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The Dawn of a New Era*

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Abstract— Generalization of fields in to isofields by R.M.Santilli has stimulated a corresponding generalization of all of 20th century mathematics and its application to mechanics. Santilli discovered new realizations of the abstract axioms of numeric fields with characteristic zero, based on an axiom-preserving generalization of conventional associative product and consequent positive-definite generalization of the multiplicative unit, and the resulting novel numeric fields are known as Santilli isofields and the elements are called as Isonumbers. Santilli Isomathematics and Isomechanics has been successfully applied to the representation of extended-deformable particles moving within physical media under Hamiltonian as well as contact non-Hamiltonian interactions. Additionally, Santilli discovered a second realization of the abstract axioms of a numeric field, with arbitrary (non-singular) negative definite generalized unit and related multiplication, today known as Santilli isodual isonumber that have stimulated a second generalization of the 20th century mathematics and mechanics known as Santilli isodual isomathematics and isodual isomechanics. The latter methods are used for the classical as well as operator form of antimatter in full democracy with the study of matter. In this paper, we present a comprehensive study of Santilli's epoch making discoveries of isonumbers and their isoduals along with their application to isomechanics and its isodual for matter and antimatter, respectively.

Keywords— Isonum, Isofield, Isonumber, Isodual Number, Isodual-Isonumber, Genonumber.

I. INTRODUCTION

While the scientific discoveries and mathematical knowledge were moving hand in hand, towards the end of 20th century there were few mathematically unexplained physical phenomena in Quantum Physics and Quantum Chemistry. These new physical situations could not be faithfully described by the existing mathematical structures and called for more generalized mathematical structures.

It was Enrico Fermi, [3] beginning of chapter VI, p.111 said “...... there are some doubts as to whether the usual concepts of geometry hold for such small region of space.” His inspiring doubts on the exact validity of quantum mechanics for the nuclear structure led to the genesis of a whole new kind of generalized mathematics, called isomathematics and generalized mechanics, called as Hadronic mechanics.

In fact, the prevailing Newtonian and Einsteinian ‘Dynamical systems’ called as ‘Exterior Dynamical systems’ which are characterized as ‘local’, ‘linear’ ‘Lagrangian’ and ‘Hamiltonian’ could not accommodate these obscure situations. Thus it was the pressing demand of time to formulate new mathematical theory which could deal with the obscure phenomena and develop a new physical theory. This stupendous task was taken up by the Italian-American theoretical physicist Ruggero Maria Santilli, President of Institute for Basic Research, Palm Harbor, Florida, USA and did the pioneering work by defining axiom-preserving, nonlinear, nonlocal and noncanonical isotopies of conventional mathematical structures, including units, fields, vector spaces, transformation theory, algebras, groups, geometries, Hilbert spaces etc. while at Department of Mathematics of Harvard University in the early 80’s. Prof. Santilli has rightly said: “There can not be really new physical theories without really new mathematics, and there can not be really new mathematics without new numbers.”

The founders of analytic mechanics, such as Lagrange, Hamilton [4] and others classified dynamical systems in to two kinds. First one is the ‘Exterior Dynamical system’ and the second one is the more complex but generalized ‘Interior Dynamical system’. It was Prof. Santilli who at the Department of Mathematics of Harvard University, for the first time, drew attention of the scientific community towards the crucial distinction between exterior and interior dynamical systems and presented insufficiencies of prevailing mathematical and physical theories by submitting the so-called axiom-preserving, nonlinear, nonlocal, and noncanonical isotopies of Lie’s theory [5] under the name Lie Isotopic theory. Further generalization as Lie-admissible theory [6,7] was also achieved by him.

II. DYNAMIC SYSTEMS

Exterior Dynamical Systems: In this system Point-like particles are moving in a homogeneous and isotropic vacuum
with local-differential and potential-canonical equations of motion. These are linear, local, Newtonian Lagrangian and Hamiltonian. Conventional Mathematical structures such as Algebras, Geometries, Analytical Mechanics, Lie Theory can faithfully represent these systems.

**Interior Dynamical Systems:** In this system we consider extended non-spherical deformable particles moving within non-homogeneous anisotropic physical medium. These are non-linear, non-local, non-Newtonian, non-Lagrangian and non-Hamiltonian. The mathematical structures needed to describe these systems are most general possible which are axiom preserving; non-algebraic number theory [14].

Starting with the definition of ‘isonumbers’, starting with the definition of ‘isonumbers’, we present the theory of isonumbers, pseudoisonumbers, “hidden numbers” and their isoduals. Genonumbers, pseudogenonumbers and their isoduals are also of fundamental importance. Resulting iso-algebras have tremendous applications in generalising prevailing concepts in Quantum Physics and Quantum Chemistry.

In his study Santilli has taken into account the four normed algebras over reals as given in the above theorem. The isotopic lifting of these algebras give rise to isotopies of normed algebras with multiplicative unit of dimension 1,2,4 and 8 which includes realization of ‘isorreal numbers’, ‘isocomplex numbers’, ‘isoquaternions’, ‘isooctonions’. Isodualities of these structures give isodial isonumbers.

In a nutshell, the theory of isonumbers is at the foundation of current studies of nonlinear-nonlocal-nonhamiltonian systems in nuclear, particle and statistical physics, superconductivity and other fields.

### III. Origin of Isonumbers

The concept of ‘Isotopy’ plays a vital role in the development of this new age mathematics ref. R.J. Bruck [2] and [19].

The first and foremost algebraic structure defined by Santilli is ‘isofield’. Elements of an isofield are called as ‘isonumbers’. The conversion of unit $1$ to the isounit $\hat{1}$ is of paramount importance for further development of ‘Isonomathematics’.

The reader should be aware that there are various definitions of “fields” in the mathematical literature [20], [21], [22] and [14] with stronger or weaker conditions depending on the given situation. Often “fields” are assumed to be associative over the multiplication. I.e.

$$ax(bxc) = (axb)xc \quad \forall a,b,c \in F$$

We formally define an isofield [23], [24] as follows.
Definition 1.1  Given a “field” \( F \), here defined as a ring with with elements \( a,b,c \), sum \( a+b \), multiplication \( ab \), which is commutative and associative under the operation of conventional addition + and (generally nonassociative but) alternative under the operation of conventional multiplication \( \times \) and respective units 0 and 1. Santilli’s “isofields” are rings of elements \( \hat{a} = a\hat{1} \) where \( a \) are elements of \( F \) and \( \hat{1} = T^{-1} \) is a positive definite \( n \times n \) matrix generally outside \( F \) equipped with the same sum \( \hat{a} + \hat{b} \) of \( F \) with related additive unit \( \hat{0} = 0 \) and a new multiplication \( \hat{a} \hat{\times} \hat{b} = \hat{a} \hat{\Theta} \hat{b} \), under which \( \hat{1} = T^{-1} \) is the new left and right unit of \( F \) in which case \( \hat{F} \) satisfies all axioms of the original field.

\( T \) is called the isoelement. In the above definitions we have assumed “fields” to be alternative, i.e.

\[
a \times (b \times b) = (a \times b) \times b, \quad (a \times a) \times b = a \times (a \times b) \quad \forall a,b \in F.\]

Thus, “isofields” as per above definition are not in general isoassociative, i.e. they generally violate the isoassociative law of the multiplications, i.e.

\[
\hat{a} \hat{\times} (\hat{b} \hat{\times} \hat{c}) = (\hat{a} \hat{\times} \hat{b}) \hat{\times} \hat{c}, \quad \forall \hat{a},\hat{b},\hat{c} \in \hat{F}
\]

but rather satisfy isoalternative laws.

The specific need to generalize the definition of “number” to ‘real numbers’, complex numbers, ‘quaternions’ and ‘octonians’ suggested the above definition. The resulting new numbers are ‘isoreal numbers’, isocomplex numbers, ‘isouaternions’ and ‘isooctonians’ respectively, where ‘isooctonians’ are alternative but not associative.

The ‘isofields’ \( \hat{F} = \hat{F}(\hat{a},\hat{\times},\hat{\psi}) \) are given by elements \( \hat{a},\hat{\psi},\hat{\psi} \) characterized by the definition of the “number” to ‘real numbers’, complex numbers, ‘quaternions’ and ‘octonians’ suggested the above definition. The resulting new numbers are ‘isoreal numbers’, isocomplex numbers, ‘isouaternions’ and ‘isooctonians’ respectively, where ‘isooctonians’ are alternative but not associative.

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Santilli has shown that the transition from exterior dynamical system to interior dynamical system can be effectively represented via the isotopy of conventional multiplication of numbers \( a \) and \( b \) from its simple possible associative form \( ab \) in to the isotopic multiplication, or isomultiplication for short, as introduced in [8].

Isomultiplication then lifts the conventional unit 1 defined by \( 1 \times a = a \times 1 = a \) to the multiplicative isounit \( \hat{1} \) defined by

\[
\hat{1} \times a = a \hat{1} = a, \text{where} \quad \hat{1} = T^{-1}
\] (2)

Under the condition that \( \hat{1} \) preserves all the axioms of 1 the lifting 1 \( \rightarrow \hat{1} \) is an isotopy, i.e. the conventional unit 1 and the iso unit \( \hat{1} \) (as well as the conventional product \( ab \) and its isotropic form \( \hat{a} \hat{\times} \hat{b} \) have the same basic axioms and coincide at the abstract level by conception. The isounit \( \hat{1} \) is so chosen that it follows the axioms of the unit 1 namely; boundedness, smoothness, nowhere degeneracy, hermiticity and positive-definitness. This ensures that the lifting 1 \( \rightarrow \hat{1} \) is an isotopy and conventional unit 1 and the isounit \( \hat{1} \) coincide at the abstract level of conception.

Thus, the isonumbers are the generalization of the conventional numbers characterized by the isounit and the isoproduct as defined above.

The liftings \( a \rightarrow \hat{a} \), and \( x \rightarrow \hat{x} \) can be used jointly or individually.

It is important to note that unlike isotopy of multiplication \( x \rightarrow \hat{x} \), the lifting of the addition \( + \rightarrow \hat{+} \) implies general loss of left and right distributive laws. Hence the study of such a lifting is the question of independent mathematical investigation.

The first generalization was introduced by Prof. Santilli when he generalized the real, complex and quaternion numbers [23], [24] based on the lifting of the unit 1 into isounit \( \hat{1} \) as defined above. Resulting numbers are called isoreal numbers, isocomplex numbers and isoquaternion numbers.

In fact, this lifting leads to a variety of algebraic structures which are often used in physics. The following flowchart is self explanatory.

Isonumbers \( \rightarrow \) Isofields \( \rightarrow \) Isospaces \( \rightarrow \)

Isotransformations \( \rightarrow \) Isoalgebras \( \rightarrow \) Isogroups \( \rightarrow \)

Isoalgebras \( \rightarrow \) Isoalgebras \( \rightarrow \) Isoalgebras \( \rightarrow \)

Is mosquito \( \rightarrow \) Isogeometry etc.

The isounit is generally assumed to be outside the original field with all the possible compatible conditions imposed on it. For rudiments of isomathematics reader can refer to [1, 6, 7, 25].

The lifting of unit 1 to isounit \( \hat{1} \) may be represented as \( I \rightarrow \hat{I}(t,r,r,p,T,\psi,\psi^1,\partial\psi,\partial\psi^1,\cdots) \), where \( t \) is time, \( r \) is the position vector, \( p \) is the momentum vector, \( \psi \) is the wave function and \( \psi^1 \) are the corresponding partial differentials. The positive definiteness of the isounit \( \hat{1} \) is assured by \( \hat{I}(t,r,r,p,T,\psi,\psi^1,\partial\psi,\partial\psi^1,\cdots) \) \( \frac{\hat{1}}{I} > 0 \) where \( \hat{T} \) is called the isotopic element, a positive definite quantity.

The isonumbers are generated as, \( \hat{a} = \hat{a} \) \( n=0,1,2,3,\cdots \). 

Isofields are of two types, isofield of first kind, wherein the
isounit does not belong to the original field, and isofield of second kind, wherein the isounit belongs to the original field. The elements of the isofield are called as isonumbers. This leads to number of new terms and parallel developments of conventional mathematics.

IV. Isoounits and their Isoduals

Prof. Santilli further, introduced isodual isounumbers [26, 27, 28] by lifting the isounit into the form
\[ \hat{1} \hat{n} \hat{n}' a = \hat{a} \hat{n} \hat{n}' \hat{1}, \text{ where } \hat{1}' := -\hat{1} \]
(3)
called the isodial isounit following lifting of iso-multiplication defined in (1) into the isodial multiplication called isoduality
\[ \hat{a} \hat{n} \hat{n}' \hat{b} := \hat{a} \hat{n} \hat{n}' (\hat{a} \hat{n} \hat{n}' \hat{b}) = -\hat{a} \hat{n} \hat{n}' \hat{b} = -\hat{a} \hat{n} \hat{n}' \hat{b} \]
where \( T^d = T \).

The isodial isounits were first conceived as characterized by isodial multiplication (4) with respect to the multiplicative isodial isounit \( \hat{1}' = -\hat{1} \).

The significance of isonumbers and isodual isounumbers lies in fulfilling the specific physical needs refs [18, 29, 30, 31] as given below;

- In the exterior dynamical system ordinary particles moving in the vacuum are characterized by conventional numbers.
- In the interior dynamical system ordinary particles moving in the physical medium are characterized by isonumbers.
- In the exterior dynamical system ordinary antiparticles moving in vacuum are characterized by isodual isounits.
- In the interior dynamical system the antiparticles moving in the physical medium are characterized by isodual isounits.

Interpretation of isonumbers is customary characterization of antiparticles via negative-energy solutions of Dirac’s equations behave in an un-physical way when interpreted with respect to the same numbers and unit \( \hat{1} \) of particles, forcing various hypothetical assumptions and postulates, where as, reinterpretation of antiparticles with same negative energy solutions when interpreted as belonging to the field of isodual numbers behave in a fully physical way ref [1]. This treatment of antiparticles with isodual numbers also leads to intriguing geometrical implications which predict another universe, called as isodial universe, interconnected to our universe via isoduality and identified by the isodualities of Riemannian geometry and their isounits refs.[31, 24, 32]. Thus, the isodial theory emerged from the identification of negative units in the antiparticle component of the conventional Dirac equation and the reconstruction of the theory with respect to this new negative unit. Hence isoduality provides a mere reinterpretation of Dirac’s original notion of antiparticle leaving all numerical predictions of electro-weak interactions essentially unchanged.

In view of the definition of an isofield [1], we can say that an isofield is an additive abelian group equipped with a new unit (called isounit) and isomultiplication defined appropriately so that the resulting structure becomes a field.

If the original field is alternative then the isofield also satisfies weaker isosolutionar laws as follows.

\[ \hat{a} \hat{x} (\hat{b} \hat{x} \hat{b}) = (\hat{a} \hat{x} \hat{b}) \hat{x} \hat{b} \text{ and } \hat{a} \hat{x} (\hat{a} \hat{x} \hat{b}) = (\hat{a} \hat{x} \hat{a}) \hat{x} \hat{b}. \]

We mention two important propositions by Santilli.

**Proposition 2.1** The necessary and sufficient condition for the lifting (where the multiplication is lifted but elements not the elements) \( \hat{F}(a, +, x) \rightarrow \hat{F}(\hat{a}, +, \hat{x}) \hat{x} = \hat{x} \hat{T} \hat{x}, \hat{1} = T^{-1} \) to be an isotopy (that is for \( \hat{F} \) to verify all axioms of the original field \( F \)) is that \( T \) is a non-null element of the original field \( F \).

**Proposition 2.2** The lifting (where both the multiplication and the elements are lifted) \( \hat{F}(a, +, x) \rightarrow \hat{F}(\hat{a}, +, \hat{x}), \hat{a} = a \hat{x} \hat{1}, \hat{x} = \hat{x} \hat{T} \hat{x}, \hat{1} = T^{-1} \) constitutes an isounit even when the multiplicative isounit \( \hat{1} \) is not an element of the original field.

The above proposition guarantees the physically fundamental capability of generating Plank’s unit \( v \) of quantum mechanics into an integro-differential operator \( \hat{1} \) for quantitative treatment of nonlocal interactions [33].

As the first application of the isointerpretations of numbers Santilli considers the set \( S = \{ n \mid n > 0 \} \), the set of all purely imaginary numbers. This set is not closed (\( i^2 = -1 \) \( \in S \)). On the other hand, the same set \( S \) represented as \( \hat{S}(\hat{n}, +, \hat{x}) \) with \( \hat{n} = in \) constitutes an isofield. i.e. it verifies all the axioms of a field including closure under isomultiplication because \( T = i^{-1} \) and \( \hat{n} \hat{x} \hat{m} = in \hat{x} \hat{m} = imm \in \hat{S} \).

This illustrates an important fact that, even when a given set does not constitute a field, there may exist an isofield under which it verifies the axioms of a field.

As stated earlier the lifting of \( + \) to \( + \) does not necessarily produce an isounit of a given field. This lifting does not preserve the distributivity in the resulting set as stated in the following proposition 2.3.

**Proposition 2.3** The lifting \( F(a, +, x) \rightarrow \hat{F}(\hat{a}, +, \hat{x}) \) where
\[ \hat{a} = a \hat{x} \hat{1}, \hat{1} = T^{-1} \]
\( K \) is an element of the original field \( F \) and \( T \) is an arbitrary invertible quantity, is not an isofield for all nontrivial values of the quantity \( \hat{K} = F(0) \), because it preserves all the axioms of proposition 2.1 except the distributive law.

Based on the failure of distributivity Santilli defines “pseudoisofields” as follows.
Definition 2.1 Let \( \hat{F}(\hat{a},+\hat{x}) \) be an isofield as defined above. Then the “pseudoisofields” \( \hat{F}((\hat{a},+\hat{x}) \) are given by the images of \( \hat{F}(\hat{a},+\hat{x}) \) under all possible liftings of the addition \( + \rightarrow \hat{+} = +K \), with additive isounit \( \hat{0} = -K_x = K \hat{1}, K \neq 0 \) in which case the elements \( \hat{a} \) are called the “pseudoisounumbers”.

For the algebra of isounumbers and isodual numbers readers are advised to refer [1, 34].

Images of field, isofield and pseudoisofield under the change of sign of the isounit \( 1 \rightarrow \hat{1}^\dag = -1 \) is called the Isotopic conjugation or isoduality ref. [28, 29, 30].

Definition 2.2 Let \( \hat{F}(\hat{a},+\hat{x}) \) be a field as per definition 1.1. Then the isofield \( F^d(\hat{a}^d, +\hat{x}^d) \) is constituted by the elements called “isodual numbers”
\[
a^d := a \times 1^d = -a
\] (5)
defined with respect to the “isodual multiplication” and related “isodual unit”
\[
x^d := x \times 1^d = -x, \quad 1^d = -1.
\] (6)

Definition 2.3 Let \( \hat{F}(\hat{a},+\hat{x}) \) be an isofield as per definition 1.1. Then the isofield \( F^d(\hat{a}^d, +\hat{x}^d) \) is constituted by the elements called “isodual isounumbers”
\[
\hat{a}^d := a^d \times 1^d = -a^d \times 1
\] (7)
where \( a^d \) is the conventional conjugation of \( F \) (e.g. complex conjugation) defined in terms of the “isodual isomultiplication”
\[
\hat{x}^d := x \times T^d = -\hat{x}, \quad T^d = -T.
\] (8)

Definition 2.4 Let \( \hat{F}(\hat{a},+\hat{x}) \) be a pseudofield \( \hat{F}((\hat{a},+\hat{x}) \) as per definition 2.1. Then the “isodual pseudofield” \( F(\hat{a}^d, +\hat{x}^d) \) is given by the image of the original isofield under isodualities (6) and (7) plus the additional isoduality
\[
\hat{0} \rightarrow \hat{0}^d = 0
\] (9)
and its elements \( \hat{a}^d \) are called “isodual pseudounumbers”

V. Iso numbers and their Isoduals

2.5 Isoreal numbers and their isoduals

2.5.1 Real numbers:

Real numbers constitute a one-dimensional normative associative and commutative normed algebra \( U^d(1) \) which is anti-isomorphic to \( U(1) \) ref.[1].

Real numbers are the conventional numbers \( n \) defined with respect to the isodual unit \( 1^d = -1 \). The isodual conjugation of real numbers is then written as
\[
n = n_x \times 1^d = n_x 1^d = -n.
\] (14)

Note that, such a sign inversion occurs when the isodual real numbers are projected in the field of conventional real numbers. As a result, all the numerical values change sign under isoduality.

The one-dimensional real isodual Euclidean space \( E^d(\hat{x}, \hat{0}^d, R^d(n^d, +x^d)) \) is a straight line, with conventional additive unit \( 0 \), and isodual multiplicative unit \( 1^d = -1 \). The \( R^d(n^d, +x^d) \) represents the Euclidean space \( E^d(\hat{x}, \hat{0}^d, R^d(n^d, +x^d)) \). Also, the isodual dilations are defined by
\[
x^d = n_x^d \times x = n_x x.
\] (15)

This establishes an isomorphism between \( R^d(n^d, +x^d) \) and the isodual group of dilations \( G^d(1) \) (the conventional group reformulated according to the multiplicative unit \( 1^d \)). Santilli points out that \( E^d(\hat{x}, \hat{0}^d, R^d) \) and \( E^d(\hat{x}, \hat{0}^d, R^d) \) are antisymmetric and the same property holds for \( G^d(1) \) and \( G^d(1) \). Also, the isodual dilations coincide with dilations as defined above. Santilli further says that “this could be the a reason for the lack of detection of isodual numbers until then.”

In the isodual case, the isodual basis is given by
\[
e^d = 1^d
\] (16)
with isodual norm.
| n| \equiv (\sigma \times n)^{\frac{1}{2}} \times |n| = -|n| < 0 \quad (17) \\
\text{satisfying the axioms} \\
|n^d| \times |n^d| = |n^d| = |n^d| \\
(18)

2.5.3 Isoreal numbers:

Isoreal numbers constitute a one-dimensional, isonormed isoassociative and isocommutative isoalgebra \( U(1) = U(1) \) ref.\[1\].

Isoreal numbers are the numbers \( \hat{n} = n \times \hat{T} \) of an isofield of Class I, with isomultiplication defined by \( \hat{x} = x \times \hat{T} \) and isounit \( \hat{1} = T^{-1} > 0 \), generally outside the original field \( R(n, +, \times) \). These can be represented as the isoeuclidean spaces \( \hat{E}_{i,1} = (x, \hat{\delta}, \hat{R}(\hat{n}, +, \hat{x})) \) with \( \hat{\delta} = T \hat{\delta} \), over \( \hat{R}(\hat{n}, +, \hat{x}) \) the isotopes of conventional one-dimensional Euclidean spaces \( E_{i,1} = (x, \delta, R) \).

Some of the important remarks are as follows.

- The conventional Euclidean space \( E_{i,1} = (x, \delta, R) \) and its isoassociative isoisocommutative isoalgebra \( \hat{E}_{i,1} = (x, \hat{\delta}, \hat{R}) \) are locally isomorphic due to the joint liftings \( \hat{\delta} = T \delta \) and \( \hat{1} = T^{-1} \).
- The isoeuclidean spaces \( \hat{E}_{i,1} = (x, \hat{\delta}, \hat{R}) \) is not a Riemannian space because of the intrinsic dependence of the isometric \( \hat{\delta} \) on the derivatives \( x, \hat{x}, \ldots \) as well as the fact that the basic unit is not the conventional quantity \( 1 \).
- However, \( \hat{E}_{i,1} = (x, \hat{\delta}, \hat{R}) \) is a simple, yet bona-fide isosystem of dimension one with coordinate unit \( \hat{1} \).

In fact, the one-dimensional isospace \( \hat{E}_{i,1} = (x, \hat{\delta}, \hat{R}) \) represents a one-dimensional generalization of conventional straight line, called as \textit{isoline}. This is because of its intrinsic nonlinearity, nonlocal and noncanonical metric \( \hat{\delta}(t, x, \hat{x}, \ldots) \) with multiplicative isounit \( \hat{1} = T \delta = 1 \). The isodilation \( \hat{R}(\hat{n}, +, \hat{x}) \) can be realized via isodilations on \( \hat{E}_{i,1} = (x, \hat{\delta}, \hat{R}) \) as;

\[ \hat{x}' = \hat{n} \hat{x} \times = n \times x, \quad (19) \]

which is isodual dilation and represents one-dimensional isogroup of isodilations \( \hat{G}(1) \) same as the group \( G(1) \) realized with respect to isounit \( \hat{1} \).

Again, the isobasis is given by

\[ \hat{e} = 1 \]

with isonorm defined as;

\[ \left\| \hat{n} \right\| = (n \times n)^{\frac{1}{2}} \times \hat{1} = |n| \times \hat{1} \]

which is the conventional norm only rescaled to the new unit \( \hat{1} \). We then also have

\[ \left\| \hat{n} \hat{x} \right\| = \left\| \hat{n} \right\| \times \left\| \hat{x} \right\| \]

2.5.4 Isodual Isoreal numbers:

The isodual isoreal numbers are the realization of the one-dimensional isodual, isonormed, isoassociative and isocommutative isoalgebra \( \hat{U}(1) = U(1) \) ref.\[1\].

These are the isoreal numbers \( \hat{n} = n \times \hat{T} \), \( \hat{1}' = -\hat{1} \) in the isodual isofield \( \hat{R}(\hat{n}, +, \hat{x}) \). These correspond to \( \hat{E}_{i,1} = (x, \hat{\delta}, \hat{R}) \) the isoeuclidean space of Class II \( \hat{E}_{i,1} = (x, \hat{\delta}, \hat{R}) \) of dimension one with isodual isocommutative isogonal group \( \hat{G}'(1) \). The underlying isomorphism

\[ E_{i,1} = (x, \delta, R(n', +, x')) = \hat{E}_{i,1} = (x, \hat{\delta}, \hat{R}(\hat{n}, +, \hat{x})) \]

implies the \( \hat{G}'(1) = G'(1) \). The isodual isobasis is defined by

\[ \hat{e}' = \hat{e} \]

The isodual isonorm

\[ \left\| \hat{n} \right\| = (n \times n)^{\frac{1}{2}} \times \hat{1}' = \left\| \hat{n} \right\| \]

verifies the axioms

\[ \left\| \hat{n} \hat{x} \right\| = \left\| \hat{n} \right\| \times \left\| \hat{x} \right\| \]

2.6 Isocomplex numbers and their isoduals

2.6.1 Complex numbers:

Complex numbers constitute a two-dimensional, normed associative and commutative algebra \( U(2) \) ref.\[1\].

Complex numbers \( C = n_0 + n_1 i \) where \( n_0 \) and \( n_1 \) are real numbers and \( i \) is an imaginary unit, are represented in a Gauss plane which is a realization of two-dimensional Euclidean space \( E_{i,2} = (x, \delta, R(n, +, x)) \) satisfying
\[ x^2 = x \cdot x = x_i^2 + x_j^2 \in \mathbb{R}(n, +, \times) \] (28)

whose group of isometries is one dimensional Lie Group \( O(2) \), the invariance of the circle. Hence, complex numbers can be represented via fundamental representation of \( O(2) \) as follows.

A one-to-one correspondence between complex numbers and points in the Gauss plane can be obtained by following dilative rotations

\[ z' = (x_i + x_j) = c \circ z = (n_0 + n_i) \circ (x_i + x_j) \]

and multiplication

\[ c \circ z = (n_0, n_i) \circ (x_i, x_j) = (n_0 x_i - n_i x_j, n_0 x_j + n_i x_i) \]

which preserve all the properties of a field.

Representation of a complex number via matrices has the following form

\[ c := n_0 \times I_0 + n_i \times I_i = \begin{pmatrix} n_0 & n_i x_i \\ n_i & n_0 \end{pmatrix} \]

where

\[ I_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad I_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

which are well known as the identity and fundamental representation of \( O(2) \).

Norm can also be defined as;

\[ \| c \| = \| n_0 + n_i x_i \| = (\text{Det} c)^{\frac{1}{2}} = (n_0^2 + n_i^2)^{\frac{1}{2}} \] (33)

Also, the identification of basis in terms of matrices is \( e_1 = I_0 \) and \( e_2 = I_i \).

2.6.2 Isodual complex numbers:

Isodual complex numbers constitute a two-dimensional isodual, normed, associative and commutative algebra \( U^d \) \textsuperscript{2} anti-isomorphic to \( U(2) \) ref.[1].

Isodual complex numbers are given by

\[ C^d = \{ (c^d, x^d) | x^d = -c^d \} = \mathbb{E}(x^d) = \mathbb{E}(\mathbb{C}) \]

where \( \mathbb{E} \) is the complex conjugation. Thus, given a complex number \( c = n_0 + n_i x_i \), its isodual is given by

\[ c^d = -n_0 + n_i x_i \]

Considering the group of isometries, the one-dimensional isodual Lie group \( O^d(2) \) i.e. the image of \( O_2 \) under the lifting \( L = \text{diag}(1, 1) \rightarrow L^d = \text{diag}(\text{I}, -1, -1) \) of the two-dimensional isodual Euclidean space \( E^d_2 (x, \delta^d, R^d(n^d, +, x^d)) \) with basic invariant

\[ x^{2d} = x^d \cdot x = x_i^d x_j^d = x_i^2 + x_j^2 = x_i^2 \cdot x_j^2 + x_j^2 = x_i^2 \cdot x_j^2 + x_j^2 \in R^d(n^d, +, x^d) \]

isodual complex numbers can be characterized by the isorepresentation of \( O^d(2) \).

Now, the image of the conventional plane under isoduality is the isodual Gauss plane. Also, a one-to-one correspondence between the points \( P = (x_i, x_j) \) and complex numbers can be defined by isodual dilative rotations as

\[ z' = (x_i + x_j) = c^d \circ z = (n_0 + n_i x_i) c^d (x_i + x_j) \]

following the multiplication rules

\[ c^d \circ z = (-n_0, n_i) c^d (x_i, x_j) = (-n_0 x_i - n_i x_j, n_0 x_j + n_i x_i) \]

which preserve all the properties of a field.

Isodual transformations form an isodual group \( G^d(2) \) antiisomorphic to \( G(2) \). Even the one-to-one correspondence between complex numbers and Gauss plane continues under isoduality.

Matrix representation of isodual complex numbers can be defined as

\[ c^d := n_0^d \times I_0^d + n_i^d \times I_i^d = \begin{pmatrix} n_0 & n_i x_i \\ n_i & n_0 \end{pmatrix} \]

\[ I_0^d = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad I_i^d = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

with the isodual unit and isodual representations of \( O^d(2) \) respectively.

The isodual norm can be defined as

\[ |c^d| = |n_0 + n_i x_i|^d := |\text{Det} c^d|^\frac{1}{2} = (n_0^2 + n_i^2)^{\frac{1}{2}} \]

which may be written as

\[ |c^d| = |c \times x^d|^d = (n_0^2 + n_i^2)^{\frac{1}{2}} \times |I_0^d|^d = (n_0^2 + n_i^2)^{\frac{1}{2}} \times I_0^d \]

and verifies the axioms

\[ |c^d \circ c^d|^d = |c^d|^d \times x^d|^d \times |c^d| 

\[ c^d, c^d \in C^d \].

The isodual basis in terms of matrices is given by

\[ e_1^d = I_0^d, \quad e_2^d = I_i^d \].

2.6.3 Isocomplex numbers

Isocomplex numbers constitute a two-dimensional, isonormed, isoassociative and isoalgebras over the isoreals \( \hat{U}(2) = \hat{U(2)} \) ref.[1].

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In this case we consider the isofield of isocomplex numbers
\[
\hat{C} = \{(\hat{c}, +, \hat{x}) | \hat{x} = xT \hat{x}, \hat{T} = T^{-1}, \hat{c} = c \times \hat{1}\},
\tag{43}
\]
with generic element \(\hat{c} = \hat{n}_h + \hat{n}_h \times i\). Here we need the two-dimensional isoeuclidean space of class I, \(E_{t,2}(x, \hat{\delta}, \hat{R}(\hat{n}_h, +, \hat{x}))\). The most important realization used in the physical literature has the diagonalized and positive-definite isotropic element and isounit
\[
T = \text{diag}(\hat{b}_1, \hat{b}_2), \hat{1} = \text{diag}(\hat{b}_1^{-1}, \hat{b}_2^{-1}), \hat{b}_k > 0, k = 1, 2,
\tag{44}
\]
with basic isoseparation
\[
x^2 = (x^2 \hat{\delta}c)\hat{x}\hat{1} = (x_1 \hat{b}_1^2 \hat{x}_1 + x_2 \hat{b}_2^2 \hat{x}_2)\hat{1} \in \hat{R}(\hat{n}_h, +, i, \hat{\delta}, c)\hat{c} \hat{O} \hat{1} \hat{c}^{-1} \hat{1}. \tag{45}
\]
The group of isometries of this space is the Lie group \(\hat{O}(2) \equiv O(2)\), the group constructed with respect to the multiplicative isounit \(\hat{1} = \text{diag}(\hat{b}_1^{-1}, \hat{b}_2^{-1})\) which provides the invariance of all possible ellipses with semi-axes \(a = \hat{b}_1^{-1}, b = \hat{b}_2^{-1}\) as the infinitely possible deformation of the circle \(x^2 = x_1^2 + x_2^2 \in \hat{R}(\hat{n}_h, +, i, \hat{\delta}, c)\hat{c}\). Thus, isocomplex numbers are characterize via fundamental representation of \(\hat{O}(2)\).

Isocomplex numbers \(\hat{c} = (\hat{n}_h, \hat{n}_h)\) can also be characterized to the set of points \(P = (\hat{x}_1, \hat{x}_2)\) on the isogauus plane on \(E_{t,2}(x, \hat{\delta}, \hat{R}(\hat{n}_h, +, \hat{x}))\).

In fact, a one-to-one correspondence between isocomplex numbers \(\hat{C}(\hat{c}, +, \hat{x})\) and the points on the isogauus plane can be defined via following isodiotative isorotations
\[
z^1 = (x_1 + x_2 \times i) = \hat{c} \hat{z}^2 \hat{z} \tag{46}
\]
characterized by the isomultiplication defined as
\[
\hat{c} \hat{z}^2 \hat{z} = (\hat{n}_h, \hat{n}_h) \hat{z} (x_1, x_2) = \left(\left[(\hat{n}_h \times x_1) \hat{1} - \hat{\Delta}^2 (\hat{n}_h \times x_2) \times \hat{1}\right], \left[(\hat{n}_h \times x_2) \hat{1} + (\hat{n}_h \times x_2) \times \hat{1}\right]\right), \tag{47}
\]
with
\[
\hat{\Delta} = \det T = b_1^2 \times b_2^2
\tag{48}
\]
Isocomplex numbers also admit following two-by-two matrix representation:
\[
\hat{c} = \hat{n}_h \times \hat{i} + n_i \hat{i} = \begin{pmatrix}
\hat{n}_h \times b_1^2 & i \times n_i \times b_1^2 \times \hat{\Delta}_{2}^2 \\
 i \times n_i \times b_2^2 \times \hat{\Delta}_{2}^2 & \hat{n}_h \times b_2^2
\end{pmatrix}
\tag{49}
\]
where
\[
\hat{i} = \hat{i}_0 = \begin{pmatrix}
\hat{b}_1^2 & 0 \\
0 & \hat{b}_2^2
\end{pmatrix}, \hat{\Delta} = \text{Det} T = \hat{b}_1^2 \times \hat{b}_2^2
\tag{50}
\]
which characterize the isounit and the fundamental (adjoint) representation of \(\hat{O}(2)\) respectively.

The set of matrices (63) is closed under addition and multiplication. Also, each element possesses the isoinverse
\[
\hat{c}^{-1} = \hat{c}^{-1} \hat{1}
\tag{51}
\]
where \(\hat{c}^{-1}\) is the ordinary inverse. As a result, \(\hat{S}(\hat{c}, +, \hat{x})\) is an isofield with the local isomorphism \(\hat{S}(\hat{c}, +, \hat{x}) = \hat{C}(\hat{c}, +, \hat{x})\). We note that the one-to-one correspondence between complex numbers and Gauss plane is preserved under isotopy. It is important know that the realization of complex numbers as matrices is not unique. The isonorm is defined as
\[
\|\hat{c}\| = \text{Det} (\hat{c} \times T) = \hat{1} \times \hat{I}_0 = (n_0^2 + \hat{\Delta} n_2^2) \hat{1} \times \hat{I}_0
\tag{52}
\]
which readily verifies the axiom
\[
\|\hat{c} \times \hat{c}'\| = \|\hat{c}\| \|\hat{c}'\| \in \hat{R}, \quad \hat{c}, \hat{c}' \in \hat{C}.
\tag{53}
\]
The isobasis is given by
\[
\hat{e}_1 = \hat{I}_0, \hat{e}_2 = \hat{1}.
\tag{54}
\]

2.6.4 Isodual isocomplex numbers:
The isodual isocomplex numbers constitute a two-dimensional, isodual, isonormed, isosassociative and isocommutative isolgebras over the isodual isoratals \(\tilde{U}^d(2) = U^d(2)\) ref.[1].

Now the isodual isocomplex numbers are defined as
\[
\tilde{c} = \hat{n}_h \times \hat{i} + n_i \hat{i} = \begin{pmatrix}
\hat{n}_h \times b_2^2 & i \times n_i \times b_2^2 \times \hat{\Delta}^2 \\
 i \times n_i \times b_2^2 \times \hat{\Delta}^2 & \hat{n}_h \times b_2^2
\end{pmatrix}
\tag{55}
\]
with generic element \(\tilde{c}^d = \hat{n}_2 \times \hat{i} + n_i \times \hat{i}\).

Here we need a two-dimensional isodual isoeuclidean space \(E_{t,2}(x, \hat{\delta}, \hat{R}(n_2^d, +, \hat{x}^d))\) with the realization
\[
T^d = \text{diag}(-b_1^2, -b_2^2), \hat{T^d} = \text{diag}(-b_1^2, -b_2^2), \hat{b}_k > 0, k = 1, 2,
\tag{56}
\]
with basic isoseparation...
The isodual isogauas plane is defined as the set of points 
\[ P = (\tilde{x}_1, x_2) \] on \[ \tilde{E}_{12}(x, \delta^d, \tilde{R}^d (\tilde{n}^d, \tilde{x}^d)) \] which characterize the isocomplex numbers \( \tilde{c} = (-\tilde{n}_3, \tilde{n}_4) \).

This satisfies isomultiplication rule (74) characterizing the isocomplex numbers, which verifies
\[ [\tilde{c}] \cdot [\tilde{c}'] = [\tilde{c}] [\tilde{c}'] \in \tilde{R}^d, \quad \tilde{c}^d, \tilde{c}_i^d \in \tilde{C}^d. \] (64)

The isodual isobasis is given by
\[ \tilde{c}_1^d = \tilde{I}_0^d, \quad \tilde{c}_2^d = \tilde{i}. \] (65)

The correspondence between the isodual isocomplex numbers \( \tilde{C}^d (\tilde{c}, +, x^d) \) and the isodual gauss plane can be made one-to-one by the isodual isodilative isorotations
\[ z^d = (x + x_i x) = \tilde{c}^d \tilde{c} z \] (58)

having rule for multiplication as
\[ \tilde{c} \circ \tilde{c}' = (\tilde{n}_3, \tilde{n}_4) \tilde{c} (x, x_i) = \]
\[ = [(\tilde{n}_3 x_i x) \tilde{I} + \Delta^2 \times (\tilde{n}_4 x_i) \tilde{I}], \]
\[ = (-\tilde{n}_3 x_i x) \tilde{I} + (\tilde{n}_4 x_i) \tilde{I}. \]

Isoodual isogauas planes characterize isodual isofield.

Also the isodual isormutations forms an isodual isogroup \( \tilde{G}^d (2) = G^d (2) \).

Isodual isocomplex numbers also admit the following two-by-two matrix representation.
\[ \tilde{c}^d = \tilde{n}_3 \tilde{I} + \tilde{n}_4 \tilde{i} \tilde{I}^d = \begin{pmatrix} -\tilde{n}_3 x_i & i \times n_i + \tilde{n}_3 x_i \times \Delta^2 \times \\tilde{i}_3 \\ i \times n_i + \tilde{n}_3 x_i \times \Delta^2 & -\tilde{n}_4 x_i \hat{b}_2^2 \end{pmatrix} \]

where
\[ \tilde{i} = \tilde{i}_3 \begin{pmatrix} -\tilde{h}_3 & 0 \\ 0 & -\tilde{h}_3 \end{pmatrix}, \]
\[ \tilde{i}^d = \begin{pmatrix} 0 & i \times \tilde{h}_3 \times \Delta^2 \\ -i \times \tilde{h}_3 \times \Delta^2 & 0 \end{pmatrix}. \] (61)

This satisfies isomultiplication rule (74) characterizing the isodual isocomplex numbers of \( \tilde{G}^d (2) \).

The set of matrices representing isodual isocomplex numbers \( \tilde{S}^d (\tilde{c}, +, x^d) \) is closed under addition and isomultiplication. Each element possesses the isodual isoidentity
\[ (\tilde{c}^{-1})^d = (\tilde{c}^d)^{-1} \tilde{i}^d. \] (62)

As a result we get a local isomorphism
\[ \tilde{S}_d (\tilde{c}, +, x^d) = \tilde{C}^d (\tilde{c}, +, x^d). \]

Now, the isodual iosenorm can be defined as
\[ [\tilde{c}]^d = [\tilde{D}_{2d}(\tilde{c} \times \tilde{I}^d)]^{\dagger} \tilde{I}_0^d = (\tilde{n}_3 + \Delta^2 \tilde{i}_3) \times \tilde{i}^d, \]

which verifies
\[ [\tilde{c} \cdot \tilde{c}']^d = [\tilde{c}^d] \times [\tilde{c}'^d] \in \tilde{R}^d, \quad \tilde{c}^d, \tilde{c}'^d \in \tilde{C}^d. \] (63)
\[ I_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \]
\[ i_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad i_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \] (71)

with fundamental relations
\[ i_x i_y = -i_y i_x, \quad n = m, \quad n, m = 1, 2, 3. \] (72)

where \( E_{\text{iso}} \) is the tensor of rank three. The norm of the quaternion can be defined as
\[ |q|^2 = (q^T q) = \sum_{k=1}^{3} n_k^2, \] (73)
satisfying
\[ |q \cdot q'| = |q| \cdot |q'| \in \mathbb{R}, \quad q, q' \in \mathbb{Q}. \] (74)
The basis is defined by
\[ e_i = I_0, \quad e_{i\kappa} = i_\kappa, \quad \kappa = 1, 2, 3. \] (75)

2.7.2 Isodual quaternions:

Isodual quaternions constitute an isodual four-dimensional, normed associative and noncommutative algebra over the isodual reals \( U^d(4) \) which is anti-isomorphic to \( U(4) \) ref. [1].

Isodual quaternions \( q^d \in \mathbb{Q}^d(q^d, +, \times^d) \) can be represented via the isodual Hermitian Euclidean space \( \mathbb{E}^2_d(z^d, \delta^d, C^d(c^d, +, \times^d)) \) with
\[ (z^d, \delta^d, z^d) \times I^d = (-z^d, 1) \times I^d \in \mathbb{R}^d. \] (76)

Isodual complex numbers can also be realized via pairs of isodual complex numbers as
\[ q^d = (c^d, 2c^d), \quad c^d, c^d \in \mathbb{C}^d. \]

Also, the isodual Hermitian dilative rotation on \( \mathbb{E}^2_d(z^d, \delta^d, C^d(c^d, +, \times^d)) \) leaving invariant \( z^d+\delta^d z^d \) is given by
\[ z^d = c^d \delta^d, \quad \delta^d = c^d z^d, \] (77)

where the dilation is represented by the value
\[ c_1^d \delta^d = c_2^d \delta^d = -1. \]

These transformations form an associative but noncommutative isodual group \( G^d(4) \) which is in one-to-one correspondence with isodual quaternions \( Q^d(q^d, +, \times^d) \).

As a result there is a matrix representation of isodual complex numbers over the field of isodual complex numbers \( C^d(c^d, +, \times^d) \) as

\[ q^d = \begin{pmatrix} c^d & -\bar{c}^d \\ c_2^d & \bar{c}_2^d \end{pmatrix} \] (78)

under the condition
\[ c_1^d = -n_1 + n_2 \times i, \quad c_2^d = -n_0 + n_2 \times i \] (79)

We can represent \( q^d \) as
\[ q^d = n_1^d \times I^d + n_2^d \times i^d + n_2^d \times i^d + n_2^d \times i^d \] (80)

where \( i^d \) are the Pauli’s matrices. Note that Pauli’s matrices change sign under isoduality although their product with isodual numbers is isoselfdual.

\[ \text{Isodual norm} \text{ is then defined as} \]
\[ |q^d| = |\text{Det}_d(q^d \times T^d)| \times I^d = (-\sum_{k=1}^{3} n_k^d)^2 \times I^d \] (81)

satisfying
\[ |q^d \times q^d| = |q^d|^2 \times |q^d|^2 \in \mathbb{R}^d, \] (82)

\[ q^d, q^d \in \mathbb{Q}^d. \]

The isodual basis is defined as
\[ c_1^d = I_0^d, \quad c_{i\kappa}^d = i_\kappa, \quad \kappa = 1, 2, 3. \] (83)

2.7.3 Isoquaternions:

Isoquaternions constitute a four-dimensional, isometric, isocommutative, non-isocommutative isospace algebra over the isospace \( \hat{U}(4) \equiv U(4) \), ref. [1].

Isoquaternions \( q \in \hat{U}(\hat{q}, \hat{\times}) \) can be represented using two-dimensional, complex Hermitian isospace Euclidean space of class I,

\[ \hat{E}_{12}(\hat{z}, \hat{\delta}, \hat{\zeta}), \hat{z}^\dagger = \hat{z}, \hat{\delta} = \hat{\delta}, \hat{\zeta}^\dagger = \hat{\zeta}, \] (84)

with basic isotopic element and isounit
\[ T = \text{Diag}(b_1^2, b_2^2), \quad \hat{1} = \text{Diag}(b_1^2, b_2^2), b_i > 0. \] (85)

The (unimodular) invariance group of this space is the Lie-isotopic group \( S\hat{U}(2) \). Isoquaternions can also be characterized by fundamental representation of \( S\hat{U}(2) \) algebra. A Hermitian isodilative isotolation on \( \hat{E}_{12}((\hat{z}, \hat{\delta}, \hat{\zeta})) \) is given by
\[ \hat{z}^\dagger = \hat{c}_1 \hat{c}_2 \hat{z}^\dagger + \hat{c}_2 \hat{z}^\dagger, \quad \hat{z}^\dagger = -\hat{c}_1 \hat{c}_2 \hat{z}^\dagger + \hat{c}_1 \hat{z}^\dagger, \] (86)

where the dilation is represented by the value
\[ \hat{c}_1 \hat{c}_2 \hat{c}_1 + \hat{c}_2 \hat{c}_2 \neq 1. \] Representation of isoquaternions into
two-by-two matrices on \( \mathcal{C}(\mathcal{L},+,\mathcal{X}) \) is characterized by the isorepresentations of the Lie-isotropic algebra \( \mathcal{S}(\mathcal{U}) \) ref. [40, 41, 42]. These can be expressed in terms of the basic isounit
\[
\hat{I} = \hat{I}_0 = \begin{pmatrix} b_1^2 & 0 \\ 0 & b_2^2 \end{pmatrix}
\]  
and fundamental representation of \( \mathcal{S}(\mathcal{U}) \) as
\[
\hat{I} = \Delta^{-\frac{1}{2}} \begin{pmatrix} 0 & ib_2 \\ ib_2 & 0 \end{pmatrix}, \quad \hat{I} = \Delta^{-\frac{1}{2}} \begin{pmatrix} 0 & -b_2 \\ -b_2 & 0 \end{pmatrix},
\]  

\[
\hat{I} = \Delta^{-\frac{1}{2}} \begin{pmatrix} ib_2 & 0 \\ 0 & -ib_1 \end{pmatrix}.
\]  

\[(87)\]

Note that the matrices above satisfy the properties of isotropic image
\[
\hat{I}_n \hat{I}_m = \hat{I}, \quad n \neq m,
\]  

\[(88)\]

and hence are closed under commutators, which is a necessary condition for the existence of an isotopy. This results into a Lie-isotropic \( \mathcal{S}(\mathcal{U}) \) algebra
\[
\begin{pmatrix} \hat{I}_n, \hat{I}_m \end{pmatrix} := \hat{I}_n \hat{I}_m - \hat{I}_m \hat{I}_n = -2\Delta^{-\frac{1}{2}} e_{nm} \hat{I}_n.
\]  

\[(90)\]

Isounit quaternions can be represented in the form
\[
\hat{q} = n_0 I_0 + n_1 \hat{I}_1 + n_2 \hat{I}_2 + n_3 \hat{I}_3 = \begin{pmatrix} (n_0 h_1^2 + \Delta^{-\frac{1}{2}} ) + \Delta^{-\frac{1}{2}} \left( -n_2 + in_1 h_2^2 \right) \\ \Delta^{-\frac{1}{2}} \left( n_0 h_2^2 - \Delta^{-\frac{1}{2}} \right) \end{pmatrix}
\]  

\[(106)\]

Note that the set \( \mathcal{S}(\hat{q},+,\mathcal{X}) \) is a four dimensional vector space over the isoreals \( \mathcal{R}(\mathcal{h},+,\mathcal{X}) \) which is closed under the operation of conventional addition and isomultiplication and hence, is an isofield. Thus, \( \mathcal{S}(\hat{q},+,\mathcal{X}) = \mathcal{Q}(\hat{q},+,\mathcal{X}) \).

The isounit of the isounit quaternion is defined as follows
\[
\|\hat{q}\| = \left| \det_{\hat{q}}(\hat{q} \hat{T}) \right|^{\frac{1}{2}} \hat{I}_0,
\]  

\[(91)\]

and may be written as
\[
\|\hat{q}\| = |n_0| + \Delta(n_1^2 + n_2^2 + n_3^2)|I_0|,
\]  

\[(92)\]

and then
\[
\|\hat{q}, \hat{q}'\| = \|\hat{q}\| \|\hat{q}'\| \in \mathbb{R}, \quad \hat{q}, \hat{q}', \in \hat{Q}.
\]  

\[(93)\]

The isobasis is defined as
\[
\hat{e}_1 = \hat{I}_0, \quad \hat{e}_{k+1} = \hat{I}_k, \quad k = 1, 2, 3.
\]  

\[(94)\]

2.7.4 Isounit isounit quaternion:

The isounit isounit quaternions constitute a four-dimensional, isounit, isounomized, isounassoociative, non-isocommutative isounideal algebra over the isounit isoreals \( \mathcal{U}(4) = \mathcal{U}^d(4) \) ref. [1].

The isounit quaternions \( \hat{q}^d \in \hat{Q}^d(\hat{q}^d,+,\hat{X}^d) \) by a two-dimensional isounit complex Hermitian isounclidean space of class II over the isounit isocomplex field as
\[
E_{2,\mathcal{U}}^d(\hat{Z}^d, \mathcal{C}^d(\mathcal{C},+,\mathcal{X}^d))  : \hat{Z}^d \hat{\delta}^d \hat{Z}^d = \hat{Z}^{-d} \hat{X}^d \hat{Z}^{-d} + \hat{Z}^{-d} \hat{X}^d \hat{Z}^{-d} = -z^{-d} \hat{X}^d \hat{Z}^{-d} - z^{-d} \hat{X}^d \hat{Z}^{-d}.
\]  

\[(95)\]

having basic isounit isotopic element and isounit isounit \( T^d = \text{Diag}(\hat{h},-\hat{h}) \), \( Y^d = \text{Diag}(\hat{h},-\hat{h}) \)

\[(96)\]

having invariance as the isounit Lie-isounit group \( \mathcal{S}(\mathcal{U})^d \). An isounit Hermitian isounit dilatation on the isounit isounit isounideal algebra over the \( \mathcal{S}(\mathcal{U})^d(\hat{Z}^d, \mathcal{C}^d(\mathcal{C},+,\mathcal{X}^d)) \) is given by
\[
\hat{Z}^{d^2} = \hat{c}^d \hat{Z}^{d^2} \hat{c}^d - \hat{c}^{d^2} \hat{Z}^{d^2} + \hat{c}^{d^2} \hat{Z}^{d^2} + \hat{c}^{d^2} \hat{Z}^{d^2}
\]  

\[(97)\]

where dilatation is represented by
\[
\hat{c}^d = \hat{c}^d \hat{c}^d = \hat{c}^d \hat{c}^d = \hat{c}^d \hat{c}^d = \hat{c}^d \hat{c}^d = \hat{c}^d \hat{c}^d = \hat{c}^d \hat{c}^d = \hat{c}^d \hat{c}^d = \hat{c}^d \hat{c}^d.
\]  

\[(98)\]

Note that the set of all the matrices \( \hat{q}^d(\hat{q}^d,+,\hat{X}^d) \) is an isofield and hence \( \mathcal{S}(\hat{q}^d,+,\mathcal{X}^d) = \mathcal{Q}^d(\hat{q}^d,+,\hat{X}^d) \).

The isounit isounorm is defined as
\[
\hat{q}^d \hat{q}^d = \left| \det_{\hat{q}}(\hat{q} \hat{T}^d) \right|^{\frac{1}{2}} \hat{I}^d_0 = \left| n_0^d - \Delta(n_1^d + n_2^d + n_3^d) \right| \hat{I}^d_0
\]  

\[(99)\]

and
\[
\|\hat{q}^d, \hat{q}^{d^2}\| = \|\hat{q}^d\| \|\hat{q}^{d^2}\| \in \mathbb{R},
\]  

\[(100)\]

The isounit isounbasis is defined as
\[
\hat{e}^{d}_1 = \hat{I}^d_0, \quad \hat{e}^{d}_{k+1} = \hat{I}^d_k, \quad k = 1, 2, 3.
\]  

\[(101)\]
\[ \hat{e}_1^d = \hat{I}_n \hat{e}_1^d, \quad \hat{e}_k^d = \hat{I}_n \hat{e}_k^d, \quad k = 1,2,3. \]  

2.8 Isooctonians and their isoduals

2.8.1 Octonians:

Octonians constitute and eight-dimensional normed, non-associative and non-commutative alternative algebra \( U(8) \) over the field of reals \( \mathbb{R}(n,+,\times) \) ref.\[20,21\].

Octonians \( \mathcal{O} \subset (\mathbb{O}(\mathbb{R},+,\times)) \) can be realized as two-dimensional quaternions \( \mathcal{O} = (q_1,q_2) \) with multiplication rules

\[ o o' = (q_1,q_2)(q'_1,q'_2) = (q_1 q'_1 + q_1 q'_2 - q_2 q'_1 + q_2 q'_2). \]  

The anti-automorphic conjugation of an octonian is defined as \( \bar{o} = (\bar{q}_1,-q_2) \).

The norm of an octonian is defined as

\[ |o| = (\sigma o \sigma)^{\frac{1}{2}} = |q_1| + |q_2|, \]

with the basic axioms

\[ o o' = |o| x |o' \subset \mathbb{R}, \quad o,o' \subset \mathbb{O}. \]

It is important to note that Octonians do not constitute a realization of the abstract axioms of a numeric field and, therefore, they do not constitute numbers as conventionally known in mathematics due to the non-associative character of their multiplication (see Ref.\[1\]).

2.8.2 Isooctonians:

The isodual octonians constitute an eight-dimensional isodual, normed, non-associative and non-commutative alternative algebra \( \hat{U}(8) \) over the isodual real numbers \( \hat{R}^d(n',+,\times') \) ref.\[1\].

Isooctonians are defined as \( o' = (q_1',q_2') \) over the isodual reals \( \hat{R}^d(n',+,\times') \). The isodual multiplication of isodual octonians is defined by

\[ o o'^d = (q_1',q_2')(q'_1',q'_2') = (q_1 q'_1 + q_1' q_2 - q_2 q'_1 + q_2' q_2). \]

The isodual anti-automorphic conjugation of an octonian is defined as

\[ \bar{o}' = (\bar{q}_1',-q_2'). \]

The isodual norm of an octonian is defined as

\[ |o'| = (\sigma o \sigma')^{\frac{1}{2}} = |q_1' + q_2'|, \]

with the basic axioms

\[ o^d o'^d = |o|^d x |o'^d \subset \hat{R}^d, \quad o^d,o'^d \subset \hat{O}. \]

Isooctonians form an eight-dimensional isodual, isonormed, non-isoassociative, non-iso-commutative, but isoalternative iso-algebra \( \hat{U}(8) \) over the isodual iso-field \( \hat{R}^d(n',+,\times') \), ref.\[43\].

Isooctonians \( \hat{o}' \in \hat{O}'(\hat{o}'^d,+,\times') \) can be defined as the pair of isoquaternions \( \hat{o}' = (\hat{q}_1',\hat{q}_2') \) over the isodual iso-fields \( \hat{R}^d(n',\times') \) with the multiplication rule

\[ \hat{o}'^d \hat{o}''^d = (\hat{q}_1',\hat{q}_2')(\hat{q}_1'',\hat{q}_2'') = (\hat{q}_1' \hat{q}_1'' - \hat{q}_1'' \hat{q}_2',\hat{q}_2' \hat{q}_1'' + \hat{q}_1'' \hat{q}_2''). \]

The isodual isoantiautomorphism is defined as \( \hat{o}'^d = (\hat{q}_2',-\hat{q}_1') \).

The isodual iso-norm is defined as

\[ \|\hat{o}'\|^d = \|\hat{o}'\| x \hat{1}' = |\hat{q}_1'| + |\hat{q}_2'|, \]

which readily verifies

\[ \|\hat{o}'^d \hat{o}''^d\|^d = \|\hat{o}'^d\| \|\hat{o}''^d\| \in \hat{R}^d, \quad \hat{o}',\hat{o}'' \in \hat{O}'. \]

Again it is important to note that Isooctonians do not constitute a realization of the abstract axioms of a numeric field and, therefore, they do not constitute numbers as conventionally known in mathematics due to the non-associative character of their multiplication (see Ref.\[1\]).

3 Grand Unification of Numeric Fields

Isotopic generalization has brought about a grand unification of the conventional numbers into one single, abstract notion of isonumber. It is important to note that the unification of all numbers was conjectured by Prof. Santilli in numerous publications through out his research for many years. Finally it was proved by Kadeisvile, Kamiya and Santilli ref.\[40\]. The following theorem is the main result in this regard.

**Theorem 3.1** Let \( \mathbb{R},\mathbb{C},\mathbb{Q} \) be the fields of real numbers, complex numbers and quaternions, respectively, \( \mathbb{R}^d,\mathbb{C}^d,\mathbb{Q}^d \) the isofields, \( \hat{\mathbb{R}},\hat{\mathbb{C}},\hat{\mathbb{Q}} \) the isofields and \( \hat{\mathbb{R}}^d,\hat{\mathbb{C}}^d,\hat{\mathbb{Q}}^d \) the isodual isofields as defined in the preceding section. Then all these fields can be constructed with the same methods for the construction of \( \hat{\mathbb{R}} \) from \( \mathbb{R} \), under the relaxation of the condition of positive definiteness of the isounit, thus achieving a unification of all the fields, isofields and their isoduals into the single abstract isofield of Class III, denoted by \( \hat{\mathbb{R}} \).

Genonumbers and their isoduals

We have seen that the two degrees of freedom due to isotopic lifting of addition and multiplication give rise to
isofields and pseudoisofields respectively. These fields are at the foundation of the Lie-isotopic theory [8, 9, 18].

Also, there exists a third degree of freedom caused by the ordering of the above operations which leads to further generalization of a field which is at the foundation of Lie-admissible algebras [8, 9, 18].

Given a field \( F(a_*, +, \times) \) of ordinary numbers with generic elements \( a, b, c, \ldots \), with addition \( a + b = b + a \) and multiplication \( a \times b \), we can define the following.

**Genoaddition**: Addition of \( a \) to \( b \) from the left, denoted by \( a \times b \), and addition of \( b \) to \( a \) from the right denoted by \( a \times b \) are called genoadditions.

**Genomultiplication**: Multiplication of \( a \) times \( b \) from the left denoted by \( a \times b \), and multiplication \( b \) times \( a \) from the right denoted by \( a \times b \) are called genomultiplications.

It is worthwhile to note that ordering of multiplication is fully compatible with its basic axioms, such as commutativity for real and complex numbers, associativity for quaternions, and alternativity for the octonions. In the case of real and complex numbers we will have

\[
a \times b = b \times a, \quad a \times b = b \times a
\]  

(115)

The identity of multiplication from left and right can be different i.e.

\[
a \times b \neq a \times b
\]  

(116)

with realization,

\[
a \times b := aRb, \quad a \times b := aSb, \quad R \neq S,
\]  

(117)

where \( R \) and \( S \) are fixed isotopic elements, called the genotopic elements. These are sufficiently smooth, bounded and nowhere singular (not necessarily Hermitian) outside the original field.

The left and right generalised genounits can be defined in the following manner

**left unit** \( \hat{1} = R^{-1} \) : \( \hat{1} \times a = a \times \hat{1} = a \)

**right unit** \( \check{1} = S^{-1} \) : \( \check{1} \times a = a \times \check{1} = a \)

(118)

Note that all the axioms and properties of the original field are preserved under the mentioned left or right multiplication and multiplicative units under the appropriate ordering for all the dimensions 1, 2, 4, 8. This procedure leads to new fields called as genofield denoted by \( \hat{F}(\hat{a}, +, \times) \) (right genofield) or \( \check{F}(\check{a}, +, \times) \) (left genofield) or \( \hat{F}(\hat{a^*}, +, \times) \). Also, isodual genofields are defined by the antiautomatic conjugations

\[
R \rightarrow R^\ast = -R, \quad S \rightarrow S^d = -S
\]  

(119)

denoted by \( \hat{F}^\ast(\hat{a}, +, \times) \) and \( \check{F}^\ast(\check{a}, +, \times) \).

Note that isofields are the particular case of genofields where the genotopic elements coincide, i.e.

\[
\hat{F}^\ast(\hat{a}, +, \times) = F(\hat{a}, +, \times)
\]  

(120)

**R-S mutation of the Lie product**: is defined as

\[
(A, B) = ARB - BSA
\]  

(121)

which is Lie-admissible via the attached antisymmetric product

\[
[A, B] = (A, B) - (B, A) = ATB - BTA, T = R - S
\]  

(122)

which is Lie-isotopic.

The lifting \( [A, B] \rightarrow [A, \hat{B}] \) is called an isotopy. The lifting \( [A, B] \rightarrow (A, B) \) is called a genotopy, ref. [8, 1].

The Lie-isotopic algebras are defined by one single isotopy of the enveloping associative algebra and related unit

\[
AB = A \times B \rightarrow A \times B = ATB, \quad 1 \rightarrow \hat{1} = T^{-1}.
\]  

(123)

For the consistent formulation of Lie-isotopic algebras they must be defined over an isofield \( \hat{F}(\hat{a}, +, \times) \) with isounits \( \hat{1} = T^{-1} \).

Note that for the conventional multiplication \( \times \) there is no ordering as \( \hat{1} = 1 = \check{1} \). The above ordering can be defined for isomultiplication \( \times \) wherein we can have different isounits.

The Lie-admissible algebras can be generated by two different isotopies of the original associative algebra using left and right isounits with corresponding isotopies as

\[
AB \rightarrow ARB := A \times B, \quad 1 \rightarrow \hat{1} = R^{-1},
\]  

(124)

\[
BA \rightarrow BSA := B \times A, \quad 1 \rightarrow \check{1} = S^{-1}
\]  

(125)

which must be defined over the genofields \( \hat{F}(\hat{a}^*, +, \times) \) with isounits \( \hat{1} \). Here, the isounits related with the left and right isomultiplication are disjoint and can indeed be Hermitian and real-valued, which admit Kadeisville classification into classes I, II, III, IV and V.

However, in physics the isounits (left and right) used have a real physical significance when they are inter-related by a Hermitian conjugation as

\[
\hat{1} = \check{1} = (\hat{1})^\dagger
\]  

(126)

This representation of the genounits (and hence genofields) provides approximation of irreversibility ref.[18].

It is important to note that conventional addition admits no meaningful ordering as \( 0 = 0 \neq 0 \). However, the ordering exists for the isoaddition \( \hat{+} = +K + \) as \( \hat{+} = +K + \) with \( K \neq K^* \). But there is loss of distributive law for the resulting genofield under genoadditions \( \hat{+} \hat{+} \).

All the above discussion leads to a broadest generalization of the existing theory of numbers through
1. **Pseudogenofields** \( \hat{F}^e (\hat{a}^e, \hat{b}^e, \hat{c}^e, \hat{x}^e) \) defined via genotopies of all aspects of conventional fields \( F(\hat{a}, \hat{+}, \hat{x}) \) and

2. **Isodual pseudogenofields** \( \hat{F}^{ed} (\hat{a}^{ed}, \hat{b}^{ed}, \hat{c}^{ed}, \hat{x}^{ed}) \) defined via isoduality of pseudogenofields.

This new generalization of the conventional numbers leads to the following categorization of numbers:
- Conventional numbers of dimension 1,2,4,8 and their isoduals
- Isonumbers of the same dimension and their isoduals
- Genonumbers of the same dimensions and their isoduals
- Pseudoisonumbers of the same dimension and their isoduals
- "Hidden pseudoisonumbers" of dimension 3,4,5,7 and their osioduals
- "Hidden pseudogenonumbers" of dimension 3,4,5,7 and their isoduals.

Note that each of these can be defined for the fields of characteristic 0 or for \( p \neq 0 \).

In addition to above generalization, we can have an ordered set of values for the multiplicative unit such as

\[ \{2, 4, 6, \ldots\} \]

This possibility leads to the new numbers called as hyper-Santillian numbers. These include hyper-real, hyper complex, hyper-quaternon numbers which have vast applications in biological sciences.

In the further generalization, the multiplicative unit can very well have non-zero negative values. This leads to a new class of numbers called iso-dual Santillian numbers. This further leads to a new kinds of conventional iso-dual numbers called as iso-topic isodual numbers, geno-topic iso-dual numbers and hyper-structural isodual. These numbers have applications for antimatter.

The above generalization of the conventional numbers gives us, in all, eleven classes of new numbers namely, the iso-topic numbers, genotopic to the right and left, right and left hyper-structural numbers, iso-dual conventional numbers, iso-dual iso-topic numbers, iso-dual geno-topic to the right and left numbers and hyper-structural isodual to the right and left numbers. Each class is applicable to the real, complex and quaternon numbers where each of the applications have infinite number of possible units.

**Applications and advances**

Quantum mechanics was sufficient to deal with 'Exterior Dynamical systems' which are linear, local, lagrangian and hamiltonian. The main purpose of formulating the new generalized mathematics was to deal with the insufficiencies in the modern mathematics to describe 'Interior Dynamical systems' which are intrinsically non-linear, non-local, non-hamiltonian and non-lagrangian. The axiom-preserving generalization of quantum mechanics which can also deal with non-linear, non-local non-hamiltonian and non-lagrangian systems is called the **Hadronic Mechanics**. The mechanics, built specifically to deal with 'hadrons' (airline interacting particles) ref. [18]. Prof. Santilli, in 1978 when at Harvard University, proposed 'Hadronic mechanics' under the support from U. S. Department of Energy, which was subsequently studied by number of mathematicians, theoreticians and experimentalists. Hadronic mechanics is directly universal; that is, capable of representing all possible nonlinear, nonlocal, nonhamiltonian, continuous or discrete, inhomogeneous and anisotropic systems (universal), directly in the frame of the experimenter (direct universality). In particular the hadronic mechanics has shown that quantum mechanics is completely inapplicable to the synthesis of neutron [46], as mass of the neutron is greater than the sum of the masses of proton and electron (called "mass defect") of which it is made. In this case quantum equations are completely inconsistent. Hadronic mechanics has achieved numerically exact results in the cases in which quantum mechanics results are not valid. For further details of isonumber theory we recommend refs. [47, 1, 48, 46, 49].

As far as mathematics is concerned, one of the major applications of isonumber theory is in Cryptography, ref. [50].

Cryptograms can be lifted to iso-cryptograms which render highest security for a given crypto-system. Isonumbers, hypernumbers and their pseudo-formulations can be used effectively for the highest security via new disciplines, isocryptology, genocryptology, hypercryptology, pseudocryptology etc. More complex cryptograms can be achieved using pseudocryptograms in which we have the additional hidden selection of addition and multiplication to the left and those to the right whose results are generally different among themselves. Yet more complex pseudocryptograms can be achieved in which the result of each individual operations of addition and multiplication is given by a set of numbers [50]. Santillian iso-crypto systems have maximum security due to a large variety of isonumbers which can be changed automatically and continuously, achieving maximum possible security needed for the modern age banking and other systems related with information technology.

Reformulations of conventional numbers to the most generalized isonumbers and subsequently to genonumbers and hypernumbers led to a vast variety of parallel developments in the conventional mathematics including hyperstructures [51] and its various branches such as 'iso-functional analysis' ref [35], iso-calculus ref [52], iso-cryptography [50] etc.

Isogalois fields [53], Iso-permutation groups [54, 53] have been defined by this author, which can play an important role in cryptography and other branches of mathematics where finite fields are used. Investigations are underway.

Isomathematics can also explain complex biological structures and hence has applications in Fractal geometry. Further applications in Neuroscience and Genetics can provide new insight in these disciplines.
References

[1] R. M. Santilli, isounumbers and genounumbers of dimension 1,2,4,8, their isoduals and pseudoduals, and “hidden numbers” of dimension 3,5,6,7, Algebras, Groups and Geometries 10, 273-322 (1993).


