Invariant Lie-Admissible Formulation of Quantum Deformations

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In this note we outline the history of q-deformations, indicate their physical shortcomings, suggest their apparent resolution via an invariant Lie-admissible formulation based on a new mathematics of genetopic type, and point out their expected physical significance.

1. INTRODUCTION

In 1948 Albert\(^{(1)}\) introduced the notions of Lie-admissible and Jordan-admissible algebras as generally nonassociative algebras \(U\) with elements \(a, b, c\), and abstract product \(ab\) which are such that the attached algebras \(U^-\) and \(U^+\), which are the same vector spaces as \(U\) equipped with the products \([a, b]_\nu = ab - ba\) and \(\{a, b\}_\nu = ab + ba\), are Lie and Jordan algebras, respectively. Albert then studied the algebra with product

\[
(A, B) = p \times A \times B + (1 - p) \times B \times A
\]

where \(p\) is a parameter, \(A, B\) are matrices or operators (hereon assumed to be Hermitian), and \(A \times B\) is the conventional associative product.

It is easy to see that the above product is indeed jointly Lie- and Jordan-admissible because \([A, B]_\nu = (1 - 2p) \times (A \times B - B \times A)\) and \(\{A, B\}_\nu = A \times B + B \times A\). However, there exist no (finite) value of \(p\) under which product (1) recovers the Lie product. As a result, product (1) cannot be used for possible coverings of current physical theories.

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\(^{(1)}\) This paper is dedicated to the memory of Larry Biedenharn, my teacher of the rotational symmetry.

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Note that the attached products \([ A, B ]_\nu = ( A, B ) - ( B, A ) = A \times T \times B - B \times T \times A, T = P + Q,\) and \([ A, B ]_\nu = ( A, B ) + ( B, A ) = A \times T \times B + B \times T \times A, T = P - Q,\) are still Lie and (commutative) Jordan, respectively, although of a more general type called isotropic.\(^{12, 6}\)

Note also that the \(P\) and \(Q\) operators must be sandwiched in between the elements \(A\) and \(B\) to characterize an algebra as commonly understood in mathematics.\(^{12, 6}\) It should be finally indicated that, when properly written, Hamilton's equations with external terms possess precisely a Lie-admissible structure.

Following these studies I had the opportunity of discussing Albert's Lie-admissibility with Larry Biedenharn in a number of occasions, including a visit to his department at Duke University in North Carolina in spring 1981. Subsequently, our scientific contacts were interrupted for several years.

In 1989 Biedenharn\(^7\) and, independently, Macfarlane\(^8\) introduced the so-called \(q\)-deformations, with a structure of the type

\[
A \times B - B \times A \rightarrow A \times B - q \times A \times B
\]

which are an evident particular case of structures \((2)\), and which were followed by a number of papers so large as to discourage an outline (see, e.g., representative papers\(^9\)). More recently, other types of deformations of relativistic quantum formulations appeared in the literature under the name of \(k\)-deformations (see, e.g., Refs. 10, quantum groups (see, e.g., Refs. 11), and other generalizations.

I saw Larry Biedenharn for the last time at the Third Wigner Symposium held at Oxford University, England, in September 1993. During that occasion, I communicated to him the existence of a number of physical shortcomings of the Lie-admissible models in general, and of the \(q\)-deformations in particular, which our group had identified following our last meeting of 1981, on which shortcomings he agreed immediately.

I then indicated to Larry Biedenharn, also at the Third Wigner Symposium, new lines of inquiries which apparently permit the resolution of the problematic aspects of Lie-admissible and \(q\)-deformations via their invariant formulation on generalized spaces and fields. He expressed interest and requested copies of our forthcoming papers in the field. I explained that this would take some time because the resolution of the physical shortcomings requires a new mathematics, called \textit{genomathematics}, with new numbers, new Hilbert spaces, new geometries, etc., which had to be studied in mathematical journals prior to any possible physical application.

Memoirs\(^{12, 6}\) on the new genomathematics was published only recently and I regret to have been unable to send a copy to Larry Biedenharn because of his, for me, unexpected departure.

\[1\]
2. ELEMENTS OF GENOMATHMATICS

The main idea of the Lie-admissible theory is that its structure is inherent in the conventional Lie theory. In fact, a one-parameter connected Lie group realized via Hermitian operators \( X = X^\dagger \) on a Hilbert space \( \mathcal{H} \) has in reality the structure of a bimodule (also called in non-associative algebras split-null extension; see, e.g., Ref. 13).

In nontechnical terms, the structure of a Lie group as a bimodule is essentially characterized by an action from the left \( U^> \) and an action from the right \( {}^\ast U \) with explicit realization and interconnecting conjugation

\[
A'(w) = U^> \times Q(0) \times < U = \{ e^{iX^\ast X} \} > A(0) < \{ e^{-iwX < X} \} = (I^> + iX^> \times w + \cdots) > A(0) < (I^< - iX^< \times X + \cdots) \quad (7)
\]

\[
U^> = (U^>)^t = U, \quad X^< = (X^>)^t = X, \quad {}^\ast I = < I = I
\]

(where \( w \) is a Lie parameter and the multiplications > and < represent conventional associative products ordered to the right and to the left, respectively). The infinitesimal version in the neighborhood of the unit then acquires the familiar form

\[
[i(A'dw - A(0))/dw = A < X - X > A = A \times X - X \times A \quad (8)
\]

which clarifies that in the product \( Ax = A < X(A = X < A) >, X \) in actuality acts from the right (from the left).

The bimodule structure is generally ignored in the conventional formulation of Lie's theory because it is unnecessary. In fact, in a Lie bimodule \( \{ \mathcal{H}, \mathcal{M} \} \), where \( \mathcal{H} = \mathcal{M}^\ast = \mathcal{H} \) is a conventional Hilbert space, the modular action to the right and to the left are interconnected with the simple bimodular rules

\[
X^> \psi^> = X \psi = - \psi^< X = - \psi X, \quad \psi < X = \psi^> X
\]

is an element of the universal enveloping associative algebra \( \xi^> (L) \) of the considered Lie algebra. Lie's action \( (\xi^>(L))^{-} \) for the action to the right, and \( \xi^> \in \xi^> (L) \). Since \( \mathcal{H} = \mathcal{M} = \mathcal{H}^\ast \), \( \xi^> (L) \) over \( \xi^> (L) \) can be effectively reduced to the one-sided representations, or just representations for short, of \( \xi^> (L) \) as well known. However, as we shall see shortly, the original manifold structure of Lie's theory is no longer trivial for broader realizations of axioms (7).

Lie-admissible structure (5) was proposed on the basis of the mere observation that the abstract axioms of the bimodular structure (7) do not essentially require that the multiplications > and < must be conventional, because they can also be generalized, provided that they remain...
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Moreover, it is suggested that $P + Q$ and $P - Q$ be nonsingular to preserve a well-defined Lie and Jordan content, respectively. To differentiate forms (9) from the isotopic ones, I called them genotopic in Ref. 5, to denote their character of inducing a more general realization. $I^>$ and $I^<$ are then called genotopic units and $P$ and $Q$ the genotopic elements.

Broader products and units (9) characterize the following more general realization of the abstract axioms (7) I tentatively called Lie-admissible transformation group \cite{5, 6, 12, 20}:

$$ A(w) = U^> > A(0) < U = \{e^{ixXw} \} > A(0) < \{e^{-ixXw} \} $$

$$ = \{e^{ixXw} \} \times A(0) \times \{e^{-ixXw} \} $$

$$ = (I + iX + Xw + \ldots) \times A(0) \times (I - iX + Xw - \ldots) $$

$$ = (I^> + iX + Xw + \ldots) > A(0) < (I^> - iX + Xw - \ldots) $$

(10)

$$ U^> = (U^>)^t, \quad X^> = (X^>)^t = X = X, \quad P^> = P = (Q^>)^t = Q^t $$

$$ I^> = P^{-1} = (I^>)^t = (Q^>)^t $$

with infinitesimal version in the neighborhood of the genounits characterized by the general Lie-admissible algebra (loc. cit.)

$$ i \times [A(dw) - A(0)]/dw = A < X - X > A = A \times P \times X \times Q \times A $$

(11)

where we have used the genoexponentiation to the right and to the left \cite{12, 19}:

$$ e^{ixXw} = (I^>)^t + iX + Xw + \ldots \times A \times (iX + Xw + \ldots)^2 ! + \ldots $$

$$ = \{e^{ixXw} \} \times A $$

$$ e^{ixXw} = (I^>)^t + iX + Xw + \ldots \times A \times (iX + Xw + \ldots)^2 ! + \ldots $$

$$ = (I^>)^t \times \{e^{ixXw} \} $$

(12)

It is at this point where the essential bimodular character of axioms (7) acquire their full light because they are no longer effectively reducible to a one-sided form. It is evident that realization (10) and (11) of the conventional Lie axioms (7) coincides with the Lie-admissible equations (4) and (5). For this reason, realizations (9)–(12) are assumed as the foundation of the Lie-admissible theory under study in this section.

The central assumption we are studying herein is the bimodular lifting of the unit of Lie's theory $I \rightarrow \{I^>, I^<\}$, $I^> = (I^>)^t$. To achieve consistency, the entirety of the Lie theory must be lifted into a dual genotopic form, with no known exception. A rudimentary review of the emerging
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genotopic mathematics or genomathematics for short of Ref. 12, plus unpublished aspects is the following.

Definition 1\textsuperscript{[21, 12]. Let } \( F = (a, +, \times) \) be a conventional field of \((\text{real } R, \text{ complex } C, \text{ or quaternionic } Q)\) numbers \( a \) with additive unit 0, multiplicative unit \( 1 = 1 \), sum \( a + b \), and product \( a \times b \). The \textit{genofields} to the right \( F^r = (a^r, +^r, \times^r) \) are rings with elements \( a^r = a \times 1^r \) called \textit{genonumbers}, where \( a \) is an element of \( F \), \( \times \) is the multiplication in \( F \), and \( 1^r = P^{-1} \) is a well-behaved, everywhere invertible and non-Hermitian quantity generally outside \( F \), equipped with all operations ordered to the right, \( i.e., \), the ordered genosum to the right, ordered genoprod to the right, etc.,

\[
\begin{align*}
(a^r) +^r (b^r) &= (a + b) \times 1^r, \\
(a^r) \times^r (b^r) &= (a \times b) \times 1^r.
\end{align*}
\]

(13)

genoadditve unit to the right \( 0^r = 0 \) and multiplicative genounit to the right \( 1^r \). The \textit{genofields} to the left \( F^l = (a, +, \times) \) are rings with \textit{genonumbers} \( a^l = 1 \times a \), all operations ordered to the left, such as genosum \( (a^l) + (b^l) = 1 \times (a + b) \), genoprod \( (a^l) \times (b^l) = (a \times b) \times 1 \), etc., with additive genounit to the left \( 0^l = 0 \) and multiplicative genounit to the left \( 1^l = Q^{-1} \) which is generally different from the genounit \( I^r \) to the right. A \textit{bigenofield} is the structure \( \{ F, F^r \} \) with corresponding bigenouns, bigenoprod, etc. holding jointly to the left and right under the condition \( I^r = (\langle I \rangle) \).

Lemma 1\textsuperscript{[21]. Each individual genofield to the right \( F^r \) or to the left \( F^l \) is a field isomorphic to the original field \( F \). Thus, the liftings \( F \rightarrow F^r, F \rightarrow F^l \), and \( \{ F, F \} \rightarrow \{ F, F^r, F^l \} \) are axiom-preserving.

Remarks. In the definition of fields (and isofields\textsuperscript{[21]) there is no ordering of the multiplication in the sense that in the products \( a \times b \) and \( a \times b \), one can either select a multiplying \( b \) from the left, \( a \times b \), or \( b \) multiplying \( a \) from the right, \( a \times b \), because \( a \times b = a \times b \) (even for noncommutative isofields such as the isouaternions). A genofield requires that all multiplications and related operations (division, moduli, etc.) be ordered \textit{either} to the right \textit{or} to the left because now, for a commutative field \( F = R \) or \( C \), we have the properties \( a = b = a \times a = a \times b \), but in general \( a \times b = a \times b \not= a \times b = a \times a \otimes b \). Note that in the field the \textit{genounit} is the \textit{left} and \textit{right} unit, Eqs. (9). The important advances of Ref. 21 are therefore the identification, first, that the axioms of a field remain valid when the multiplication is ordered to the right or to the left, and, second, each ordered multiplication can be generalized, provided that it remains associative. The above mathematical occurrences permit the axiomatization of irreversibility beginning with the most fundamental quantities, units and numbers. In fact, the unit and product to the right, \( l^r \) and \( >^r \), characterize motion forward in \textit{time} while the conjugate quantities \( <^r \) and \( <^r \) characterize motion \textit{backward} in \textit{time}. Irreversibility is then assumed under the condition \( l^r \not= <^r \) because all subsequent mathematical structures, being always built on numbers, must preserve the same axiomatization of irreversibility, as a necessary condition for consistency.

Definition 2\textsuperscript{[12]. Let } \( S = S(r, g, R) \) be a conventional \( n \)-dimensional metric or pseudo-metric space with local chart \( r = \{ r^k \}, k = 1, 2, \ldots, n \), nowhere singular, real-valued and symmetric metric \( g = g(r, \ldots) \) and invariant \( r^2 = r^t \times g \times r \) (where \( t \) denotes transpose) over a conventional real field \( R = (a, +, \times) \). The \( n \)-dimensional \textit{genospaces} to the right \( S^r = S^r(r^r, g^r, R^r) \) are vector spaces with local genocoordinates to the right \( r^r = r^t \times I^r \), genometric \( G^r = P \times g \times I^r = (g^r) \times I^r, g^r = P \times g, \) and genoinvariant to the right

\[
(r^r)^{2r} = (r^r)^t (g^r) ^> r^r = [r^t \times (g^r) \times r] \times I^r \in R^r
\]

which, for consistency, must be a genoscalar to the right with structure \( n \times I^r \) and be an element of the genofield \( R^r \) with common genounit to the right \( I^r = P^{-1} \), where \( P \) is given by an everywhere invertible, real-valued, nonsymmetric \( n \times n \) matrix. The \( n \)-dimensional \textit{genospaces} to the left \( S^l = S^l(r^l, r, +, F) \) are genospaces over genofields with all operations ordered to the left and a common \( n \times n \)-dimensional genounit to the left \( I^l = Q^{-1} \) which is generally different from that to the right but verifying the interconnecting condition \( P = Q^t \). The \textit{bigenospaces} are the structures \( \{ S, S^r \} \) with bigenocoordinates, etc., defined over the bigenofield \( \{ R, R^r \} \) under the condition \( I^r = (\langle I \rangle)^r \).

Lemma 2\textsuperscript{[12]. Genospaces to the right \( S^r \) and, independently, those to the left \( S^l \) (thus bigenospaces \( \{ S^r, S^r \} \) ) are locally isomorphic to the original spaces \( S = \{ S, S \} \).

Proof. The original metric \( g \) is lifted in the form \( P \rightarrow P \times g \), but the unit is lifted by the \textit{inverse} amount \( I \rightarrow I^r = P^{-1} \) thus preserving the original axioms because the invariant is \( (length)^2 \times (unit)^2 \), and the same occurs for the other cases.

Remarks. The best way to see the local isomorphism between conventional and genospaces is by nothing that the latter are the results of the
following novel degree of freedom of the former (here expressed for the case of a scalar complex function $P$)

\[
\begin{align*}
\mathcal{L} & = \{ x \times g \times x \times I \equiv (r^x g_r x r) \times I \} \\
& \equiv \{ P^{-1} \times P \times (r^x g_r x r) \times I \} \equiv (r^x g_r x r) \times I \\
\end{align*}
\]

which is another illustration of the structure of the basic invariant of metric spaces (length)$^2 \times$ (unit)$^2$.

**Definition 3.** The genodifferential calculus to the right on a genospace $S^r (r^x, R^x)$ over $R^x$ is the image of the conventional differential calculus characterized by the expressions (where we have ignored for notational simplicity the multiplication to the right by $I^r$)

\[
\begin{align*}
\partial r^k \to \partial r^k & = (I^x)^r_{i} \times dr^i, \partial r_k \to \partial r_k = P^r_k \times dr^i \\
\partial r^k \to \partial r^k & = P^r_k \times \partial r^i, \partial r_k \to \partial r_k = I^r \times \partial r^i \\
\end{align*}
\]

with all operations ordered to the right and main properties

\[
\begin{align*}
\partial r^k \partial r^j = 0, \quad \partial r^k \partial r^j = 0, \\
\end{align*}
\]

The genodifferential calculus to the left is the conjugate of the preceding one for the genounit to the left $S^{I \neq I}$. The bigenodifferential calculus is that acting on $S^r (S^r)$ over $S^{I \neq I}$ for $I^r = (S^r)^i$.

**Lemma 3.** The genocalculus to the right and, independently, that to the left on genospaces over genofields, preserve all original properties, such as commutativity of the second-order derivative, etc.

**Remarks.** A important advance of Ref. 12 is the identification of an insidious lack of invariance where one would expect it the least, in the conventional differential calculus, because it is traditionally formulated without indicating its dependence on the selected unit. As a result, all generalized equations of motion expressed in terms of conventional derivatives, e.g., $dA/dt$, are not invariant.

**Definition 4.** The genogeometries to the right, or to the left or the bigenogeometries are the geometries of the corresponding genospaces when entirely expressed via the applicable genomathematics, including the genodifferential calculus.

**Lemma 4.** The genoeuclidean, genominkovskian, genometrician, and genosymplectic geometries to the right and, independently, to the left and their combined bimodular form, are locally isomorphic to the original geometries (i.e., they verify their abstract axioms).

**Remarks.** Another intriguing property identified in memoir$^{12}$ is that the Riemannian axioms do not necessarily need symmetric metrics because the metrics can also be nonsymmetric with structure $g^x = P \times g_{-} \times P_{-}$ real-valued but nonsymmetric, provided that the geometry is formulated on a genofield with genounit given by the inverse of the nonsymmetric part, $I^r = P^{-1}$ and the same occurs for the case to the left. This property has permitted the first quantitative studies on the irreversibility of interior gravitational problems via the conventional Riemannian axioms$^{12}$ e.g., the geometrization of the irreversible black hole model by Ellis, Nonopoulos, and Mavromatos$^{22}$ which has precisely a Lie-admissible structure, and other models. These remarks are important to begin to see the physical relevance of quantum deformations when written in an axiomatically correct form.

**Definition 5.** Let $\mathcal{H}$ be a conventional Hilbert space with states $|\phi>, \langle \psi|$, inner product $\langle \psi| \times |\phi>$ over the field $C = C(c, +, \times, 0)$ of complex numbers and normalization $\langle \psi| \times |\phi> = 0$. A genohilbert space to the right $\mathcal{H}^r$ is a right genolinear space with genostates $|\phi^r>, \langle \psi^r|$, genoinner product and genonomalization to the right

\[
\begin{align*}
\langle \phi^r| \times |\psi^r> = & <\phi^r| \times P \times |\psi^r> \times I^r \in C^r(c^+, +, \times ), \\
\langle \phi^r| \times |\psi^r> = & I^r \\
\end{align*}
\]

defined over a genocomplex field to the right $C^r(c^+, +, \times )$ with a common genounit $I^r = P^{-1}$. A genohilbert space to the left $\mathcal{H}^l$ is the left conjugate of $\mathcal{H}^r$ with left genounit $<Q^{-1} \times I^r$ generally different from $I^r$. A bigenohilbert space is the bistructure $\{ \mathcal{H}, \mathcal{H}^r \}$ over the bigenofield $\{ C, C^r \}$ under the conjugation $I^r = (I^r)^i$.

**Lemma 5.** The right- and left-spaces are locally isomorphic to the original space $\mathcal{H}$.

**Proof.** The original inner product is lifted by the amount $\langle |x| > \rightarrow <x \times P \times x| >$, but the underlying unit is lifted by the inverse amount, $1 \rightarrow P^{-1}$, thus leaving the original axiomatic structure unchanged.

**Remark.** The understanding of genooperator theory requires the knowledge that it is a consequence of the following, hitherto unknown
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Remarks. The reader should be aware that the entire theory of linear operators on a Hilbert spaces must be lifted into a genotypic form for consistency. For instance, conventional operations, such as Tr X, Det X, etc. can be easily proved to be inapplicable for genomathematics, and must be replaced with the corresponding genoforms. The same happens for all conventional and special functions and transforms. A systematic study of the theory of genoalinear operators will be conducted elsewhere.

We are now equipped to present, apparently for the first time, the central notion of this note which consists of the old notion of Lie-admissibility(5) upgraded with the systematic use of genomathematics.

Definition 7. Consider the conventional Lie theory with ordered $N$-dimensional basis of Hermitian operators $X = \{X_k\}$, parameters $w = \{w_k\}$, universal enveloping associative algebra $\xi = \xi(L)$, Lie algebra $L \simeq [\zeta(L)]^{-1}$, and corresponding (connected) Lie transformation group $G$ on a space $S(r, F)$ with local coordinates $r = \{r^k\}$ over a field $F$.

The Lie-admissible theory (also called Lie–Santilli genotheory(15–18)) is here defined as a step-by-step bimodular lifting of the conventional Lie theory defined on bigenspaces over bigenfields, and includes:

(7A) The universal genoenvoloping associative algebra to the right $\xi^r(L)$ of an $N$-dimensional Lie algebra $L$ with ordered basis $X^r = \{X_k\}$, $k = 1, 2, \ldots, N$, genounit $I^r = Q^{-1}$, genoassociative product $X_k > X_j = X_k \times Q \times X_j$ and infinite-dimensional genobasis characterized by the genotypic Poincaré–Birkhoff–Witt theorem to the right

$$I^r = Q^{-1}, X_k, X_j > X_k(i \leq j), X_i > X_j > X_k(i \leq j \leq k), \ldots$$

and genoexponentiation (12); the universal genoassociative algebra to the left $\xi^{\ell}(L)$ with genounit $I = P^{-1}$ and genoproduct $X_k < X_j = X_k \times P \times X_j$, with infinite-dimensional genobasis characterized by the genotypic Poincaré–Birkhoff–Witt theorem to the left

$$< I = P^{-1}, X_k, X_j < X_j(i \leq j), X_i < X_j < X_k(i \leq j \leq k), \ldots$$

and genoexponentiation to the left (12); the bigenoenvelope is the bistructure $\{\xi^r, \xi^{\ell}\}$ defined on corresponding bigenspaces and bigenfields under the condition $I^r = (I^{\ell})^r$.

(7B) A Lie–Santilli genoalgebra is a bigenolinear bigenalgebra defined on $\{\xi^r, \xi^{\ell}\}$ over $\{F^r, F^{\ell}\}$ with Lie-admissible product

$$(X_k, X_j) = X_k < X_j - X_j > X_k = X_k \times P \times X_j - X_j \times Q \times X_k$$

and properties (20) follow.
(7.C) A (connected) Lie–Santilli genotransformation group is the biset \( \{ < G, G^> \} \) of bigenotransforms on \( \{ < S, S^> \} \) over \( \{ < F, F^> \} \) with genounits \( < I = (F^)> \)

\[
\begin{align*}
\text{r}^> &= (U^>) \times (Q \times r \times I^>) = V \times r \times I^>, \quad U^> = V \times I^> \\
\langle \text{r}^> &= \langle \text{r} < \langle \text{u}^> = \langle I \times r \times P \times (\langle \text{u}^> = \langle I \times r \times W, \langle \text{u} = \langle I \times W \\
&= \text{verifying the following conditions: genodifferentiability of the maps } G^> \times S^> \rightarrow S^> \text{ and } < S^> \times S^< < G^>, \text{ invariance of the genounits and genolinearity, with realizations } U^> = \exp_a (i \times w \times X) \text{ and } U^< = \exp_a (-i \times w \times X), \text{ genolaws}
\end{align*}
\]

\[
U^> (w^>) > U^> (w^>) = U^> (w^> + w^>)
\]

and Lie-admissible algebra in the neighborhood of the genounits \( < I, I^> \) according to rule (10).

**Lemma 8.** Lie-admissible product (24) verifies the Lie axioms when defined on \( \{ < \xi, \xi^> \} \) over \( \{ < F, F^> \} \).

**Proof.** The genoenvelopes to the left \( < \xi \) and to the right \( \xi^> \) are isomorphic to the original envelope \( \xi \), thus implying \( <_I (A < B) = (A > B)_I^> \), i.e., the value of the genoproduct \( A < B = A \times P \times B \) when measured with respect to the genounit \( < I = P^{-1} \), is equal to that of the genoproduct \( A > B = A \times Q \times B \) measured with respect to the genounit \( I^> = Q^{-1} \).

The most important property of this section, which is an evident consequence of the preceding analysis, can be expressed as follows:

**Theorem 1.** Lie-admissible groups as per Definition 7 coincide at the abstract level with the original Lie-transformation groups.

**Remarks.** Note that the generators of the original Lie algebra are not lifted under genotopies, evidently because they represent conventional physical quantities, such as energy, linear momentum, angular momentum, etc. Only the operations defined on them are lifted. Note also that, when the conjugation \( P = Q^\dagger \) is violated, the Lie axioms are lost. Note also that the genothy is highly nonlinear, because the elements \( P \) and \( Q \) in genotransforms (25) have an unrestricted functional dependence, thus including that in the local coordinates. Nevertheless, genomathematics

reconstructs linearity in genospaces over genofields. The same happens for nonlocality, noncanonicity, nonunitarity, and irreversibility. In fact, on genospaces over genofields, genothories are fully linear, local, canonical unitary and reversible. Departures from these axiomatic properties occur only in their projection over conventional spaces and fields. These are evident fundamental conditions to lift nonlinear, nonlocal, noncanonical, nonunitary, and irreversible theories into a form compatible with the notoriously linear, local, canonical, unitary, and reversible axioms of the special relativity.

Needless to say, we have been able to present in this note only the rudiments of the needed genomathematics, with the understanding that its detailed study is rather vast indeed. Also, by no means should genomathematics be considered as the most general possible form admitted by the Lie axioms. Mathematics and physics are disciplines which will never admit “final theories.” In fact, a still broader multivalued hyperrealization of Lie’s theory has already been identified in Ref. 12 and cannot be treated here for brevity.

### 3. INVARIANT FORMULATION OF QUANTUM DEFORMATIONS

We are now equipped to submit the suggested invariant formulation of the \((p, q)-)\) or q-deformations. First, we have to identify the following insufficiencies:

(I) No invariant formulation is possible for \((p, q)-\)parameters because, under the nonunitary time evolution of the theory, brackets (2) or (8) assume the general Lie-admissible form (4) for which reason the latter was submitted in the first place,

\[
U \times (A, B) \times U^\dagger = p \times U \times A \times B \times U^\dagger - q \times U \times B \times A \times U^\dagger
\]

\[
= A' \times P \times B' \times Q \times A'
\]

\[
P = p \times (U \times U^\dagger)^{-1},
\]

\[
Q = q \times (U \times U^\dagger)^{-1},
\]

\[
A' = U \times A \times U^\dagger,
\]

\[
B' = U \times B \times U^\dagger
\]

(II) Despite such a generality, the formulation are still not physically acceptable because they generally violate the crucial conjugation
\( P = Q^\dagger \), without which there is the loss of the Lie axioms (Sec. 2) with consequential problems for invariance, causality, etc. The condition \( P = Q^\dagger \) is therefore assumed hereon.

(III) Brackets \((A, B) = A \times P \times B - B \times Q \times A, P = Q^\dagger \) on conventional spaces and fields are still not invariant and, therefore, they have all problematic aspects (1)-(5) of the \((p, q)\)- and \(q\)-deformations (Sec. 1). In fact, under an additional (necessarily) nonunitary transform we have

\[
U \times (A, B) \times U^\dagger = U \times A \times P \times B \times U^\dagger - U \times B \times A \times U^\dagger \\
= A' \times P' \times B' - B' \times Q' \times A',
\]

\[
P' = U^\dagger \times P \times U^{-1},
\]

\[
Q' = U^\dagger \times Q \times U^{-1},
\]

\[
A' = U \times A \times U^\dagger,
\]

\[
B' = U \times B \times U^\dagger.
\]

This implies the lack of invariance of the fundamental genounits \( I^\circ = P^{-1} \) and \( <I = Q^{-1} \), with consequential ambiguous physical applications.

The only possible resolution of the above problematic aspects known to this author is the formulation of the \(q\)-parameter deformations in the operator \((P, Q)\)-deformations formulated via the genonmathematics of Sec. 2, i.e., on bigenofield, bigenospaces, bigenoomgebra, etc.

In fact, it is easy to see that each structure to the right is invariant under the action of the genogroup to the right, e.g., \( U^\circ > I^\circ > U^\circ = I^\circ, \)

\( U^\circ (A > B) > U^\circ = A' > B' \); the initial genohermiticity to the right can be proved to remain invariant under the action of a genogroup to the right, etc.

From these grounds, genominkowskian spaces, the genopointcaré symmetry, and the genspecial relativity are expected to coincide at abstract level with the conventional corresponding structures, with the understanding that the detailed study of this expectation will be predictably long and cannot possibly be done in this note.

We close with a simple rule for the explicit construction of invariant \((P, Q)\)-deformations and related genonmathematics. It is based on the systematic use of two nonunitary transforms for the characterization of motion forward and backward in time,

\[
A \times A^\dagger \neq I, \quad B \times B^\dagger \neq I, \quad A \times B^\dagger = I^\circ = Q^{-1}
\]

\[
B \times A^\dagger = <I = (I^\circ)^\dagger
\]
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