, 776 (1963). Some parts of our calculations are analogous to those of LÉVY-LEBLOND.

⁽¹⁴⁾ See Appendix A of ref. (1).

operators corresponding to the composition law (2.2) is

$$(2.5) \quad \mathcal{U}_{g_2} \mathcal{U}_{g_1} = \omega(g_2, g_1) \mathcal{U}_{g_2 g_1}.$$

As follows from the general theory of nontrivial central extension of groups (?), the phase ω in (2.5) is essential and cannot be eliminated by a redefinition $\mathcal{U}_g \rightarrow \tau(g) \mathcal{U}_g$, $|\tau(g)| = 1$, of the operators \mathcal{U}_g .

The explicit determination of ω is done by applying (2.5) onto the state vector of the one-dimensional representation. This will be shown at the end of the Appendix. The result of the calculation is that f (defined by (2.3)) is given by

$$(2.6) \quad f(g_2, g_1) = -l^{-1}(b_2 A_2 a_1 + \frac{1}{2} b_2^2 \sigma_1),$$

where l is the constant appearing in (1.4) and (1.5a). Thus, f or ω depends only on the translation part of g_1 and on the « homogeneous » part of g_2 .

Finally, we note that the unit element of $\tilde{\mathcal{G}}_5$ is $(1; 0, 0, 0, 1)$ and hence the inverse element g^{-1} is given by

$$(2.7) \quad g^{-1} = (\exp[-i(\theta + \hat{f})]; -\sigma, -A^{-1}(a - b\sigma), -A^{-1}b, A^{-1}),$$

where

$$(2.7a) \quad \hat{f} \equiv f(g, g^{-1}) = -l^{-1}(-ba + \frac{1}{2} b^2 \sigma).$$

The unitary representations of $\tilde{\mathcal{G}}_5$ furnished by the homomorphism (2.4) and the multiplication law (2.5) (with ω given by (2.3) and (2.6)) are called *unitary projective* (or *ray*) *representations* (?). It is these ray representations (which cannot be reduced to the true representations of \mathcal{G}_5) that will be constructed in the following. The first step in this program is the decomposition of $\tilde{\mathcal{G}}_5$ into the semi-direct product of a suitably chosen invariant Abelian subgroup N and a remainder H . The coset space $\Gamma = \tilde{\mathcal{G}}_5/H$ will then be taken as the representation space.

2'2. Invariant Abelian subgroup. - Consider the invariant Abelian subgroup

$$(2.8) \quad N = T_1^\sigma \times T_4^a$$

of \mathcal{G}_5 and introduce the notation

$$(2.9) \quad H = \{T_4^b \otimes (SL_{2,c} \times T_1^0)\}.$$

Then $\tilde{\mathcal{G}}_5$ can be written as the semi-direct product

$$(2.10) \quad \tilde{\mathcal{G}}_5 = N \otimes H.$$

The semi-direct product structure is realized by the automorphisms π_h of N ,

$$(2.11) \quad n \rightarrow \pi_h(n) \equiv hnh^{-1}, \quad n \in N, \quad h \in H.$$

Indeed, the mapping

$$(2.12) \quad h \rightarrow \pi_h, \quad h \in H,$$

defines a homomorphism of H into the group of all automorphisms of N . Thus, every element g of $\tilde{\mathcal{G}}_5$ can be uniquely represented by a pair,

$$(2.13) \quad g = (n; h), \quad n \in N, \quad h \in H,$$

in terms of which the composition law of $\tilde{\mathcal{G}}_5$ becomes

$$(2.14) \quad g_2 g_1 = (n_2; h_2)(n_1; h_1) = (n_2 \pi_{h_2}(n_1); h_2 h_1).$$

We now turn to the irreducible unitary representations of the Abelian group N . They are, of course, one dimensional, and have the form $\mathcal{U}_n = \exp[i(r\sigma + pa)]1$. Here r is a real scalar and p a real four-vector⁽¹⁵⁾. For convenience (and to emphasize the dual role of the parameters σ , a and representation labels r , p), we introduce the notation

$$(2.15) \quad (\sigma, a|r, p) = \exp[i(r\sigma + pa)].$$

The pair of labels $[r, p]$ is called the *character* of the representation.

The set of all representations $(\sigma, a|r, p)$ forms a group \hat{N} , usually called the *character group* of N . For each $h \in H$, the automorphism π_h defines a one-to-one mapping of \hat{N} into itself, because the transformed form of (2.15) induced by π_h is again a unitary irreducible representation of N , so that it belongs to \hat{N} .

2'3. Orbits. - Let $n = (1; \sigma, a, 0, 1) \in N$ and let $h = (\exp[i\theta]; 0, 0, b, A) \in H$. The automorphism (2.11) is explicitly given by

$$(2.16) \quad n \rightarrow \pi_h(n) = (\omega(h, n); \sigma, Aa + b\sigma, 0, 1),$$

where, according to (2.3) and (2.6),

$$(2.16a) \quad \omega(h, n) = \exp[-i\theta^{-1}(bAa + \frac{1}{2}b^2\sigma)].$$

⁽¹⁵⁾ Since S and P_μ are the generators of T_1^σ and T_4^a , respectively, the numbers r and p_μ are obviously the corresponding eigenvalues.

Because of the dualism between (σ, a) and (r, p) in (2.15), eq. (2.16) implies that the character $[r; p]$ evaluated at $\pi_h(n)$ is equal to a transformed character $[r'; p']$ evaluated at n . In other words,

$$(2.17a) \quad h(\sigma, a|r, p) h^{-1} = \omega(h, n)(\sigma, \Lambda a + b\sigma|r, p) \equiv (\sigma, a|r', p').$$

In a similar manner we get

$$(2.17b) \quad h^{-1}(\sigma, a|r, p) h = \omega(h^{-1}, n)(\sigma, \Lambda^{-1}a - \Lambda^{-1}b\sigma|r, p) \equiv (\sigma, a|r'', p'').$$

Thus, h and h^{-1} induce automorphic transformations of the characters. We write, somewhat symbolically,

$$(2.18a) \quad h^{-1}[r, p] \equiv [r', p'] = \left[r + pb - \frac{1}{2l} b^2, \Lambda^{-1} \left(p - \frac{1}{l} b \right) \right],$$

$$(2.18b) \quad h[r, p] \equiv [r'', p''] = \left[r - \Lambda pb - \frac{1}{2l} b^2, \Lambda p + \frac{1}{l} b \right].$$

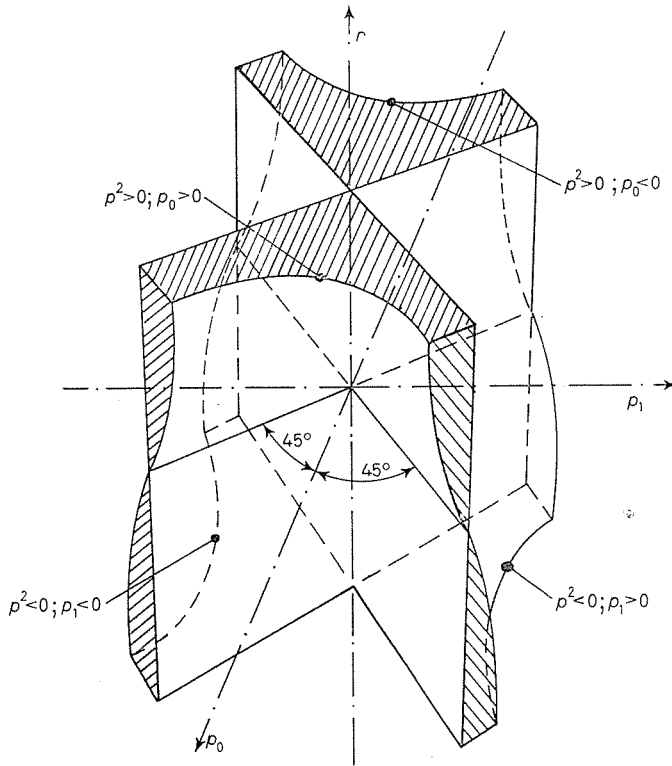


Fig. 1. - The orbit surface $p^2 + 2l^{-1}r = \mathcal{D}$ for $\mathcal{D} = 0$, $l < 0$. Two spatial directions (p_2 and p_3) are suppressed. Cuts parallel to the (p_0, p_1) plane give all Poincaré orbits.

It is easily verified that under these transformations the value of the invariant \mathcal{D} (cf. eq. (1.5a)) is left unchanged, *i.e.*

$$(2.19) \quad p^2 + 2l^{-1}r = p'^2 + 2l^{-1}r' = p''^2 + 2l^{-1}r'' = \mathcal{D}.$$

Thus, the automorphisms π_h of N define *orbits* in N . Each orbit is characterized by some standard character $[\hat{r}, \hat{p}]_{\mathcal{D}, l}$. (The subscripts \mathcal{D} and l were used to emphasize that each orbit is fixed when the invariants \mathcal{D} and l are given. In the following, however, we shall suppress these subscripts.) If $[r_1, p_1]$ and $[r_2, p_2]$ belong to the same orbit then there exists an element $h \in H$ such that

$$(2.20) \quad [r_2, p_2] = h[r_1, p_1].$$

The orbit (2.19) is graphically represented in Fig. 1.

2.4. Stability group. — By definition⁽¹¹⁾, the *stabilizer* (*stability group*, or *little group*) H_0 of an orbit $[\hat{r}, \hat{p}]$ is a subgroup of H such that for every element $h_0 \in H_0 \subset H$, any given point of the orbit remains fixed, *i.e.*

$$(2.21) \quad h_0[\hat{r}, \hat{p}] = [\hat{r}, \hat{p}].$$

From (2.18a) and (2.21) we obtain the conditions

$$(2.22a) \quad \hat{r} = \hat{r} - A\hat{p}b - \frac{1}{2l}b^2,$$

$$(2.22b) \quad \hat{p} = A\hat{p} + \frac{1}{l}b.$$

These are not independent, because using (2.22b) in (2.22a), we simply get

$$(2.23) \quad A\hat{p}b + \frac{1}{2l}b^2 = \frac{l}{2}[(A\hat{p} + l^{-1}b)^2 - \hat{p}^2] = 0.$$

This tells us that, selecting an arbitrary A , we only have to choose

$$(2.24) \quad b = l(\hat{p} - A\hat{p}),$$

whence both conditions become satisfied. Thus, an arbitrary element $h_0 \in H_0$ has the form

$$(2.25) \quad h_0(\theta; A) \equiv (\exp[i\theta]; 0, 0, l(\hat{p} - A\hat{p}), A).$$

We easily verify the combination law

$$(2.26) \quad h_0(\theta''; A'')h_0(\theta'; A') = h_0(\theta'' + \theta'; A''A').$$

Thus, we see that the little group is isomorphic to the direct product of a phase group with an $SL_{2,\sigma}$ group ⁽¹⁶⁾:

$$(2.27) \quad H_0 \approx T_1^\theta \times SL_{2,\sigma}.$$

At this point we wish to make a remark. Instead of (2.8), we could have chosen the *maximal* Abelian subgroup of $\tilde{\mathfrak{G}}_5$, *viz.* $T_1^\theta \times T_1^\sigma \times T_4^a$. Then, instead of (2.9), we would have had $T_4^a \rtimes SL_{2,\sigma}$, but (2.10) would still have held. If we had done so, the characters would have been $\exp [i(\beta\theta + r\sigma + pa)] \equiv \equiv (\theta, \sigma, a|\beta, r, a)$, and the parameter β would have explicitly occurred in the transformations (2.18a), (2.18b). The orbits would no longer have been the invariants (2.19) of $\tilde{\mathfrak{G}}_5$; instead we would have had $p^2 + 2l^{-1}\beta^{-1}r = \text{const}$ as their equation ⁽¹⁷⁾. On the other hand, (2.25) would have been replaced by the simpler term $h_0(A) = (1; 0, 0, l(\hat{p} - A\hat{p}), A)$, and the little group would have been simply $H_0 \approx SL_{2,\sigma}$. However, we feel that our treatment is more satisfactory, because, as pointed out above, our orbits have a more direct interpretation. The inconvenience of having a somewhat more complicated little group is only trivial, since (2.27) is a *direct* product.

Returning to our main subject, we now wish to find a relation between elements of our little group (2.27), and arbitrary elements of H . Let us consider an orbit $[\hat{r}, \hat{p}]$ and choose for a specific point $[r, p]$ of this orbit an element $h_{r,p}$ of H such that ⁽¹⁸⁾

$$(2.28) \quad h_{r,p}[\hat{r}, \hat{p}] = [r, p].$$

Let now $h(\theta; b, A) \equiv (\exp [i\theta]; 0, 0, b, A)$ be an arbitrary element of H and write

$$(2.29) \quad h^{-1}(\theta; b, A)[r, p] = [r', p'],$$

where the r.h.s. is given by eq. (2.18a). Combining the last two equations, we get the identity

$$h_{r',p'}^{-1} h^{-1}(\theta; b, A) h_{r,p}[\hat{r}, \hat{p}] = [\hat{r}, \hat{p}],$$

⁽¹⁶⁾ From (2.25) it is clear that the $SL_{2,\sigma}$ which appears in H_0 is *not* the $SL_{2,\sigma}$ subgroup of $\tilde{\mathfrak{G}}_5$. We shall come back to this point later, in Sect. 3'5.

⁽¹⁷⁾ In his work on the ray representations of the nonrelativistic Galilei group, VOISIN actually proceeds in a manner as now sketched, and obtains the orbits $E - \mathbf{p}^2/2Mq_0 = \text{const}$ (cf. eq. (14) of ref. ⁽¹²⁾, first paper), instead of the more desirable $E - \mathbf{p}^2/2M = \text{const}$ parabolas. At a later point, he then sets $q_0 = 1$ which, even though it seems to be an artificial choice, apparently does not lead to loss of generality.

⁽¹⁸⁾ In view of (2.20), we are assured of the existence of such an element of H . Actually, $h_{r,p}$ is not even unique.

which, because of (2.21), implies that the product of the three elements on the l.h.s. is an element of the little group H_0 . We denote this particular element by h_0^{-1} , so that we have

$$(2.30) \quad h_0 = h_{r,p}^{-1} h(\theta; b, A) h_{r',p'}.$$

Conversely, we have

$$(2.31) \quad h(\theta; b, A) = h_{r,p} h_0 h_{r',p'}^{-1}.$$

The meaning of this equation is that given an arbitrary element $h \in H$, it can be expressed in terms of *some* element $h_0 \in H_0$ associated with the orbit $[\hat{r}, \hat{p}]$. The transformer $h_{r,p}$ is defined by (2.28). The point $[r', p']$ that labels the transformer on the right is related to $[r, p]$ by eq. (2.18a).

2'5. *Representation space.* — We choose for our representation space the coset space

$$(2.32) \quad \Gamma = \tilde{\mathcal{G}}_5/H.$$

In view of eq. (2.10), Γ is isomorphic to the Abelian group N . Consequently, there is a one-to-one correspondence between the elements of Γ and the characters $[r, p]$ of the orbits $[\hat{r}, \hat{p}]$ in \hat{N} . Therefore, the elements of Γ can be labeled by the character of an orbit $[\hat{r}, \hat{p}]$, *i.e.* by all numbers r, p which obey the relation

$$p^2 + 2l^{-1}r = \hat{p}^2 + 2l^{-1}\hat{r} = \mathcal{D}.$$

The basis of our representation will be a set of functions $\psi_\xi(r, p)$, which depend on the character $[r, p]$ as specified above. The additional collective label ξ stands to distinguish further the states within a given representation. (Thus, in general, the functions ψ_ξ are vector valued.) We introduce in Γ an invariant measure by defining

$$(2.33) \quad d\Omega(r, p) = dr d^4p \delta(p^2 + 2l^{-1}r - \mathcal{D}),$$

and require that the functions ψ_ξ belong to the space $\mathcal{L}^2(\Gamma, d\Omega)$ of square-integrable functions. Thus, our representation space is the Hilbert space $\mathcal{H}(\Gamma)$ with the inner product defined by

$$(2.34) \quad \langle \psi | \phi \rangle = \int dr d^4p \delta(p^2 + 2l^{-1}r - \mathcal{D}) \psi_\xi^\dagger(r, p) \phi_\xi(r, p).$$

Here summation over ξ is understood.

If \mathcal{U}_n is a unitary representation of N , the basis ψ_ξ will transform under

the action of this representation according to

$$(2.35) \quad \mathcal{U}_n \psi_{\xi}(r, p) = \exp [i(r\sigma + pa)] \psi_{\xi}(r, p) .$$

Our problem is now to find the transformation law of the basis under the unitary irreducible projective representations \mathcal{U}_g of \mathfrak{G}_5 .

3. – The unitary irreducible projective representations of \mathfrak{G}_5 .

In order to check the applicability of the subsequent mathematical construction, let us summarize the relevant properties of our group $\tilde{\mathfrak{G}}_5$:

i) $\tilde{\mathfrak{G}}_5$ is a separable locally compact group ⁽¹⁹⁾,

ii) it can be written as the semi-direct product of the invariant Abelian subgroup $N' = T_1^\sigma \times T_4^a \times T_1^0 = N \times T_1^0$ and the noninvariant subgroup $H' = T_4^b \rtimes \rtimes SL_{2,\sigma}$,

iii) both factors N' and H' are closed subgroups ⁽²⁰⁾.

As MACKEY has shown ⁽¹¹⁾, the fulfilment of these criteria ensures that the method of induced representations will furnish all irreducible representations of the group if the irreducible representations of the stabilizer are known.

3.1. Representations of H_0 . – We start with the study of the representations of the stability group H_0 . Because of its direct-product structure (exhibited by (2.27)), all irreducible representations of H_0 will have the form

$$(3.1) \quad \mathcal{U}_{h_0} = \exp [i\beta\theta] D(\hat{h}_0) .$$

Here, the first factor (with β arbitrary and real) is a representation of T_1^0 and the second factor stands for a representation of $SL_{2,\sigma}$. In view of (2.27) and (2.26), the group element \hat{h}_0 in the argument of D denotes the element (2.25) with θ set equal to zero, *i.e.*

$$(3.1\alpha) \quad \hat{h}_0 = h_0(0; A) = (1; 0, 0, i(\hat{p} - A\hat{p}), A) .$$

⁽¹⁹⁾ This is obvious from the parametrization of the group elements.

⁽²⁰⁾ Since N is Abelian, its closedness is obvious. To see that H is closed, we note that the first factor in (2.9) is isomorphic to the covering of a Poincaré group which is known to be closed, and the second factor is an Abelian phase group, likewise closed.

The irreducible representations D of $SL_{2,c}$ are well known⁽²¹⁾. They are labeled by a pair of indices⁽²²⁾ k, c and we shall write D^{kc} to symbolize a specific irreducible representation. There are the following cases:

a) $k = 0, c = 1$. This is the *trivial one-dimensional* (and obviously unitary) representation.

b) k and c simultaneously integral or half-integral⁽²³⁾ and $|c| > |k|$. These representations are *finite dimensional and nonunitary*.

c) $k = 0, \frac{1}{2}, 1, \dots$ and $c = i\varphi$, with $-\infty < \varphi < +\infty$. These representations are infinite dimensional and unitary. They are said to belong to the *principal series*.

d) $k = 0$, and c is a real number such that $0 < c < 1$. These are also infinite-dimensional unitary representations and are said to belong to the *supplementary series*.

We shall not be interested in the nonunitary representations of case *b)*, and will discuss the simple case of the representation *a)* separately in the Appendix. For the unitary infinite-dimensional representations *c)* and *d)*, each state of a given representation is characterized by two numbers⁽²⁴⁾ s and s_3 . For any given representation D^{kc} , s can take on the infinite sequence of discrete values

$$(3.2a) \quad s = k, k + 1, k + 2, \dots$$

For any specified s , the s_3 then can assume the $2s + 1$ values

$$(3.2b) \quad s_3 = -s, -s + 1, \dots, s - 1, s.$$

We mention that the representations D^{kc} and D^{-k-c} are equivalent. Finally, we recall that the representation D^{k-c} is conjugate to D^{kc} .

3.2. Induced representations of $\tilde{\mathfrak{G}}_5$. — We are now in a position to determine the labels which are needed to specify a representation of $\tilde{\mathfrak{G}}_5$. They are as

⁽²¹⁾ See, for example, I. M. GELFAND, R. A. MINLOS and Z. YA. SHAPIRO: *Representations of the Rotation and Lorentz Group* (New York, 1963), especially p. 200 and p. 188. See also M. A. NAIMARK: *Linear Representations of the Lorentz Group* (New York, 1964).

⁽²²⁾ The numbers k and c are related to the Casimir operators of $SL_{2,c}$; see eqs. (3.26) and (3.27) below.

⁽²³⁾ Case *a)* is a special case of Case *b)*, but for obvious reasons has been treated separately.

⁽²⁴⁾ These numbers are related to the Casimir operators occurring in the chain $SL_{2c} \supset SU_2 \supset SO_2$.

follows:

- i) an arbitrary real number l ,
- ii) an arbitrary real number \mathcal{D} ,
- iii) two numbers k and c (as given above).

Here l and \mathcal{D} are necessary to specify an orbit, and k, c are needed to specify the representation of the little group associated with the orbit ⁽²⁵⁾. A representation of $\tilde{\mathcal{G}}_3$ will be denoted by the symbol $(l|\mathcal{D}, k, c)$. Each state of a given representation is labeled (apart from r and p , selecting a point of the orbit) by two supplementary labels ⁽²⁶⁾ s and s_3 . For the relevant unitary irreducible representations (Cases c) and d) above) the possible values of s and s_3 are given by ⁽²⁷⁾ (3.2a), (3.2b). In view of these comments, the complete labeling of the basis functions $\psi_\xi(r, p)$ (introduced in Subsect. 2'5) belonging to an irreducible representation of $\tilde{\mathcal{G}}_3$ is given by the notation

$$\psi = \psi_{s_3 s_2}^{l\mathcal{D}kc}(r, p).$$

Let us now consider a representation $\mathcal{U}_{h_0} = \exp [i\beta\theta] D^{kc}(\hat{h}_0)$ of H_0 . The transformation law of our basis under \mathcal{U}_{h_0} is

$$(3.3) \quad \mathcal{U}_{h_0} \psi_{s_3 s_2}^{l\mathcal{D}kc}(r, p) = \exp [i\beta\theta] [D^{kc}(\hat{h}_0)]_{s' s'_3; s_2} \psi_{s' s'_3}^{l\mathcal{D}kc}(r, p),$$

where summation over s' and s'_3 is understood.

Next we consider a homomorphism $h \rightarrow \mathcal{U}_h$ from the subgroup H to a set of unitary operators. On account of eq. (2.31) we can write

$$(3.4) \quad \mathcal{U}_h = \mathcal{U}_{h_r p h_0 h_r^{-1}}.$$

Since the functions $\psi(r, p)$ carry the representation \mathcal{U}_h , we have (omitting for a moment the super- and subscripts of ψ)

$$\mathcal{U}_h \psi(r, p) \equiv \psi(h^{-1}[r, p]) = \psi(h_{r' p'} h_0^{-1} h_{r p}^{-1}[r, p]) = \psi(h_{r' p'} h_0^{-1}[\hat{r}, \hat{p}]) \equiv \mathcal{U}_{h_0 h_{r' p'}} \psi(\hat{r}, \hat{p}).$$

⁽²⁵⁾ The additional real number β , that occurs in (3.1) and which, in addition to k and c , is needed to specify a representation of the little group, is immaterial because $\exp [i\beta\theta]$ will only be an arbitrary phase factor multiplier in the representation of $\tilde{\mathcal{G}}_3$, cf. eq. (3.6) below.

⁽²⁶⁾ Thus, the additional « collective label » ξ which was introduced in Subsect. 2'5, corresponds to the pair s, s_3 .

⁽²⁷⁾ Thus, these labels run through a set of discrete, integer or half-integer numbers.

In the next to the last step we used (2.28). Now, we have ⁽²⁸⁾

$$\mathcal{U}_{h_0 h_{r', p'}^{-1}} = \mathcal{U}_{h_0} \mathcal{U}_{h_{r', p'}^{-1}}.$$

Hence, we can continue the previous equation as

$$\begin{aligned} \mathcal{U}_{h_0 h_{r', p'}^{-1}} \psi(r, p) &= \mathcal{U}_{h_0} \mathcal{U}_{h_{r', p'}^{-1}} \psi(\hat{r}, \hat{p}) = \mathcal{U}_{h_0} \psi(h_{r', p'}[r, p]) = \\ &= \mathcal{U}_{h_0} \psi(r', p') = \exp [i\beta\theta] D^{kc}(\hat{h}_0) \psi(r', p'). \end{aligned}$$

In the penultimate step we used again (2.28) and in the last step we utilized (3.3). Thus, in detail we have the transformation law

$$(3.5) \quad \mathcal{U}_h \psi_{ss_2}^{i\mathcal{D}^{kc}}(r, p) = \exp [i\beta\theta] [D^{kc}(\hat{h}_0)]_{s's'_3; ss_2} \psi_{s's'_3}^{i\mathcal{D}^{kc}}(r', p').$$

In view of (2.28) and (2.18a), the arguments r' and p' are explicitly

$$(3.5a) \quad r' = r + pb - \frac{1}{2l} b^2, \quad p' = A^{-1} \left(p - \frac{1}{l} b \right).$$

Finally, let us consider an arbitrary group element $g \in \tilde{\mathfrak{G}}_5$. Because of (2.10), we have the unique decomposition ⁽²⁹⁾ $g = n\hat{h}$. In the homomorphism $g \rightarrow \mathcal{U}_g$ this means that ⁽³⁰⁾ $\mathcal{U}_g = \mathcal{U}_n \mathcal{U}_h$. The action of \mathcal{U}_h is given by (3.5), and the action of \mathcal{U}_n is shown in (2.35). Thus, putting these together, we obtain the transformation law for the irreducible unitary projective representations of \mathfrak{G}_5 as follows ^(31,32):

$$(3.6) \quad \mathcal{U}_g \psi_{ss_2}^{i\mathcal{D}^{kc}}(r, p) = \exp [i(\beta\theta + r\sigma + pa)] [D^{kc}(\hat{h}_0)]_{s's'_3; ss_2} \psi_{s's'_3}^{i\mathcal{D}^{kc}}(r', p').$$

We note that the unitarity of the representation is, of course, meant with respect to the inner product in the Hilbert space $\mathcal{H}(\Gamma)$ of $\mathcal{L}^2(\Gamma, d\Omega)$ integrable functions, as defined by (2.34). That is, for $\psi, \phi \in \mathcal{L}^2(\Gamma, d\Omega)$,

$$(3.7) \quad \langle \psi | \phi \rangle = \int dr d^4p \delta \left(p^2 + \frac{2}{l} r - \mathcal{D} \right) (\psi_{ss_2}^{i\mathcal{D}^{kc}}(r, p))^\dagger \phi_{ss_2}^{i\mathcal{D}^{kc}}(r, p)$$

⁽²⁸⁾ Since the group elements $h \in H$ have no translational part ($\alpha = \sigma = 0$), no phase factor will occur in the composition law of the representation operators \mathcal{U}_h .

⁽²⁹⁾ As is well known, this is a consequence of the representation (2.13) and composition law (2.14) of semi-direct-product groups.

⁽³⁰⁾ We do not have a phase factor, because the group element h has no translational part.

⁽³¹⁾ Equation (3.6) has been already given, without proof and without detailed discussion, in Appendix C of our first paper, ref. (1).

⁽³²⁾ We remark that, as the reader will easily verify, we would get eq. (3.6) in an unchanged form if we had used the maximal Abelian subgroup $T_1^0 \times T_1^\sigma \times T_4^a$ instead of (2.8).

(summation over s, s_3 understood). The unitarity follows trivially from that of D^{kc} . Furthermore, we emphasize that in consequence of Mackey's theorems ⁽¹¹⁾, our construction (3.6) gives all unitary irreducible representations, up to equivalence, because, as pointed out at the beginning of Sect. 3, all necessary criteria are satisfied.

For clarity's sake we summarize the notation in (3.6). \mathcal{D} and l are invariants of $\tilde{\mathcal{G}}_5$. The numbers k, c are the labels of the unitary irreducible representations of the $SL_{2,\sigma}$ part of H_0 . The labels s, s_3 characterize the state within each representation of $\tilde{\mathcal{G}}_5$, together with the labels r, p . The transformed r' and p' are given by (3.5a). The element \hat{h}_0 is given by h_0 with zero phase, where

$$(3.8) \quad h_0 = h_{r,p}^{-1} h(\theta; b, A) h_{r',p'},$$

as follows from (2.31). Here, in turn, $h_{r,p}$ is defined by (2.28), $[\hat{r}, \hat{p}]$ being an arbitrary point on the orbit selected by l and \mathcal{D} .

Finally, we note that the constant β in (3.6) is completely arbitrary. Since our representations are ray representations, the β may be taken to be one, without any loss of generality.

It appears from (3.8) as though the dependence of the operator $D^{kc}(\hat{h}_0)$ on the parameters of the group was rather complicated. However, we may take advantage of the arbitrariness of $[\hat{r}, \hat{p}]$ and simplify this dependence considerably. Let us choose, in particular, $\hat{r} = l/2\mathcal{D}, \hat{p} = 0$. (This is certainly a point of the orbit defined by l and \mathcal{D} .) Then eq. (2.38) reads

$$h_{r,p}[l/2\mathcal{D}, 0] = [r, p]$$

and one easily verifies that the simplest ⁽³³⁾ solution is

$$(3.9a) \quad h_{r,p} = (1; 0, 0, lp, 1).$$

Similarly,

$$(3.9b) \quad h_{r',p'} = (1; 0, 0, lp', 1),$$

with p' being given, of course, by (3.5a). Using (3.9a), (3.9b) in (3.8), one easily finds

$$(3.9c) \quad \hat{h}_0 = (1; 0, 0, 0, A).$$

Thus, \hat{h}_0 can be taken to be a pure $SL_{2,\sigma}$ transformation of $\tilde{\mathcal{G}}_5$. We can re-

⁽³³⁾ Cf. footnote ⁽¹⁸⁾.

write (3.6) in the *final form*

$$(3.10) \quad \mathcal{U}_g \psi_{ss_3}^{i\mathcal{D}kc}(r, p) = \exp [i(\beta\theta + r\sigma + pa)][D^{kc}(\mathcal{A})]_{s's'_3; ss_3} \psi_{s's'_3}^{i\mathcal{D}kc}(r', p'),$$

where r', p' are given by (3.5a).

3'3. An equivalence theorem. — As is well known (⁷), the concept of equivalence for projective representations is somewhat different from that which applies for true representations. Because of the appearance of ω in the composition law (2.5), we must consider \mathcal{U}'_g unitarily equivalent to \mathcal{U}_g if

$$(3.11) \quad \mathcal{U}'_g = \alpha(g) V \mathcal{U}_g V^{-1},$$

where V is a unitary operator and α is a complex function of modulus one.

Let us define now an operator V by setting

$$(3.12) \quad V \psi_{ss_3}^{i\mathcal{D}kc} \psi(r, p) = \psi_{ss_3}^{i\mathcal{D}kc} \left(r + \frac{l}{2} \mathcal{D}, p \right) \equiv \widehat{\psi}_{ss_3}^{i\mathcal{D}kc}(r, p).$$

We have, using (3.7) and (3.12), and setting $\tilde{r} = r + l/2\mathcal{D}$,

$$\begin{aligned} \langle \psi | \phi \rangle &= \int dr d^4p \delta \left(p^2 + \frac{2}{l} r - \mathcal{D} \right) \psi^\dagger(r, p) \phi(r, p) \equiv \\ &\equiv \int d\tilde{r} d^4p \delta \left(p^2 + \frac{2}{l} \tilde{r} - \mathcal{D} \right) \psi^\dagger(\tilde{r}, p) \phi(\tilde{r}, p) = \langle \widehat{\psi} | \widehat{\phi} \rangle = \langle V\psi | V\phi \rangle. \end{aligned}$$

Hence, V is a unitary operator, and provides an isomorphic mapping from the Hilbert space $\mathcal{H}(\Gamma)$ to the Hilbert space $\mathcal{H}_0(\Gamma)$. The latter is the same set of functions but is equipped with the measure

$$(3.13) \quad d\Omega = dr d^4p \delta(p^2 + 2l^{-1}r)$$

instead of (2.33). This implies that *the unitary representation $(l|\mathcal{D}, k, c)$ is unitarily equivalent to the representation $(l|0, k, c)$* . Actually, it is easily seen from (3.12) and (3.10) that if \mathcal{U}_g is a representation in $\mathcal{H}(\Gamma)$, then

$$(3.14) \quad \mathcal{U}'_g = \exp \left[i \frac{l}{2} \sigma \mathcal{D} \right] V \mathcal{U}_g V^{-1}$$

is a representation in $\mathcal{H}_0(\Gamma)$, which, in view of (3.11), bears out our statement in detail.

Therefore, without loss of generality, we can restrict ourselves to representations with $\mathcal{D} = 0$. The label \mathcal{D} may be omitted ⁽³⁴⁾.

3'4. *Conjugate representations.* – The complex conjugate of eq. (3.10) is

$$(3.15) \quad \overline{\mathcal{U}}_g \overline{\psi}_{ss_2}^{l\mathcal{D}k^c}(r, p) = \exp [i(-\beta\theta - r\sigma - pa)][D^{k^c}(A)]_{s's'_2; ss_2} \overline{\psi}_{s's'_2}^{l\mathcal{D}k^c}(r', p'),$$

where the bar means complex conjugation and where we took cognizance of $\overline{D}^{k^c} = D^{k^c}$. On the other hand, let us consider the representation $(-l|\mathcal{D}, k, -c)$ of $\tilde{\mathfrak{G}}_5$. Denoting the unitary operator which corresponds to a group element g in *this* representation by $\check{\mathcal{U}}_g$, we have

$$(3.16) \quad \check{\mathcal{U}}_g \psi_{ss_2}^{-l\mathcal{D}k^c}(r, p) = \exp [i(\beta\theta + r\sigma + pa)][D^{k^c}(A)]_{s's'_2; ss_2} \psi_{s's'_2}^{-l\mathcal{D}k^c}(\check{r}, \check{p}),$$

where now

$$(3.16a) \quad \check{r} = r + pb + \frac{1}{2l} b^2, \quad \check{p} = A^{-1} \left(p + \frac{1}{l} b \right).$$

If we introduce the operator A defined by

$$(3.17) \quad A \psi_{ss_2}^{l\mathcal{D}k^c}(r, p) = \overline{\psi}_{ss_2}^{l\mathcal{D}k^c}(-r, -p) \equiv \check{\psi}_{ss_2}^{l\mathcal{D}k^c}(r, p),$$

then, using (3.15) and (3.5a), we easily find that

$$(3.18) \quad (\overline{\mathcal{U}}_g \check{\psi})(r, p) = \overline{\mathcal{U}}_g \overline{\psi}(-r, -p) = \exp [i(-\beta\theta + r\sigma + bp)][D^{k^c}(A)] \check{\psi}(r', p')$$

with

$$(3.18a) \quad r' = r + pb + \frac{1}{2l} b^2, \quad p' = A^{-1} \left(p + \frac{1}{l} b \right).$$

Thus, comparing with (3.16) and (3.16a), we can write

$$(3.19) \quad \exp [2i\beta\theta] \overline{\mathcal{U}}_g \check{\psi}(r, p) = \check{\mathcal{U}}_g \check{\psi}(r, p).$$

Using (3.17), we find that

$$(3.20) \quad \overline{\mathcal{U}}_g = \exp [-2i\beta\theta] A \check{\mathcal{U}}_g A^{-1}.$$

⁽³⁴⁾ See, however, our subsequent discussion of the reduction of products of representations, Subsect. 5'2. Furthermore, the equivalence theorem obviously holds true only as long as l is finite.

⁽³⁵⁾ For simplicity, we suppress in this calculation all labels.

Thus, according to (3.11), the representations $\overline{\mathcal{U}}_g$ and $\check{\mathcal{U}}_g$ are equivalent ⁽³⁶⁾ in the sense of ray representations. We write symbolically

$$(3.21) \quad (\overline{l|\mathcal{D}, k, c}) \approx (-l|\mathcal{D}, k, -c).$$

It also follows from the above discussion that the basis functions $\psi_{ss_2}^{-l\mathcal{D}k-c}$ of the $(-l|\mathcal{D}, k, -c)$ representation can be expressed in terms of the basis functions $\psi_{ss_2}^{l\mathcal{D}kc}$ of the $(l|\mathcal{D}, k, c)$ representation. We have

$$(3.22) \quad \psi_{ss_2}^{-l\mathcal{D}kc}(r, p) = A\psi_{ss_2}^{l\mathcal{D}kc}(r, p) = \bar{\psi}_{ss_2}^{l\mathcal{D}kc}(-r, -p).$$

Finally, by checking the inner product, we easily verify that $\langle \psi|\phi \rangle = \langle A\phi|A\psi \rangle$, so that A is antilinear unitary.

3'5. Some properties of the basis functions. – For subsequent physical applications, it will be useful to summarize the effect of some operators of $\tilde{\mathcal{G}}_5$ on the basis functions. First, it is obvious that the ψ are eigenfunctions of P_μ and of S , and we have ⁽³⁷⁾

$$(3.23) \quad P_\mu \psi_{ss_2}^{l\mathcal{D}kc}(r, p) = p_\mu \psi_{ss_2}^{l\mathcal{D}kc}(r, p),$$

$$(3.24) \quad S\psi_{ss_2}^{l\mathcal{D}kc}(r, p) = r\psi_{ss_2}^{l\mathcal{D}kc}(r, p).$$

Next we recall that the Casimir operators of $\tilde{\mathcal{G}}_5$ are the operators $\mathcal{D}, \mathcal{J}, \mathcal{K}$ as given by (1.5a)-(1.5c). Hence, we have ⁽³⁸⁾, from (1.5a),

$$(3.25) \quad (P_\mu P^\mu + 2l^{-1}S)\psi_{ss_2}^{l\mathcal{D}kc}(r, p) = \mathcal{D}\psi_{ss_2}^{l\mathcal{D}kc}(r, p).$$

From (1.5b) and (1.5c) we obtain

$$(3.26) \quad \frac{1}{2}T_{\mu\nu}T^{\mu\nu}\psi_{ss_2}^{l\mathcal{D}kc}(r, p) = (k^2 + c^2 - 1)\psi_{ss_2}^{l\mathcal{D}kc}(r, p),$$

$$(3.27) \quad \frac{1}{4}\varepsilon_{\mu\nu\varrho\sigma}T^{\mu\nu}T^{\varrho\sigma}\psi_{ss_2}^{l\mathcal{D}kc}(r, p) = 2ikc\psi_{ss_2}^{l\mathcal{D}kc}(r, p),$$

respectively. Here we used the facts that \mathcal{J} and \mathcal{K} are Casimir operators of ⁽³⁹⁾ $SL_{2,\sigma}$ and that our basis carries a representation of $SL_{2,\sigma}$. (The relation

⁽³⁶⁾ On the other hand, it must be emphasized that the conjugate representation $(\overline{l|\mathcal{D}, k, c})$ is *not* equivalent to the original $(l|\mathcal{D}, k, c)$ representation unless $c = 0$.

⁽³⁷⁾ This follows from the fact that the representation space Γ is isomorphic to \mathcal{N} . It is also consistent, naturally, with the realizations $P_\mu = i\partial_\mu$, $S = i\partial_u$ in configuration space, cf. ref. (1). This can be seen by taking the Fourier transforms of (3.23) and (3.24).

⁽³⁸⁾ Equation (3.25) follows also from (3.23), (3.24) and (2.19).

⁽³⁹⁾ In ref. (1) we showed that the operators $T_{\mu\nu}$ generate an $SL_{2,\sigma}$ algebra.