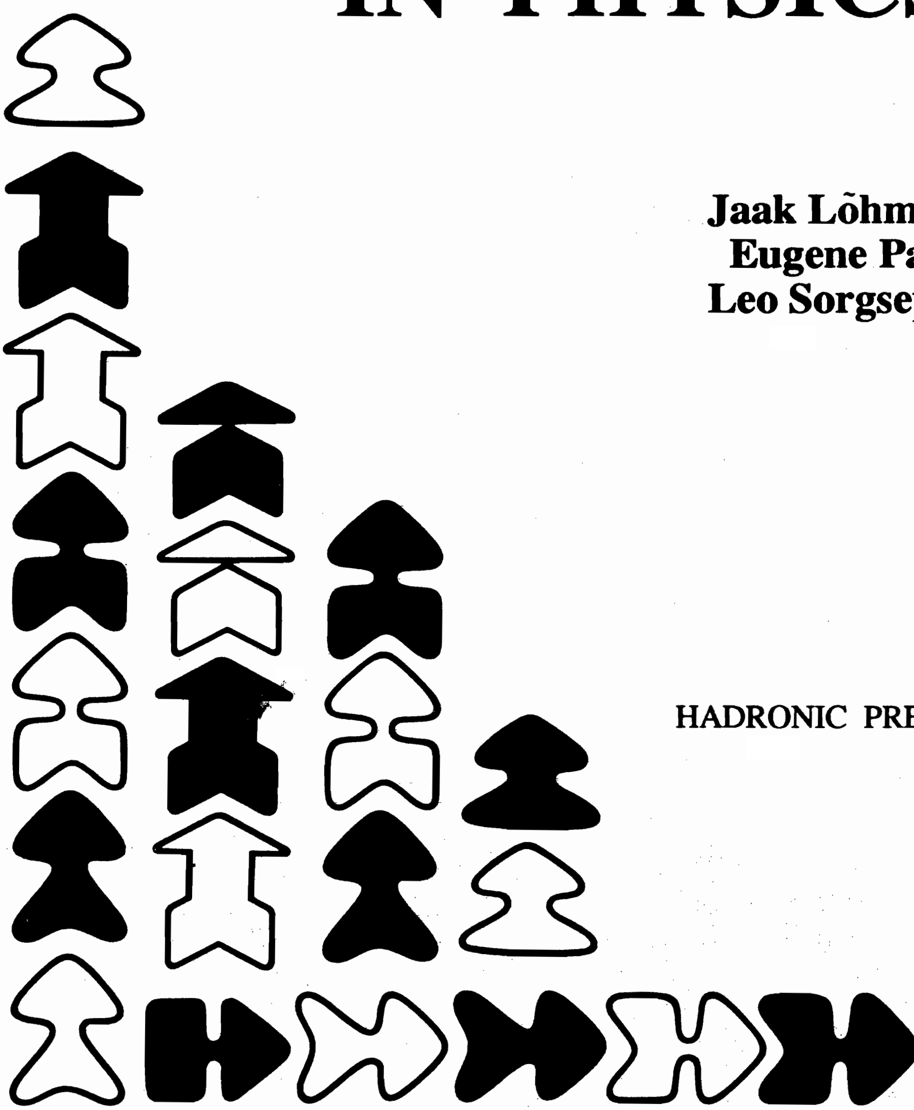
HADRONIC PRESS MONOGRAPHS IN MATHEMATICS

NONASSOCIATIVE ALGEBRAS IN PHYSICS



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NONASSOCIATIVE ALGEBRAS IN PHYSICS

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Preface

Here I have taken the courage to write this Preface myself on behalf of all the authors. It is not to diminish the roles of my co-authors but rather for the convenience of introducing their roles and contributions. As to roles, it must be said that if there is anything really interesting and new in this book, it is mainly due to my co-authors. My elder colleague Leo Sorgsepp has been the founder of studies of nonassociativity at Tartu and the generator of all main ideas. Eugen Paal, our student, is represented here by an interesting original contribution, Moufang-Mal'tsev symmetry.

This book has come out somewhat heterogeneous. At first it was planned as a research monograph about the nonassociativity studies conducted in Tartu (Estonia), and this is what it is as far as Chapters 3-6 (and partly Ch. 2) are concerned. For the introduction to the subject a review chapter (Ch. 1) was included. During the preparation of the manuscript this chapter has grown disproportionally, nevertheless on putting the full stop we became aware about a horrible fact that many interesting problems have not been discussed and many important papers have not been cited. I do hope that those who may feel touched will forgive us for the gaps, rather that a second edition is planned in some near future, where these drawbacks can hopefully be righted.

It was only when preparating the monograph that we experienced the broad frames of nonassociativity and so this book now seems to us rather narrow in scope. Although maybe with a questionable success, we have tried to do our best in the review chapter.

We have tried to present our beliefs and motivations as well as sources of enthusiasm in the Introduction and in other proper places throughout the book.

The authors are happy to express their deep gratitude to Prof. Ruggero

Maria Santilli (Institute of Basic Research, Palm Harbor, USA) and to Mr. C.G. Gandiglio (Hadronic Press Publishers, Palm Harbor, USA) for their kind interest and courtesy, and for the lucky possibility to publish the book in the Hadonic Press Monograph Series. To Prof. R.M. Santilli we owe also special thanks for undertaking the hard task of reading the manuscript and providing many valuable corrections, remarks and suggestions considerably improving the manuscript. We also wish to thank Prof. J.V. Kadeisvili and Dr. G.F. Weiss from the Institute of Basic Research for the preliminary revision of the manuscript.

We are glad to express our sincere thanks to Mrs. Carla Santilli and Mrs. Evi Vaik for the invaluable linguistic revision of the manuscript. However, as many sections have been revised and completed afterwards, they bear no responsibility for possible errors and solecisms.

This work has been partially supported by Grant N 348 of the Estonian Science Foundation.

We sincerely hope that the book will serve to the benefit of the study of the magnificent topics of nonassociative algebras.

Tartu, April 27, 1994

Jaak Lõhmus

Contents

Introduction

Every work needs some motivation and some general philosophy, the rather that such an undertaking as the analysis of a general mathematical concept with relation to its possible applications.

In what follows we should like to summarize some central ideology of the present monograph. "Why nonassociativity at all?". There is always a general answer to such questions, given without any hesitation – "Why not!" (like in a story told about a man holding a horse in his bathroom; to a friend wondering his answer was also"Why not!").

Following the historical development of the concept of number, we certainly come to four familiar division algebras. Perceiving the beauty, entirety and significance of this quadruple of algebras, we just have to exclaim "Why not octonions?" In this case the question itself is of entirely different nature and the answer presupposes some more sophisticated analysis and a more rational motivation. We have come to a conclusion that nonassociativity, represented mathematically by a huge variety of possibilities in a form of a huge variety of nonassciative algebraic systems, may appear in many very different applications, but the octonion algebra as a *unique exceptional system* may be the ultimate basis of our unique physical world – the Universe! Perhaps the most modern motivation of this idea lies in the appearance of octonionic and related structures (as E_8) in the *Theory of Everything.*

In Ch.1, a historical review of the development of algebraic systems (mainly nonassociative algebras) and their physical applications is presented beginning with the discoveries and the establishment of some important and natural types of algebras (Secs. 1.1, 1.2, and 1.4), and with their early applications (Sec. 1.3). The most effective and natural algebraic tool insofar has been the group theory, and the corresponding method of symmetry and invariance in physics. We present a short overview of this topics in Sec. 1.5.

As supersymmetric theories are already stepping beyond the frames of the classical group-theoretical method, we present a brief analysis of appearancees of exceptional systems (octonions, exceptional groups) in higher-dimensional theories (Sec. 1.6).

Some remarks about the modern status of algebras in mathematics are made in Sec. 1.7. A short excursion into the realm of infinite dimensions is made in Sec. 1.8.

Chapter 1 concludes with a review (Sec. 1.9) of three main approaches applying the idea of nonassociativity and nonassociative algebras – of the *octonionic* approach (a more detailed discussion in Sec. 1.10), of the *Lie-admissible* approach (Sec. 1.11) and the *quasigroup* approach.

Chapter 2 lies somewhat beyond the main scope of the monograph, it was included to merely demonstrate the possibilities of obtaining new (nonassociative) systems through continuous transitions, *contractions* and *deformations*. The presentation of the material is rather descriptive and out-of-date, the most fashionable topics today, *quantum groups* are not included at all.

In this chapter we present the general ideology of limit transitions between algebras (Sec. 2.1), an elementary introduction to the deformation theory of Lie algebras (Sec. 2.2) (as an ideal version of the theory, lacking for most other types of algebras), and some generalized deformations extending beyond the class of initial algebras (Sec. 2.3).

Chapters 3-6 form the core of the preliminarily planned contents of the monograph. In these chapters the introduction into Cayley-Dickson algebras and the review of octonion approach elaborated in Tartu (Ch. 3,4), a fully new concept of *Moufang-Mal'tsev symmetry* (Ch. 5), and an analysis of nonassociativity as a fundamental principle (Ch. 6) are presented.

In Ch. 3 a unified treatment of the algebras of central interest for the monograph is given with an elementary introduction to *quaternions* and *octonions* (Sec. 3.1) and *Cayley-Dickson algebras* (Sec. 3.2). We then examine the very nature of the Cayley-Dickson procedure resulting with the conclusion that in the transition from octonions to *sedenions* (Sec. 3.3) for a consistent account of nonassociativity of octonions a ternary operation is unavoidable, leading to the concept of *ternary sedenions* (Sec. 3.4). The Eilenberg-type birepresentation theory for nonassociative algebras is briefed in Sec. 3.5 (general theory), Sec. 3.6 (for alternative algebras including octonions), and in Sec. 3.7 (for ternary sedenions) with tables of regular birepresentations in Appendices 1-3.

In Ch. 4 some particular but fundamental problems are discussed applying the nonassociative-algebraic formalism of octonions and sedenions (developed in Ch. 3): the *octonionic formulation of the Dirac equation* with the introduction of *(confined) color* degrees of freedom and the construction of the *spectrum of fundamental fermions* (Sec. 4.3), and the study of *self-duality conditions* and *hypercomplex analyticity* in four and eight dimensions (Sec. 4.4 and 4.5).

In Sec. 4.4 the well-studied 4-dimensional case is treated classically in terms of quaternions, then paralleled by 8-dimensional octonion formalism, leading to the broken $SO(8)$-structure of the self-duality condition discussed also by several other authors (by different methods).

In Sec. 4.5 self-duality conditions for the Yang-Mills field strength tensor (dim=4,8) are investigated in the octonion and sedenion formalisms. In both cases the hypercomplex formalism leads to the unification of the space-time and internal space through the (generalized) t'Hooft coefficients. In case d=8, the hypercomplex formalism of ternary sedenions ensures the treatment of self-duality fully analogous to the case d=4.

In Ch. 5 a brief exposition of the concept of Moufang-Mal'tsev (MM-) symmetry is presented, beginning with an introduction and a general philosophy of the subject (Sec. 5.1). The concept of MM-symmetry represents some kind of compromise between the traditional (classical) group-theoretical symmetry method and the attempt to extend it beyond the realm of Lie groups. The MM-symmetry explores the Lie group formalism, but has also features imprinted by the non-Lie nonassociativity of a Moufang loop or its Mal'tsev algebra (as additional conservation laws, etc.). Mathematically the couple Moufang loop $\Leftrightarrow$ Mal'tsev algebra is the most economical and natural generalization of the couple Lie group $\Leftrightarrow$ Lie algebra (Sec. 5.2). For physical applications a modified (different from the Eilenberg-type) birepresentation theory is advanced (Sec. 5.3). Finally the concept of Moufang Mal'tsev symmetry is elaborated in a field theoretical context (Sec. 5.4).

In Ch. 6 some preliminary analysis of the *physical meaning of nonassociativity* is outlined. A general concept of the third *infobarrier* connected with the color-type interactions (leading to the confinement) is formulated in terms of associators (Sec. 6.1). The other two infobarriers are those of special relativity and quantum mechanics, associated with complex numbers and quaternions, respectively. The third infobarrier is associated with octonions, also a new fundamental constant, the *fundamental length* characterizing the size of the confinement region, is related with it.

There are interesting possibilities of introducing the nonassociative treatment in the most fundamental region of quantum gravity (Sec. 6.2). The spontaneous symmetry breaking due to nonassociativity enables us to deal with gravity from a completely new viewpoint where the common difficulties of the Planck region disappear. Also, some possibilities for a deep-level *associator quantization* are proposed.

Chapter 1

Nonassociativity in mathematics and physics

In this chapter, a historical review of the development of algebraic systems (mainly nonassociative algebras) and their physical applications is presented. We begin with the discoveries and establishment of some important and natural types of algebras — associative, Lie, and alternative algebras, Cayley-Dickson algebras, composition algebras, division algebras, and Mal'tsev algebras in the second half of the 19th century and in the first half of our century (Sec. 1.1, 1.2, and 1.4), and with the very beginning of applications (Sec. 1.3).

Nonassociative algebras should not be considered separately from their natural subclass of associative algebras. The most important classes of algebras for mathematics and applications are those quite near to associative algebras. Many types of algebras are obtainable from associative algebras through redefinitions (modifications) of the associative product or by some other but equally definite means.

Lie algebras and Lie groups form a mathematical apparatus for the classical group-theoretical method of invariance and symmetry which acts equally effectively in mathematics itself and in physical applications. In Sec. 1.5 we present, following the ideas of Eugene Wigner, a short overview of the method of symmetry and invariance in fundamental physics. Here also in some subsections the needs for extensions of this method become apparent.

Supersymmetric theories are already stepping beyond the frames of the classical group-theoretical method. In these theories a generalization of Lie groups and algebras into Lie supergroups and graded (super) algebras is performed (see a very short sketch in Sec. 1.6).

In higher-dimensional theories of Kaluza-Klein type and in the superstring theory in several places there appear octonions and related exceptional structures (e.g. $E_8 \times E_8$-group in the superstring theory, Sec. 1.10.4). In these theories, especially in the superstring theory, there appear some infinite-dimensional extensions of Lie algebras (Virasoro algebras, Kac-Moody algebras, etc., Sec. 1.8), which have a rich spectrum of unexpected and exotic connections with many regions of mathematics and theoretical physics related also with the exceptional concenter of octonions.

Thus we have come naturally to the main approaches applying the idea of nonassociativity and nonassociative algebras — to the octonion approach, to the Lie-admissible approach, and the quasigroup approach. A very brief introduction to these approaches is given in Sec. 1.9.

Chapter I concludes with more detailed reviews of the application of octonions (Sec. 1.10), and of Lie-admissible and -isotopic approach (Sec. 1.11).

1.1 About generalization of the concept of number in mathematics and physics

During its history algebra has been engaged in generalizations of the concept of *number*. This concept has its origin in the everyday life (but also ... *The naturals are given by God himself, everything else is human handwork*, as said by Leopold Kronecker (1823–1891)). In mathematics the main motivation for generalization of the number concept has been the need of extension of basic number field to ensure the solvability of algebraic equations (so we have come from naturals to complex numbers!). Further generalizations have come from the abstract investigation of general properties of number systems and from the appearance of mathematical entities with well-defined but extraordinary algebraic features.

In the process of historical development many different algebraic systems – quasigroups, groups, fields, rings, algebras, etc. have come into being, each of them carrying some amount of properties of the simplest common numbers, the *natural* and *real* numbers. Algebraic systems are defined through some definitory conditions, *axioms*, to be satisfied for operations in these systems. Some of them are "moderately" general and lead to some well-defined and applicable classes of systems (e.g., to the Lie, alternative, Jordan and Mal'tsev algebras).

Some very common and general axioms may be abandoned and nevertheless some very interesting generalizations of numbers may be obtained. These generalizations may seem peculiar, but they may also be interesting and some of them are fundamental. The latter means that they are not merely intellectual exercises, but represent also some fundamental features of our physical world.

Further we are interested in algebraic systems for which the axiom of *associativity* is not satisfied in general. To consider such *nonassociative algebraic systems* we have already quite an illustrative precedent with the axiom of *commutativity.*

Noncommutative algebraic systems appeared in mathematics already in the middle of the last century in the form of *quaternions* discovered by William Hamilton in 1843 and *matrices*, introduced by Arthur Cayley in 1858. In the first half of our century noncommutativity had its manifestation through *quantum mechanics* (QM). Commutativity or noncommutativity in QM is related to the simultaneous *measurability* or *nonmeasurability* of the quantum mechanical observables represented by operators. Commutation relations of QM contain a fundamental constant of Nature, the *Planck constant* — the quantum of action. A characteristic feature of QM is the *spin* angular momentum which is an essentially nonclassical entity very naturally representable by quaternions.

Now if we proceed from the well-known and applicable number systems as *real numbers* (**R**), *complex numbers* (**C**), and *quaternions* (**H**), we come in a very natural manner to a very specific and unique nonassociative system – *octonion algebra* (**O**). These four algebras together form a unique set of *composition* and *division algebras* (with accompanying split and degenerate forms) which may be got starting from real numbers by the *Cayley-Dickson doubling procedure* [52].

As for applications, the role of real numbers needs no comments here, because they constitute the basic continuum and their application is universal. Complex numbers and quaternions may also be applied to a large variety of problems, but in some contexts they have some more specific or fundamental roles. Due to the introduction of the imaginary unit complex numbers seem to have a fundamental connection with the special theory of relativity allowing to introduce the light cone and the nondetermined interval for events. In mathematics, complex numbers allow the introduction of various "parallel" compact and noncompact structures and constructions (as real forms of Lie algebras, euclidean and pseudoeuclidean spaces, etc.). This extension of the real number system has been especially fruitful for analysis because only through the introduction of complex numbers it has been possible to elaborate a theory of well-behaving *analytical functions.*

We have already remarked the significance of quaternions in QM. In these two examples – complex numbers in special relativity and quaternions in QM – there appear certain limits on the obtaining of information, the *infobarriers*, [212], expressed algebraically in the properties of complex numbers and quaternions, and related to the *fundamental constants* c and h, respectively. Now it seems very natural and straightforward to investigate the significance and the possible role of octonions (Sec. 1.10 and Ch. 3,4) and nonassociativity in general (Ch. 6). The first suggestion about the meaning of nonassociativity belongs to Pascual Jordan [163, 164], who considered a possibility of relating nonassociativity with some *elementary length.* This hypothesis has been renewed and investigated in [212, 213, 333].

Two sequental generalizations of the number concept, *isonumbers* and *genonumbers* were introduced by theoretical physicist Ruggero M. Santilli [309, 314] for specific physical needs, jointly with a new antiautomorphic conjugation, *isoduality*, leading to new numbers called *isodual isonumbers* and *isodual genonumbers* (for more details see Sec. 1.11.1).

1.2 A brief historical introduction. Formation and development of the concept of hypercomplex number system (linear algebra)

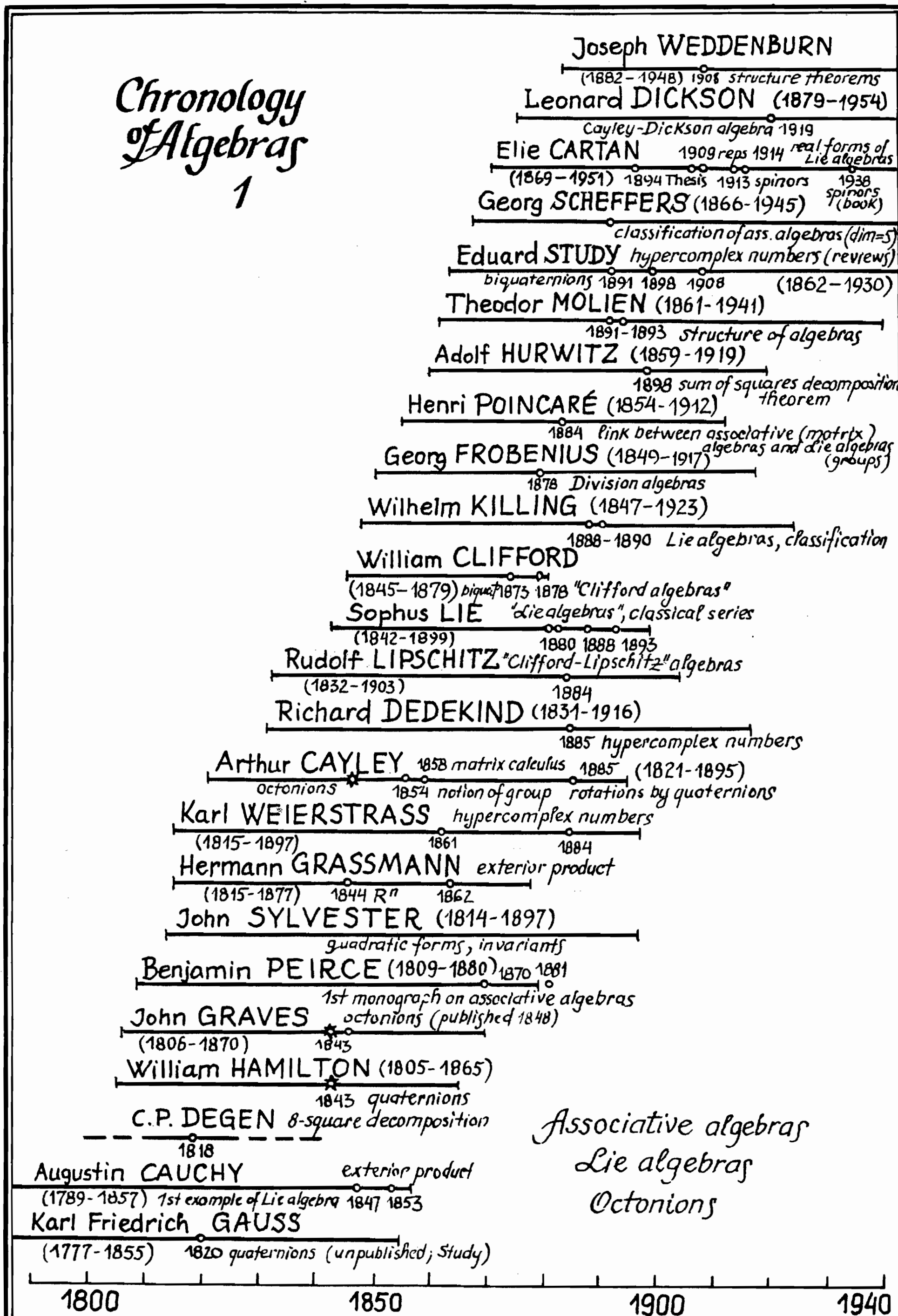
The first example of nonassociative systems was the algebra of *octonions* discovered by John Graves and A. Cayley in 1843-45 [39, 115], which plays the same role among nonassociative algebras as the quaternion algebra among associative noncommutative algebras. It is the only alternative division and composition algebra over the field of real numbers.

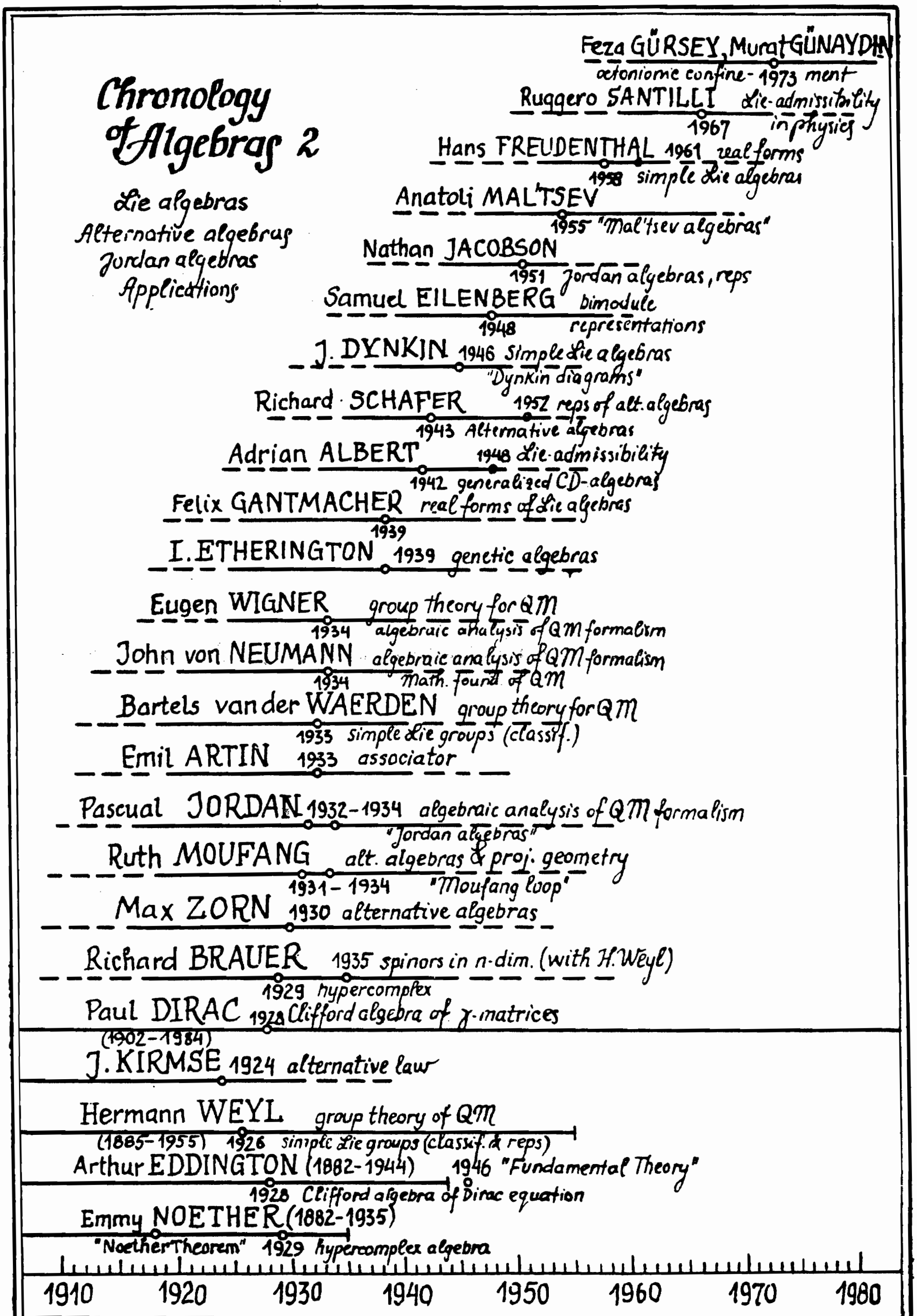
After the discoveries of quaternions and octonions the investigation of *hypercomplex numbers* (now called *linear algebras*) became popular in England, Germany, and United States. There was some renascence of ancient Pythagorean ideas about the creative and fundamental role of numbers in the very basis of world building. The existence of such beautiful systems as quaternions and octonions rose a problem of the existence of other such systems. It was an interesting and useful intellectual activity quite profitable also for the development of mathematics itself. About the formation and development of the concept of hypercomplex number (linear algebra), see reviews, books and bibliographies [274, 217, 341, 321, 2, 35, 351, 352, 354, 353, 134, 364, 317].

It was quite natural that the search for new particular systems of hypercomplex numbers was complemented by attempts to develop a *general theory* of such systems. The first monograph about hypercomplex numbers was written by Benjamin Peirce [274], further landmarks were laid by Theodor Molien [226] and John Wedderburn [372]. In this context, naturally, it should be mentioned that the general linear (associative or nonassociative) algebras without additional restrictions by some additional identities have not lead to any substantial *structural theory* of algebras.There is a characteristic replica by Adrian Albert (1942) in this connection: *"The results of nonassociative algebras in which one does not assume a type of partial associativity have almost all been of a rather primitive kind and have been scattered through the literature".*

A natural and important class of algebras in the category of all linear algebras consists of *associative algebras.* By now we have also quite a good understanding of what Albert meant under the notion of "partial associativity". We have now meaningful nontrivial theories of some classes of algebras – Lie, Jordan, and alternative algebras which are jointly termed as *almost* (or *nearly) associative algebras* [383].

In the chart "Chronology of algebras 1" we give some brief exposition of the development of *associative* and *Lie algebras* through the most distinguished investigators of the subject. Although the first example of the Lie algebras appears already in a paper of Augustin Cauchy (1847), a serious and consistent study of this important, perhaps the most important insofar class of algebras has been started and founded by Sophus Lie and his successors, where the main emphasise was laid on the continuous (group) aspect. The Lie group theory was the continuous version of the finite group theory


Chronology of Algebras 1
Joseph WEDDENBURN
(1882-1948) 1908 structure theorems
Leonard DICKSON (1879-1954)
Cayley-Dickson algebra 1919
Elie CARTAN 1909 reps 1914 real forms of Lie algebras
(1869-1951) 1894 Thesis 1913 spinors 1938 spinors (book)
Georg SCHEFFERS (1866-1945)
classification of ass. algebras (dim=5)
Eduard STUDY hypercomplex numbers (reviews)
biquaternions 1891 1898 1908 (1862-1930)
Theodor MOLIEN (1861-1941)
1891-1893 structure of algebras
Adolf HURWITZ (1859-1919)
1898 sum of squares decomposition theorem
Henri POINCARÉ (1854-1912)
1884 link between associative (matrix) algebras and Lie algebras (groups)
Georg FROBENIUS (1849-1917)
1878 Division algebras
Wilhelm KILLING (1847-1923)
1888-1890 Lie algebras, classification
William CLIFFORD
(1845-1879) biquat 1873 1878 "Clifford algebras"
Sophus LIE "Lie algebras", classical series
(1842-1899) 1880 1888 1893
Rudolf LIPSCHITZ "Clifford-Lipschitz" algebras
(1832-1903) 1884
Richard DEDEKIND (1831-1916)
1885 hypercomplex numbers
Arthur CAYLEY 1858 matrix calculus 1885 (1821-1895)
octonions 1854 notion of group rotations by quaternions
Karl WEIERSTRASS hypercomplex numbers
(1815-1897) 1861 1884
Hermann GRASSMANN exterior product
(1815-1877) 1844 R^n 1862
John SYLVESTER (1814-1897)
quadratic forms, invariants
Benjamin PEIRCE (1809-1880) 1870 1881
1st monograph on associative algebras
John GRAVES octonions (published 1848)
(1806-1870) 1843
William HAMILTON (1805-1865)
1843 quaternions
C.P. DEGEN 8-square decomposition
1818
Augustin CAUCHY exterior product
(1789-1857) 1st example of Lie algebra 1847 1853
Karl Friedrich GAUSS
(1777-1855) 1820 quaternions (unpublished; Study)
Associative algebras
Lie algebras
Octonions
1800
1850
1900
1940


Chronology of Algebras 2
Lie algebras
Alternative algebras
Jordan algebras
Applications
Feza GÜRSEY, Murat GÜNAYDIN
octonionic confine- 1973 ment
Ruggero SANTILLI Lie-admissibility
1967 in physics
Hans FREUDENTHAL 1961 real forms
1958 simple Lie algebras
Anatoli MAL'TSEV
1955 "Mal'tsev algebras"
Nathan JACOBSON
1951 Jordan algebras, reps
Samuel EILENBERG bimodule
1948 representations
J. DYNKIN 1946 Simple Lie algebras
"Dynkin diagrams"
Richard SCHAFER 1952 reps of alt. algebras
1943 Alternative algebras
Adrian ALBERT 1948 Lie-admissibility
1942 generalized CD-algebras
Felix GANTMACHER real forms of Lie algebras
1939
I. ETHERINGTON 1939 genetic algebras
Eugen WIGNER group theory for QM
1934 algebraic analysis of QM formalism
John von NEUMANN algebraic analysis of QM formalism
1934 Math. found. of QM
Bartels van der WAERDEN group theory for QM
1935 simple Lie groups (classif.)
Emil ARTIN 1933 associator
Pascual JORDAN 1932-1934 algebraic analysis of QM formalism
"Jordan algebras"
Ruth MOUFANG alt. algebras & proj. geometry
1931 - 1934 "Moufang loop"
Max ZORN 1930 alternative algebras
Richard BRAUER 1935 spinors in n-dim. (with H. Weyl)
1929 hypercomplex
Paul DIRAC 1928 Clifford algebra of γ-matrices
(1902-1984)
J. KIRMSE 1924 alternative law
Hermann WEYL group theory of QM
(1885-1955) 1926 simple Lie groups (classif. & reps)
Arthur EDDINGTON (1882-1944) 1946 "Fundamental Theory"
1928 Clifford algebra of Dirac equation
Emmy NOETHER (1882-1935)
"Noether Theorem" 1929 hypercomplex algebra
1910 1920 1930 1940 1950 1960 1970 1980

initiated by Evariste Galois, A. Cayley et. al.

Lie algebras are nonassociative, but their nonassociativity is quite a special one — all Lie algebras may be obtained from associative algebras by the transfer to *commutator algebra*, i.e. by the modification of operation $xy \to [x,y] = xy - yx$. The particular form of nonassociativity, expressed through the Jacobi identity reflects the associativity of the group operation in the Lie group. The theory of semisimple Lie groups and their representations was completed in the first half (or even in the first quarter) of our century, but the solvable part and the case of *char* $\neq 0$ are yet far from completion.

Nonassociative octonion algebra was the first and insofar the most important and exceptional example of an interesting class of alternative algebras.

In the chart "Chronology of algebras 2" one can observe the appearance of new types of nonassociative non-Lie algebras (alternative, Jordan, Lie-admissible, and Mal'tsev algebras), the further development of the theories of associative and Lie algebras, and also the beginning of various physical applications.

The general notion of *alternative algebras* appeared in mathematics in 1930, (alternative laws $(xx)y = x(xy), x(yy) = (xy)y$ were introduced by R.Kirmse (1924)). The notion of *associator* was introduced by Emil Artin (1933). E. Artin and Max Zorn conducted a preliminary study of these algebras. Alternative algebras naturally arise in the context of projective geometry (Ruth Moufang, 1931-34; there has been wide activity in this field). Historically, the first and the most outstanding alternative algebra is *octonion algebra.*

Lie algebras, (commutative) Jordan algebras (CJA), and alternative algebras may be regarded as *almost associative* algebras, because the first two classes (NB! an exception for CJA) may be obtained from associative algebras through the modification of the operation into commutator or anticommutator. In alternative algebras every two elements generate an associative subalgebra. These algebras are the most investigated ones (see also monograph [383]).

Alternative algebras in turn generate (through the transition to commutator algebras) at least a part of a very interesting class of *Mal'tsev algebras* (called *Moufang-Lie algebras* by A.I. Mal'tsev himself). Anatoli Mal'tsev introduced (1955, [218]) these algebras as infinitesimal (tangent) algebras of *analytical Moufang loops.* The class of all Mal'tsev algebras contains all Lie algebras and the Mal'tsev algebras are the very natural and closest or minimal generalizations of Lie algebras, and therefore they also provide (together with analytical Moufang loops) a very natural generalization of the classical group theoretical symmetry considerations, which may be called *Moufang-Mal'tsev symmetry* (see Ch. 5 and Refs. therein).

Considerable contributions into the theory of Mal'tsev algebras were made by Arthur Sagle and by Novosibirsk school (E.N. Kuzmin, V.T. Filippov et. al.). There is no proof yet that alternative algebras generate (through the commutator algebra) the entire class of Mal'tsev algebras.

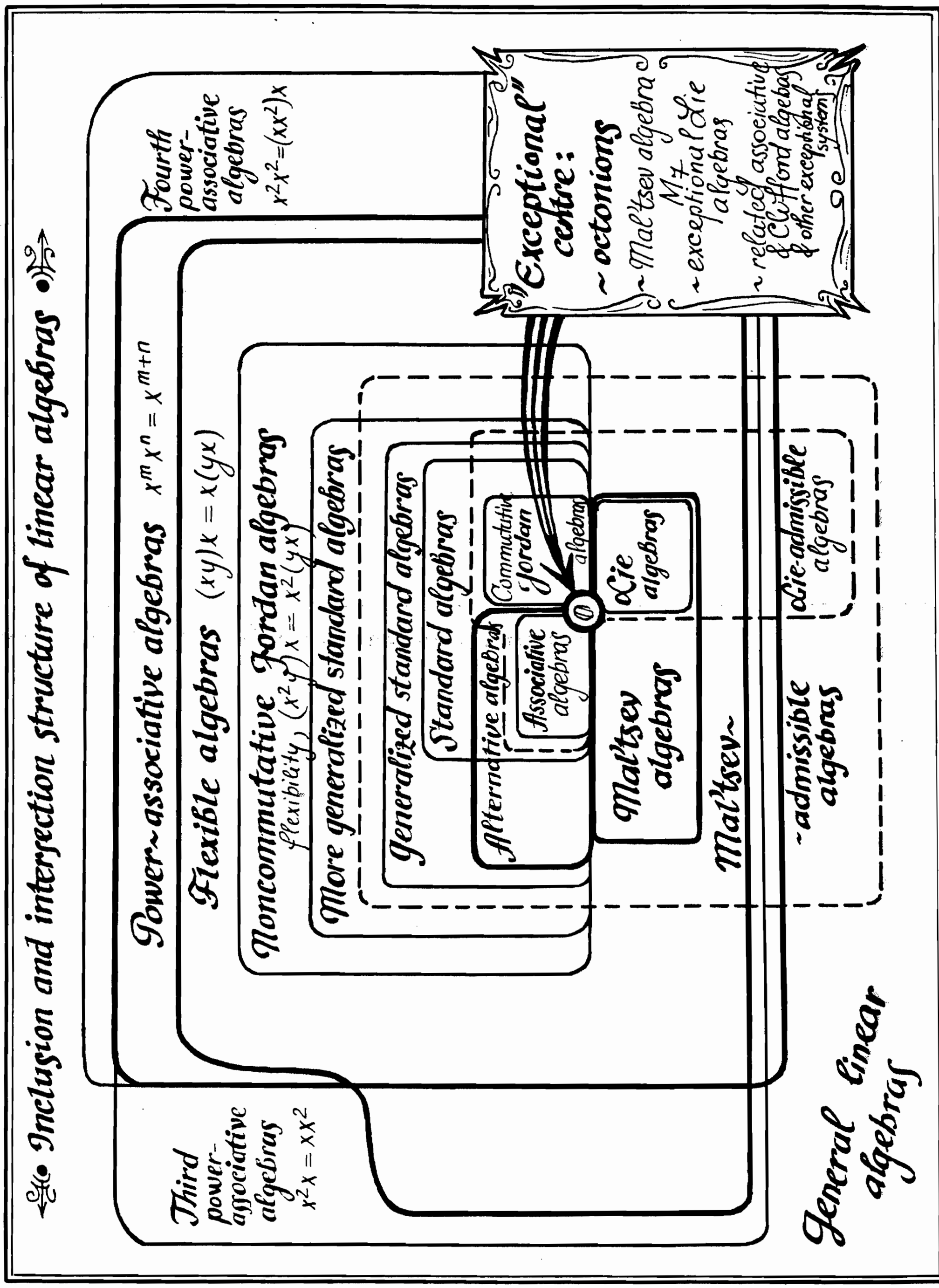
In closing this section also some new types of algebras, *isolinear* and *genolinear algebras* arising in the context of Santilli's Lie-admissible and Lie-isotopic approach should be brought to the attention (for more details see Sec. 1.11.2).

The theory of algebras is a wide subject and cannot be treated thoroughly in this limited review. We finish our presentation here with a schematic chart "Inclusion and intersection structure of linear algebras", some remarks about algebras in modern mathematics will be given in Sec. 1.7.

1.3 The beginning and development of physical applications

The application of Lie group theory to analytical mechanics is not the subject of the present review. The Lie theory became in a quite unnoticed manner a very natural part of analytical mechanics, so that speaking about applications is perhaps not the right way of presentation. If to speak about applications of algebras there were naturally some applications of quaternions in the end of the last century, see for example [19] and references therein.

Quite characteristic for the situation just after the fall of century was the so-called Princeton story, told by Freeman Dyson [63]: *"In 1910 the mathematician Oswald Weblen and the physicist James Jeans were discussing the reform of the mathematical curriculum at Princeton University. "We may as well cut out group theory," said Jeans. "That is a subject which will never be of any use in physics." It is not recorded whether Weblen disputed Jeans's point, or whether he argued for the retention of group theory on purely mathematical grounds. All what we know is that group theory continued to be taught. And Weblen's disregard for Jeans's advice turned out to be of*


Inclusion and intersection structure of linear algebras
Power~associative algebras $x^m x^n = x^{m+n}$
Flexible algebras $(xy)x = x(yx)$
Noncommutative Jordan algebras
Flexibility, $(x^2y)x = x^2(yx)$
More generalized standard algebras
Generalized standard algebras
Standard algebras
Commutative Jordan algebras
Alternative algebras
Associative algebras
Lie algebras
Mal'tsev algebras
Mal'tsev~admissible algebras
Lie-admissible algebras
0
Fourth power-associative algebras $x^2x^2=(xx^2)x$
Third power-associative algebras $x^2x = xx^2$
General linear algebras
"Exceptional" centre:
~ octonions
~ Mal'tsev algebra M_7
~ exceptional Lie algebras G_2
~ related associative & Clifford algebras & other exceptional systems

some importance to the history of science at Princeton. By an irony of fate group theory later grew into one of the central themes of physics, and it now dominates the thinking of all of us who are struggling to understand the fundamental particles of nature. It also happened by chance that Hermann Weyl and Eugene P. Wigner, who pioneered the group-theoretical point of view in physics from the 1920's to the present, were both Princeton professors."

By that time the foundations of the theory of continuous groups had been laid by S. Lie and the classification of simple continuous (Lie) groups had been given by Wilhelm Killing, and Elie Cartan. The development of the Lie theory is a wide and independent topics, and we do not intend to present it in full length. We will return to Lie algebras again when some new and decisive physical applications appeare because this will lead us also closer to the applications of other types of algebras nonassociative in general.

The pioneers of application of hypercomplex Clifford algebras in physics were Paul Dirac (1926), Arthur Eddington (1928), also the contributions of G.Temple (1930), F.Sauter (1930), and W. Franz (1935) must be noted (see the book by Arnold Sommerfeld [329] for Refs.).

The very beginning of the application of *nonassociative algebras* to physical problems is related to the papers (1932-1933) of P. Jordan and to the landmark paper by Jordan, Wigner, and John von Neumann (1934), [165]. It appeared that instead of noncommutative but associative formulation the QM may also be formulated in a nonassociative but commutative algebraic framework, now called *Jordan algebraic approach* to QM. It also became evident that the QM is essentially a *projective geometry*, or the theory of *orthomodular lattices* (in the *quantum logic* approach of Birkhoff and von Neumann [25], et.al.). These papers also cradle the origin of the mathematical theory of *Jordan algebras*, which may be obtained (with one exclusion!) from the associative algebras by the modification of multiplication $xy \rightarrow \{x, y\} = \frac{1}{2}(xy + yx)$, The theory of these algebras was developed by A. Albert, Nathan Jacobson, K. McCrimmon, and recently by the Novosibirsk school (see books and reviews [154, 34, 222, 223, 362]).

An interesting side-shoot of algebras and applications is formed by a special branch of *genetic algebras* introduced by I.Etherington (1939) (see the monograph by A. Wörz-Busekross [379]).

Insofar the remarks about physical applications have remained quite sporadic. Before proceeding with applications of nonassociative algebras we would still like to give some remarks about the further development of the classical *group theoretical method of symmetry and invariance* after the Princeton story. These remarks help to demonstrate the need for nonassociative algebras and quasigroups (loops) for physical applications.

As we have already remarked, the Lie theory has percolated the classical analytical mechanics in a very natural way, but a very important and concise landmark here is the famous theorem by Emmy Noether (1918) about the connection between *symmetry* and *conservation laws*, which became a very powerful principle for quantum theories. Due to the fundamental contributions by H. Weyl, B.L. van der Waerden, and E. Wigner the group theory became a very powerful apparatus for quantum mechanics and quantum field theory. By the 1940ies the representation theory for compact semisimple Lie groups, and for the Poincarè group (E. Wigner, V. Bargmann) was quite well established. Somewhat later, in 1940-1950ies and even later a theory of infinite dimensional unitary representations for noncompact groups was developed by the Moscow school (I.M. Gelfand, M.A. Naimark, M.I. Graev et.al.).

Werner Heisenberg's *isospin* concept (1934) and the proliferation of strange strongly interacting particles (*hadrons*) initiated the *isospin-strangeness scheme* of Murray Gell-Mann (1953) and Nakano Nishijima (1953) and started the internal symmetry boom of late 1950ies, which culminated in the *unitary "Eightfold Way" symmetry* of Gell-Mann and Yuval Ne'eman (1961), and the *quark model* of M. Gell-Mann and Georg Zweig (1964), [100].

As a valuable byproduct, the mathematical theory of *semisimple* Lie groups and their representations was rewritten for physics in detail and in a less abstract form.

Since 1965 *noncompact real forms* of classical groups have been used as *spectrum generating groups* for dynamical systems (Y. Ne'eman, E. Dothan, M. Gell-Mann [59]). The group-theoretical analysis of simple dynamical systems was brushed up and some considerations of extension of this approach to the system of hadrons was outlined [30].

Starting from 1975, simple groups of higher ($n \geq 5$) ranks were investigated as possible candidates for the *Grand Unifying Theories* (GUTs). The most popular groups here are $SU(5), SO(10)$ and E_6 [197, 325].

Higher rank classical groups were extensively used also in *supergravity theories* [365, 375] and *Kaluza-Klein type theories* [62]. In 1984 Michael Green and John Schwarz discovered the *superstring* groups $O(32)$ and $E_8 \times E_8$, about the *superstring theory* see [116].

Now we are witnessing that the classical group theory has been essentially exhausted and physicists are forced to look for generalizations and extensions of the group concept, such as *supergroups, infinite-dimensional groups, quantum groups, Lie-Santilli isotopic* and *genotopic groups* and some *nonassociative* systems (as quasigroups and nonassociative algebras).

Supergroups, which are the Lie groups with anticommuting parameters, were introduced into physics in early 1970ies to formulate the *boson-fermion (super-) symmetry.* In mathematics such a construction appeared for the first time in 1970. Supersymmetry was physically a very rich idea leading to such areas as supergravity and the superstring theory. It was very successful in getting rid of some inconsistencies of the theory, but has not yet found any direct experimental confirmation — no superpartners of common particles have been found in Nature insofar.

Also, various kinds of infinite-dimensional algebras have been introduced in connection of supersymmetric approach and also in connection of some other topics (see a brief exposition in Sec. 1.8). For example, the concept of the *Kac-Moody algebra* began to form in the latter half of the 1960ies in the Skyrme theory, Gell-Mann's quark model construction of dynamical algebra of currents, and in the vertex operator construction in the hadronic string theory (the original mathematical papers appeared in 1968). By the end of the 1970ies these topics were well formulated in terms of representations of Kac-Moody algebras. Later it became apparent that these algebras belong to a wider class of infinite-dimensional (*affine*) algebras which expose interesting relationships between almost disconnected areas in physics and mathematics, starting with sporadic simple finite groups (and the body of exotic mathematics connected with it) and proceeding to completely integrable dynamical systems, gauge field theories, and critical phenomena in two dimensions. One such algebra is the *Virasoro algebra*, the algebra of the closed string reparametrization group $Diff(S^1)$.

All the wide context of GUTs, supergravities, Kaluza-Klein and superstring theories exhibits tight connections with the nonassociativity through the exceptional alternative algebra of *octonions.* Formally it may be said that such a connection is evident from the appearance of such exceptional groups as E_6 or E_8. For the superstring theory the octonions provide a suitable formalism because the 10-dimensionality with 8 physically significant degrees of freedom is involved. Naturally it seems that this connection is not formal but lies in the very deep roots of our physical world.

This was a motivation for only one of the three main approaches to *nonassociativity in physics* – to the *octonion approach.* There are equally interesting approaches which we shall briefly discuss during our presentation.

1.4 Almost associative algebras. Division algebras. Cayley-Dickson algebras. Mal'tsev algebras

The first class of well-defined non-Lie nonassociative algebras appeared in 1920-30ies as *alternative algebras* (they include also the first nonassociative algebra of octonions discovered much earlier). In 1924 J. Kirmse formulated the alternative law (*left and right alternativity*), [176]:

$$(xx)y = x(xy)\,, \quad (xy)y = x(yy)\,. \tag{1.1}$$

Here and in what follows $x, y, \ldots$ are assumed to belong to some linear algebra $\mathcal{A}$ under consideration.

In 1935 E. Artin introduced the notion of *associator*:

$$(x, y, z) := (xy)z - x(yz)\,. \tag{1.2}$$

Artin and M. Zorn thoroughly investigated alternative algebras. According to the Artin theorem, assuming $charF = 2$, conditions (1.1) are equivalent to the alternating associator with respect of permutations of elements x, y, z, from where also originates the name of these algebras. Alternative algebras also arise from the context of *projective geometry* (R.Moufang, 1931-34 [231, 232]) as the coordinate algebras on projective planes where the little Desargue theorem is satisfied (the so-called *Moufang planes*).

Another important class of non-Lie nonassociative algebras consists of *commutative Jordan algebras* (CJA) introduced by P.Jordan (1933-34, [159, 161, 160]) in the papers devoted to the algebraic analysis of QM (for more details see Sec. 1.10.1). These algebras appeared to be very profitable and fruitful in mathematics. Today they form a big branch in the theory of algebras with many interrelations with other parts of mathematics. CJA are defined by the identities

$$xy = yx\,, \quad (x^2y)x = x^2(yx)\,. \tag{1.3}$$

Lie algebras, Jordan algebras and alternative algebras may be regarded as *almost* or *nearly associative algebras* (A.I.Shirshov 1958, [322], see also [383]). Today these algebras form the most studied class of algebras. Lie algebras and (*special*) commutative Jordan algebras may be constructed from associative algebras by some redefinition (modification) of the product operation xy into commutator $[x, y] = xy - yx$, or into (half) anticommutator $\{x, y\} = \frac{1}{2}(xy + yx)$. In alternative algebras every two elements generate an associative subalgebra.

There is a much broader class of algebras which also may be regarded as almost associative in a very certain sense. It is the class of *power-associative (monoassociative)* algebras, introduced by Albert (1948, [3]), where every single element generates an associative subalgebra. In these algebras a general law

$$x^m x^n = x^{m+n} \tag{1.4}$$

is satisfied.

All the well-known "good" algebras are almost associative (and naturally power-associative). At first there are the common fundamental number algebras (and sources of scalars for algebras), *real* and *complex numbers* **R** and **C**. And then there are *quaternion* and *octonion* algebras **H** and **O**. All these four algebras, **R, C, H, O**, have some very good properties. At first they all are *division algebras*, i.e. algebras without zero divisors. Then, secondly, for the elements of every algebra, **R, C, H, O**, positive length or *norm* $N(X)$ can be defined as a generalization of the notion of absolute value for real numbers and the module of complex numbers. In all four algebras, **R, C, H, O**, the norm $N(X)$ satisfies the *composition law*

$$N(XY) = N(X)N(Y)\,. \tag{1.5}$$

It is a property relating the product and norm operations. The algebras satisfying (1.5) are called *composition algebras*, [383, 175]. Due to the theorem by A.Hurwitz (1898, [145, 200]) and the generalized theorem of Frobenius [191], the algebras **R, C, H, O** are unique, they are the only algebras (over R) with "good" properties of numbers, and with division and norm composition. The algebras **C, H, O** may be constructed from the algebra **R** by means of a doubling procedure introduced by L.Dickson (1919, [52]) and afterwards called the *Cayley-Dickson procedure*. Dickson succeeded in inventing a formula which defines octonions as pairs of quaternions. This formula also fits to introduce complex numbers as pairs of real numbers and quaternions as pairs of complex numbers. Dickson's formula contains a special (real continuous) parameter giving the accompanying modifications of **C**, **H**, **O** with nondefinite and degenerate norms. All 8-dimensional algebras, obtained on the third step $\mathbf{H} \to \mathbf{O}$, are called *Cayley-Dickson algebras* (CDA) (see, e.g. [383]). In what follows we sometimes use a more loose terminology and call all algebras **C**, **H**, **O** and their modifications in the CD-procedure CDA.

Cayley-Dickson algebras will be treated in more detail in Ch. 3.

With alternative algebras *Mal'tsev algebras* (1955, [218]; called *Moufang-Lie algebras* by Mal'tsev himself) are connected in a very natural way. These algebras appear as infinitesimal (tangent) algebras of analytic Moufang loops (see Ch. 5), but, on the other hand, commutator algebra for any alternative algebra is a Mal'tsev algebra. The class of all Mal'tsev algebras contains all Lie algebras and, seeking for some natural generalization of Lie algebras, it is Mal'tsev algebras that seem to be most suitable. One possible version of such generalization (the *Moufang-Mal'tsev symmetry* is discussed in Ch. 5). About Mal'tsev algebras see also remarks at the end of Sec. 1.2.

In closing it should be remarked that all notions of this section admit consistent nonlinear-integral generalizations as *isotopies, genotopies* and *isodualities* [309]. All *isofields* and *genofields* and their *isoduals* [314] preserve the original properties of four division algebras with the corresponding modified *isocomposition law* (see Sec. 1.11.2 for more details).

1.5 Group-theoretical method of invariance and symmetry

Symmetry, as wide or as narrow as you may define it, is an idea by which man through the ages has tried to comprehend and create order, beauty, and perfection. (Hermann Weyl, [376]).

The notion of symmetry made out already in very ancient times as a simile of proportionality and harmony, has got its precise formulation in mathematics, in geometry particularly. The exact and convenient mathematical apparatus incarnating the idea of symmetry is the *group theory.* While treating the problem of solvability of algebraic equations in radicals, Evariste Galois in fact investigated the symmetry of equations with respect to some transformations (substitutions of roots), so he was the first to reach to the notion of a group. In the second half of the last century this notion was developed into theory (cf. the chart "Chronology of groups and symmetries"). The most expressive results of group theory are the *Erlangen program* in geometry (F.Klein 1872, [180]) and the *Noether theorem* about conservation laws (E.Noether 1918, [246]).

BIBLIOGRAPHY: two good bibliographical sources – resource letters [271, 287] on symmetry and group theory in physics should be recommended.

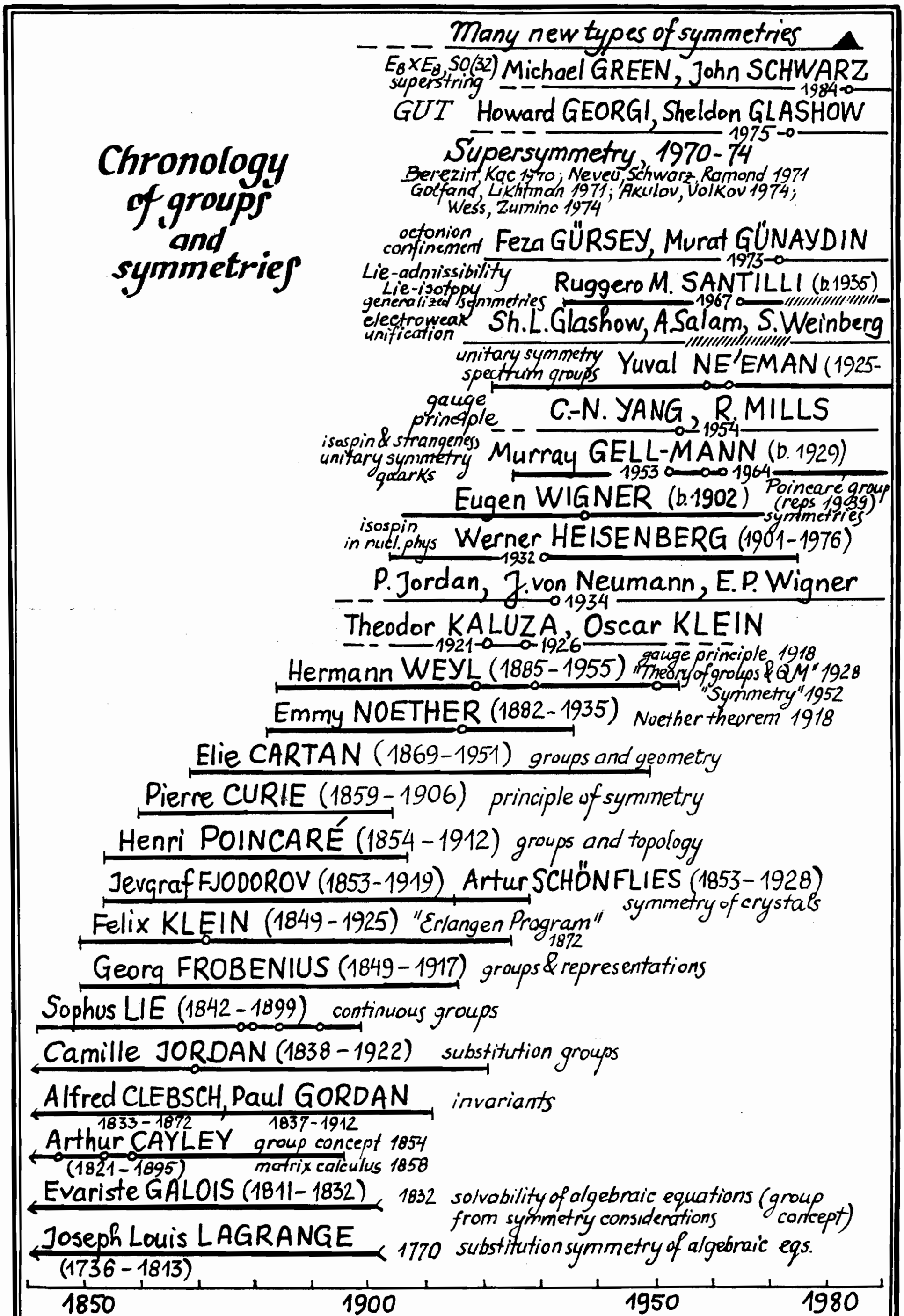
1.5.1 Events, laws of nature, invariance principles, and conservation laws

In 1964-65 interesting methodological papers [378, 377, 143] were published by E.Wigner et al., where a deep methodological analysis of interrelations between events, laws of nature and invariance principles was performed. In [203] this analysis was complemented by some aspects of dynamical symmetries and symmetry breaking, and by some analysis of dynamical invariances. Also some recent interesting generalizations in the framework of Lie-admissible and Lie-isotopic approach should be noted (see Sec. 1.11.3).

To get a smooth passage to the problems of the theory of fundamental particles (and perhaps also to the problem of *nonassociativity*) we present here a brief review of above quoted papers.

Let us start with the *laws of Nature* and with *events*. An event is something that can be related with the point of space-time, i.e. with a moment of time and with a place in the space. A law of Nature is some well-defined regularity in the sequences of events. Laws of Nature are *correlations* between events. These correlations may occur as strictly deterministic (as in classical mechanics) or probabilistic (as in QM). In physics, the laws of Nature are commonly formulated in the form of differential or integral equations.

Events are raw material for the laws of Nature. From the point of view of human cognition, the laws of Nature discovered by man represent maximally compressed information which has been accumulated by our predecessors. However we get some profit of these laws if we may be sure that they are applicable always and everywhere, i.e. if they may be translated in space-time. The possibility of such translations is expressed by *invariance principle* or *symmetry*. The laws of Nature must be invariant under


Chronology of groups and symmetries
Many new types of symmetries
$E_8 \times E_8$, SO(32) superstring Michael GREEN, John SCHWARZ 1984
GUT Howard GEORGI, Sheldon GLASHOW 1975
Supersymmetry, 1970-74
Berezin, Kac 1970; Neveu, Schwarz, Ramond 1971
Golfand, Likhtman 1971; Akulov, Volkov 1974;
Wess, Zumino 1974
octonion confinement Feza GÜRSEY, Murat GÜNAYDIN 1973
Lie-admissibility Lie-isotopy generalized symmetries Ruggero M. SANTILLI (b.1935) 1967
electroweak unification Sh.L.Glashow, A.Salam, S.Weinberg
unitary symmetry spectrum groups Yuval NE'EMAN (1925-
gauge principle C.-N. YANG, R. MILLS 1954
isospin & strangeness unitary symmetry quarks Murray GELL-MANN (b.1929) 1953 1964
Eugen WIGNER (b.1902) Poincaré group (reps 1939) symmetries
isospin in nucl. phys Werner HEISENBERG (1901-1976) 1932
P. Jordan, J. von Neumann, E.P. Wigner 1934
Theodor KALUZA, Oscar KLEIN 1921 1926
Hermann WEYL (1885-1955) gauge principle 1918 "Theory of groups & QM" 1928 "Symmetry" 1952
Emmy NOETHER (1882-1935) Noether theorem 1918
Elie CARTAN (1869-1951) groups and geometry
Pierre CURIE (1859-1906) principle of symmetry
Henri POINCARÉ (1854-1912) groups and topology
Jevgraf FJODOROV (1853-1919) Artur SCHÖNFLIES (1853-1928) symmetry of crystals
Felix KLEIN (1849-1925) "Erlangen Program" 1872
Georg FROBENIUS (1849-1917) groups & representations
Sophus LIE (1842-1899) continuous groups
Camille JORDAN (1838-1922) substitution groups
Alfred CLEBSCH, Paul GORDAN invariants
1833-1872 1837-1912
Arthur CAYLEY group concept 1854
(1821-1895) matrix calculus 1858
Evariste GALOIS (1811-1832) 1832 solvability of algebraic equations (group concept) from symmetry considerations
Joseph Louis LAGRANGE 1770 substitution symmetry of algebraic eqs.
(1736-1813)
1850
1900
1950
1980

space-time translations. Here we come to the interrelations between laws of Nature and invariance principles (symmetries). Laws of nature are correlations between events, but the laws of Nature expressed in different frames are in turn correlated by invariance principles (symmetries). A typical example is the relativistic invariance in the quantum field theory (QFT) demanding that all Lagrangians must be invariant and all equations of motion must be covariant under transformations of the Poincaré group, and all the physical entities must transform according to unitary (irreducible) representations of the Poincaré group. There is a whole complex of groups related to the space-time (cf. chart in Sec. 2.1).

Invariance principles, correlating the laws of Nature provide, according to the Noether theorem, corresponding conservation laws (a simple example is the invariance with respect to space-time translations giving the energy-momentum conservation laws). These conservation laws in turn may be called laws of Nature, but then some distinction should be arranged between *dynamical* laws and *conservation* laws. A dynamical law describes some change of situations, i.e. it determines (perhaps in the loose sense of QM) some sequence of events. An integral conservation law demands that some entities will be conserved during the time evolution of the (closed) system. Of a somewhat different character are invariance (symmetry) principles in the QFT and in the systematics of elementary particles. Space-time (geometric) symmetries provide us with correlations between events from point of view of different observers. Events themselves remain unchangeable. But even the simplest internal symmetry, the isospin symmetry, changes events, e.g. by "rotating" neutron into proton. These symmetries do not act upon space-time coordinates and they are called *internal*, of such type are all quantum numbers of particle systematics. The simplest of such symmetries is the *gauge group* of quantum electrodynamics (QED), the Noether theorem here gives the conservation law of the electric charge Q. In the particle systematics there are much more complicated groups (Secs. 1.5.2, 1.5.3, 1.5.5).

These symmetries do not have such a general nature as it is in the case of space-time symmetries. They are restricted to be characteristic of only some particular interactions or of some particular dynamical systems: they have a restricted area of applicability beyond which they are broken or do not hold at all. They may be of an approximate nature also in their area of validity, where they may be perturbed by other interactions (as it is the case of the isospin symmetry perturbed by electromagnetic interaction) or they are broken by some universal mechanism of spontaneous symmetry breaking, as the Higgs mechanism in the theory of electroweak interaction (Sec. 1.5.5). We can directly observe only exact *geometrical symmetry* where all the parts of the figure are within our sight. It is not the case of *internal (hidden) symmetries.* We have an example of exact internal symmetry, the *color symmetry* of quarks where exactly symmetrical objects (quarks) are nonobservable and a closer knowledge about them is possible only through indirect observations (see also Sec. 1.5.3).

It is interesting and instructive to observe the appearance of hidden symmetries of simple dynamical systems, which are extensions or completions of geometric symmetries (Sec. 1.5.4).

Let us now briefly consider the principle of *localized* or *gauge symmetry*, as it is usually referred to. If the parameters of some internal symmetry group, for example $U(1)$ or $SU(2)$, do not depend on space-time coordinates we have a situation where identical transformations are performed simultaneously in all points of space. Such a picture contradicts the relativity principle, because it presumes the existence of some means of instantaneous propagation of information. C.N. Yang and R. Mills [381] were the first to turn attention to this circumstance and to introduce internal symmetries depending on space-time points, which means that the parameters of internal symmetry transformations became functions of space-time coordinates. This proposal enriched the theory with a very interesting and profitable geometric structure of nontrivial *fibre fundle*, leading automatically to the interaction through the *coefficients of connection* (*gauge bosons* or *Yang-Mills fields* in physical terms). Now the gauging of symmetries is a universal tool for the introduction of fundamental interactions.

Two sequental generalizations of symmetries and conservation laws have been proposed by R.M. Santilli in the context of Lie-admissible and Lie-isotopic approach: *isosymmetries* and conservation laws and *genosymmetries* and *nonconservation laws* (see for more details Sec. 1.11.3).

1.5.2 Isospin, strangeness, ... and the boom of systematics. Unitary symmetry and the quark model

Before World War II, already in the Golden Age of physics in about 1932, there was a sufficient amount of particles to build up atoms and nuclei (only the agents of nuclear forces were yet absent). The experimentally discovered particles were *electron, positron, photon*, and *neutron*. A year before the existence of *neutrino* was suggested (Pauli 1931), which was necessary for the book-keeping of the beta-decay.

The next important moment was in 1947 when the problem of the agent of nuclear forces was solved by the discovery of *pions*. The list of discovered particles lengthened, including electron, positron, photon, proton, neutron, pions, and muon. Neutrino remained still hypothetical, but chances for its existence had been increased. All physicists believed in the existence of *antiparticles*. And only about the *muon* there was a question *"Who ordered that?"* (I.Rabi).

In the same 1947 there appeared mysterious new tracks in emulsions and chambers, forecasting a new era of *strange particles*. In 1953 M.Gell-Mann [97] and K.Nishijima [236] proposed a simple and elegant classification scheme, where the *strongly interacting particles*, later called *hadrons*, were grouped into *isospin multiplets* numbered by an additional quantum number of *strangeness*.

We do not intend to dwell on details about this early history of particle physics (which is very interesting in itself, [183]), here only a small table of particles discovered by this time and predicted by the first classification scheme will be presented (Table 1).

This scheme predicted the existence of Σ^0-, Ξ^0-hyperons and helped to clarify the situation with kaons, at the time quite ambiguous. It can be said that also the η-meson had been predicted. From this scheme also the famous *Gell-Mann-Nishijima formula* for the electric charge of particles originated:

$$Q = (I_3 + \frac{Y}{2})e = (I_3 + \frac{1}{2}(N_B + S))e \, . \tag{1.6}$$

This scheme (with the charge formula) was an extension of Werner Heisenberg's *isotopic spin* (later called *isospin*) concept introduced in 1934 to characterize isobar nuclides.

Table 1. **Particle classification in GMN scheme**

	I_3	I	N_B	S	Q		I_3	I	N_B	S	Q
p	$1/2$	$1/2$	1	0	1	K^+	$1/2$	$1/2$	0	1	1
n	$-1/2$				0	K^0	$-1/2$				0
Λ^0	0	0	1	-1	0	(η^0)	0	0	0	0	0
Σ^+	1				1	π^+	1				1
(Σ^0)	0	1	1	-1	0	π^0	0	1	0	0	0
Σ^-	-1				-1	π^-	-1				-1
(Ξ^0)	$1/2$	$1/2$	1	-2	0	$\bar{K}^0$	$1/2$	$1/2$	0	-1	0
Ξ^-	$-1/2$				-1	K^-	$-1/2$				-1
$\triangle$ Baryons $J^P = (1/2)^+$						$\triangle$ Mesons $J^P = 0^-$					

The mathematical apparatus of isospin is exactly the same as for the proper nonrelativistic spin, it is the $SU(2)$-group. In the phenomenological model, values of the strangeness quantum number are simply attached externally ("by hand") to the isospin multiplets to get the correct values for charges. In the strong interaction processes the 3rd component of the isospin (I_3) and strangeness (S) are additively conserved quantum numbers. Then there was naturally a temptation to extend the scheme by including also the strangeness quantum number in some rigorous manner. Mathematically it meant an extension of the isospin symmetry into some larger symmetry and the incorporation of isospin multiplets into some larger "supermultiplets" of 8 baryons (with $J^P = \frac{1}{2}^+$) and 7(?) or 8 mesons (with $J^P = 0^-$). This activity lead to various schemes of "fundamental", "general", "global", "cosmic", etc. symmetries, where the isospin symmetry group $SU(2)$ was included into some larger group of rank 2 or higher.

Here theoretical particle physics came into contact with a beautiful classification theorem of mathematics – with the complete description of all simple Lie groups – initiated already in the papers by S. Lie, W. Killing and others, and finished by E. Cartan. It remained to compare the experimental data ("symmetrical figures" drawn by experimentalists) with the ready list of such symmetrical figures composed by mathematicians (naturally the classification of groups here should be complemented with the description of irreducible representations, the true mathematical symmetry figures). Theoreticians then started to rewrite the classical mathematical papers of E. Cartan, H. Weyl, B.L. van der Waerden, Dynkin et al. From that period we have many excellent reviews, such as [150, 224, 13, 338, 348, 328]. This boom of symmetries lead to remarkable results. Again Gell-Mann [98], and independently Yuval Ne'eman [238], formulated the next classification scheme, the *Eightfold Way* with the symmetry group $SU(3)$, where all known (in 1961) hadrons with $J^P = 0^-, 1^-, \frac{1}{2}^+, \frac{3}{2}^+$ were grouped into unitary octets and decuplets (i.e. into 8- and 10-dimensional irreps of the symmetry group). A triumphal *experimentum crucis* was then the discovery of the Ω^--particle (1963), the most massive hyperon from the $J^P = \frac{3}{2}^+$ decuplet.

For all simple groups including $SU(3)$ all finite-dimensional irreducible representations may be constructed through the decomposition of direct products of the *fundamental irreps* corresponding to the simple roots of the group (its Lie algebra). The number of fundamental irreps equals to the *rank* of the group (i.e. to the number of simultaneously diagonalizable generators in the corresponding Lie algebra). The group $SU(3)$ has two fundamental irreps, 3 and $\bar{3}$ (symbolically). To the mathematical composition of irreps from fundamental irreps in physics there corresponds the construction of *composite* particles from some *fundamental* ones. It was a typical situation already well-known from the applications of group-theory to QM, nevertheless it needed some courage to formulate the composite particle scheme for hadrons by Gell-Mann [99] (*quarks!*) and Zweig [384] (*aces*). The decomposition of direct products $3 \otimes \bar{3} = 1 \oplus 8$ and $3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10$ gave the correct multiplet structure for mesons and baryons. It is quite interesting that up to now there are no hadrons with higher (≥ 2) isospins demanding higher (dim 10) irreps of the $SU(3)$-group.

This composite model – *quark model* – is now well established in scattering experiments, but the situation with quarks is not so simple, as in the previous composition levels of atoms and nuclei. Quarks are *fractionally charged*: it follows from the Gell-Mann charge formula and the definition of I_3, Y from *weight diagrams* of irreps (the more profound reason lying in the topological structure of unitary groups, see the next Sec. 1.5.3). There are more quarks than only three, in addition to the strange one there are charmed, beautiful and truth (top) quarks. The quarks have an additional exact *color symmetry* leading to their *confinement*. It seems that the more profound reason of nonobservability (confinement) of quarks is *nonassociativity* of the fundamental laws ruling the microworld on the quark level of fundamentality.

1.5.3 Charge, groups, quarks, and symmetries

After the success of unitary symmetry and the quark model comprehensive deep analysis of these new concepts was carried out. There were many who did not believe into fractionally charged particles! Although models described in the previous section were models of strongly interacting particles (hadrons), a central role there was played by the electric charge, the quantity intimately related to electromagnetic interactions. This means that electromagnetic and strong interactions are in certain mutual interrelations. Nevertheless the origins and the true nature of electric charge itself are quite mysterious. The establishment of such a characteristic as the electric charge for all elementary particles, which is an integer multiple of some elementary charge, was a result of laborious work of experimental atomic and nuclear physics. In modern theories the existence of charge and its conservation are not related with space-time symmetries but with the internal symmetry of some internal, charge space. In the localized (gauged) form this symmetry converts into dynamics and the charge acquires the role of the source of the interaction.

Nevertheless, the physico-mathematical description of the charge concept remains incomplete. From this description no fundamental concept of the quantized, elementary charge follows. Can we compare it with the existence of the quantum of action (the Planck constant)? We have no clear imagination of interrelations of space-time and charge space, though here recently has been some progress of extending the space-time at high energies by additional dimensions which should be compactified at lower energies into internal degrees of freedom.

In quantum field theory (QFT) charged particles are described by *complex*-valued functions. For particles which have several characteristics (quantum numbers) of charge type, such as isospin, strangeness, charm, etc., it is quite natural to search for some *hypercomplex* description. In fact this principle has been followed in every case when internal symmetries have been introduced, because all classical groups may be formulated in terms of three division algebras **R**, **C**, and **H**.

Lately among the internal groups also *exceptional* simple Lie groups have been considered which are intimately related to the nonassociative algebra **O** of octonions. It is quite obvious that all attempts to introduce into physics non-Lie nonassociative algebras have been related with some charge-type degrees of freedom, with some charge space. This was the primary reasoning of Jordan quantum mechanics (Sec. 1.10.1). With hopes of this kind were formulated octonion symmetry schemes and octonion equations in the earlier period of particle systematics (Sec. 1.10.2). In this lies the essence of the Gürsey-Günaydin theory of quark confinement (Sec. 1.10.3) and also of the QM on the complex octonion plane and, finally, with this aim also higher dimensional spaces and their compactifications in supergravity and other multidimensional theories were considered. It seems plausible that the investigation of nonassociative conformities or regularities bring us nearer to a deeper understanding of the charge concept.

Let us return now to the problem of quark charges. Immediately after the quark proposal there followed a mathematical identification of their fractional charges as a consequence of the topological structure of the $SU(3)$-group. For the first time it was noted by Gourdin [113], afterwards the papers [103, 102] appeared with a detailed analysis of possible fractional and integer charges in symmetry schemes with $U(1) \times SU(n)$- and $SU(n)$-groups. Due to the topological property of *connectedness* here arose various quite interesting possibilities. At the same time the modern experimental status of elementary particle physics indicates that there is a restricted number of fundamental families of quarks and leptons (according to the recent CERN-LEP experiments, there are *exactly three* families of fundamental fermions), also the simplest quark model with fractional charges is strongly supported.

Nevertheless, let us present here some basic features of the algebraic and topological analysis of the quark charge concept. It is based on a global topological property of the symmetry group, on the *multi-connectedness* of the group manifold as a topological space.

The problem of the *local structure* of a Lie group is reduced to the study of the tangent space of the group manifold – to the corresponding Lie algebra. The commutation relations (CR) between the generators fully determine the local structure of Lie groups (this means the structure in the vicinity of the unit element). For determining global properties Lie algebra is not sufficient, as to a Lie algebra there correspond several locally isomorphic but globally different Lie groups. From the topological standpoint the Lie group is a multi-connected manifold. This property is measured or characterized by the *fundamental group* of the manifold. The fundamental group of a simply connected manifold (where all loops may be stretched to a point) is a trivial group consisting of only a unit element. The difference between locally isomorphic but globally different Lie groups lies in different connectedness and, therefore, in different fundamental groups. The problem of decription of all these groups is solved by the construction of a *universal covering group* (UCG) which is simply connected and uniquely determined by the Lie algebra. All the other locally-isomorphic groups are then determined as factor groups of UCG with respect to its *discrete center* and its subgroups. So the global structure of a Lie group G is fully determined by its Lie algebra $\mathcal{L}(G)$ + discrete center $Z(\tilde{G})$ of its UCG $\tilde{G}$. Let $Z(\tilde{G})$ have the subgroups $Z_1, Z_2, \ldots, Z_k$, then locally isomorphic groups are $\tilde{G}, \tilde{G}/Z, \tilde{G}/Z_1, \ldots, \tilde{G}/Z_k$, which differ in the *rank of multiconnectedness*, i.e., in the number of elements of the fundamental group.

Multiconnectedness causes the appearance of multivalued representations of these groups. One-valued representations of UCG are m-valued representations of the m-connected group $\tilde{G}/Z_i$.

In the topology these problems have been elaborated in the classical papers by Hopf and Stiefel [140, 141, 339, 340]. For the $SU(n)$ groups UCG $\cong SU(n)$ and $Z(SU(n)) \cong Z_n$, the latter being the n-element cyclic group. For the group $U(1)$ UCG is $\mathbf{R}$ (the additive group of real numbers), and $SO(2) \cong U(1) \cong \mathbf{R}/\mathbf{N}$, where $\mathbf{N}$ is the additive group of integers. The group $U(1)$ is an example of a compact infinitely connected topological space.

For the irreducible representations (*irreps*) of $SU(n)$ with the highest weight $(\lambda_1, \lambda_2, \ldots, \lambda_{n-1})$ an entity

$$p(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}) = \sum_{i=1}^{n-1} i\lambda_i \pmod{n}, \tag{1.7}$$

the *plurality*, may be defined. It is an additive quantum number conserved (mod n). This entity characterizes the multiconnectedness, but it appears quite appropriate also in the physical analysis of charge formulas. For the *fundamental irreps* $D_1(1,0,\ldots,0), D_2(0,1,\ldots,0), \ldots, D_{n-1}(0,0,\ldots,1)$ of the $SU(n)$-group the pluralities have values $1, 2, ..., n-1$ respectively. All irreps which may be "projected" out from the direct product $D_i \times D_{n-i}$ ($i = 1, 2, \ldots, k$; $n = 2k$ or $2k+1$) have zero pluralities.

Zero plurality of $SU(n)$ irreps characterizes the uniqueness of these irreps with respect to the *regular irrep* D_{reg}, i.e. with respect to the factor group $SU(n)/Z_n$. In the case of the prime n all the irreps of nonzero pluralities are n-valued irreps of the factorgroup $SU(n)/Z_n$. This exhausts the case of the prime n. There are only two locally-isomorphic but globally different groups – $SU(n)$ (it is also UCG), and $SU(n)/Z_n$.

It is such situation that takes place for the group $SU(3)$. The plurality quantum number is then called *triality* and all irreps can be divided into three classes according to the values of the triality ($t = 0, 1, 2,$). All irreps are one-valued (unique) irreps of the UCG $SU(3)$, but for the factor group $SU(3)/Z_3$ (this is just the group of the Eightfold Way Model) one-valued irreps are those with $t = 0$. For them the GMN formula gives integer values for the electric charge.

Applying this charge formula for irreps with $t \neq 0$ we get fractional charges (in units of e), because we are using the charge operator (GMN formula (1.6)) for an irrep which, strictly speaking, is not one-valued irrep for the group $SU(3)/Z_3$ for which the charge formula was written.

Soon after the proposal of fractionally charged hypothetical quarks several authors pointed out that there may be *quarks with integer charges*, but in this case there are necessarily *more* than one fundamental multiplet. To get the integer charge a new quantum number, the *supercharge* C [265], must be introduced into the charge formula:

$$Q = (I_3 + \frac{Y}{2} + \frac{C}{3})e\,. \tag{1.8}$$

In paper [136] a successful mathematical treatment of the supercharge was proposed, where also the triality quantum number conservation $\pmod 3$ was introduced. The triality might have been identified with the supercharge but the conservation $\pmod 3$ would then imply also the undesirable conservation of the charge $\pmod 3$. Therefore it was appropriate to introduce a higher symmetry, $SU(3) \times U(1)$ or $SU(4)$, where the triality quantum number turns into a strictly conserved additive quantum number.

Considering the sequence of groups,

$$\ldots \subset SU(n-1) \subset SU(n) \subset SU(n+1) \subset \ldots\,, \tag{1.9}$$

we can ensure integer charges for all irreps of the group $SU(n-1)$ by choosing the charge formula in the form

$$Q = (I_3 + \frac{Y^{(2)}}{2} + \frac{Y^{(3)}}{3} + \ldots + \frac{Y^{(n-1)}}{n-1})\,e\,, \tag{1.10}$$

where the quantities ${}^{(i)}Y$ are strictly conserved supercharges for the groups $SU(i)$ related to pluralities.

If the $SU(n)$-symmetry actually exist, i.e., some of its irreps of nonzero plurality are filled by particles, it may also be considered as an argument for the existence of $SU(n+1)$-symmetry. In the framework of this higher symmetry, $SU(n)$-quarks gain integer charges in irreps of zero plurality. However, as these irreps contain several "quark" representations of the $SU(n)$-subgroup, there emerge several fundamental multiplets (for example, the $SU(4)$-irrep D^{15} with zero *quadrality* contains $SU(3)$-irreps $D^1, D^3, \bar{D}^3, D^8$ with two fundamental multiplets $D^3, \bar{D}^3$ of nonzero triality).

Thus, building up the hierarchy of unitary symmetries, we can eliminate fractionally charged quarks. The complete exorcism of quarks demands the consecutive infinite system of symmetries (1.9). If for some reason this sequence terminates somewhere on the nth step and the next, $SU(n+1)$-symmetry is missing, then we shall have quarks with fractional charges of multiplies of $\frac{1}{n}e$ or there is some reason of a non-group nature forbidding irreps with nonzero pluralities.

For a detailed analysis of all particular cases of charge operator in groups $U(1) \times SU(n)$ we refer to [202].

We dwelled into these matters because it is perhaps the first successful application of a topological idea in the particle theory where topology has helped to solve a group-theoretical problem effectively and completely. [And there are also some personal recollections: after acquaintance with Gourdin's paper [113], one of the authors (J.L.) was involved into these problems, elaborating some details and particular cases. Then papers [103, 102] by Gerstein and Whippman appeared where the problem was entirely solved in a very general and elegant manner.]

A fundamentally novel approach to the problem of the hadronic structure has been proposed by R.M. Santilli [313] (see Sec. 1.11.5).

1.5.4 Invariance and non-invariance groups of dynamical systems

The discovery of a huge variety of resonances in 1960ies changed drastically the overall picture of the particle world, changing also the view about the nature of *fundamental* (or *elementary*) particles. Moreover, these particles living only about 10^{-23} sec were coming in large quantities. [About "elementary particles" Enrico Fermi have said that the first half of the term characterizes our knowledge about them.]

In the earlier "preunitary" period (until 1960) when there were still few particles, theoreticians dreamed that the particle systematics would lead to some finite and simple patterns, figures or schemes involving a finite number of irreps of a symmetry group (something as perfect as Plato bodies). Special tricks were invented to limit the values of the electric charge to 0 and 1 (in units of the elementary charge e). Unitary symmetry and the quark model restored the simplicity of the fundamental structures and the whole evergrowing set of the observed hadrons became a spectrum of a dynamical system of hadrons composed from subunits (quarks). It was an analog of the hydrogen atom – the simplest system consisting of proton and electron. In earlier classification schemes hadrons were often figuratively imaged as dots or drawing-pins or small circles, now they are represented by lines as the energy levels of the spectrum of some dynamical system.

After the establishment of unitary symmetry and the proposal of the quark model there followed (about 1965) some excitement about *dynamical symmetries* when the invariance and noninvariance groups of dynamical systems were thoroughly studied. A pioneering work here was done by Gell-Mann, Ne'eman and Dothan [59]. Now that more than 25 years have passed, we have some nice reviews by Ne'eman and others [29, 239] and a monograph [30]. The term *dynamical system* itself has been used in very different sense in very different contexts or situations. The broadest initial meaning of the term (originating from the greek "dynamics" – force) concerns the systems where the parts of a system are interacting through some kind of forces. In this sense we are speaking about dynamical systems in biology, in economics, in physics, and in mathematics. In exact sciences, especially in physics and mathematics, the notion of a dynamical system becomes a precise notion of a system with a specified structure and with forces between structural elements (constituents). The exact formulation of the problem for a physical dynamical system enables its treating as a pure mathematical problem. Typical examples are simple (classical) dynamical systems as *oscillator, rotator*, and *H-atom* (the *Kepler problem* in general) with a well-established mathematical descriptions by differential equations. By solving these equations we get the spectrum of states of these systems. Here also symmetry considerations (group-theoretical methods) may be applied. We shall call such well-defined systems *rigorous*. In other cases the situation is not so clear and well determined, because there is no exact equation (as to quark systems), but only fragments of the spectrum, some particular facts about forces and some general properties (symmetries among them). All this is not sufficient for a complete description of a system. An example of such system is the hadron, a system of coupled quarks.

During the boom of dynamical systems and symmetries around 1965-1970ies all the simple systems were thoroughly investigated (see, e.g., [29, 30]). There was a hope that this analysis can be extended to the case of *nonrigorous* system of hadrons. The group-theoretical analysis of rigorous systems lead to some very interesting and essential regularities in the structure of the spectrum. It appeared that for a given dynamical system there exists a well-determined set of *invariance* and *noninvariance groups* which we call the *characteristic groups* of the dynamical system under consideration. We shall give here a brief description of these groups; in connection with contractions and deformations these groups are discussed in Sec. 2.4. The *group of geometrical invariance* G (or the spatial symmetry group) is a direct consequence of the spatial symmetry of the system. This symmetry group is usually *noncomplete*, which means that the states corresponding to some particular value of the energy belong to some *reducible* representation of the group G. The *full group of (approximate) symmetry* G_I or the *dynamical degeneracy group* (DDG), sometimes called also a *hidden symmetry group*, acts on degenerate energy levels irreducibly, i.e., through irreps. Naturally, only a part of this symmetry may be called *hidden* because $G_g \subset G_I$ as a subgroup (a hidden part of the DDG for the hydrogen atom was first discovered by Pauli [272] and Fock [81]. DDG are *redundant* in the sense that they have "superfluous" irreps which do not correspond to energy levels (and at the same time all subgroups are incomplete (reducible)). About *approximate symmetries* we must speak in the case of nonrigorous systems. The flavour symmetry groups in particle systematics are examples of approximate symmetry groups. In the gauge theory of interactions we meet a characteristic situation of the *spontaneously broken symmetry* where the ground state (vacuum) has a lower symmetry compared to the initial symmetry of the Lagrangian.

In addition to invariance (symmetry) groups there are groups which connect irreducibly states of different energy levels and even the whole spectrum in some suitable unitary infinite-dimensional irrep. These are *spectrum gen-*

erating groups (SGG, for algebras SGA) first introduced by Barut, Bohm, Dothan, Gell-Mann and Ne'eman [10, 59] more than 25 years ago. There are *compact*, *noncompact*, and *inhomogeneous* groups of spectrum connected with interesting inclusion and limit relations (see Sec. 2.4).

There are also dynamical groups of transitions or *transition operator groups* (TOG), but these will remain beyond our interests here. SGG and DGT (TOG) are called also *noninvariance groups* of a dynamical system. This algebraic approach is quite effective because it determines the level structure of a spectrum. In connection with dynamical systems it would be useful to be acquainted with some general propositions of the *mathematical theory of dynamical systems* (MTDS) which takes its origins from qualitative theory of differential equations and classical mechanics [23]. The main object here is a 1-parameter group acting in the phase space of the system. Passing over from classical mechanics to the mechanics of media, the MTDS acquires an ergodic character and the phase space is tracted as a topological space with a measure.

In a most general case MTDS is a general theory of transformations with invariant measure, it has also wide connections with other parts of mathematics [367]. From such general theories one can see some (otherways nonunderstandable) facts about particular systems. For example, it is a well known fact that symmetry groups of known stationary or quasistationary systems, and also their noninvariance groups (let us forget about IG) are *simple* or *semisimple* groups. This fact may be explained from the standpoint of general theories, where the simple groups correspond to periodical motions [8].

1.5.5 Interaction-dependent systematics of particles. Standard Model and Grand Unifying Theories

The most serious shortage of the earlier particle systematics schemes was the property that they were essentially classification schemes for the *strongly interacting particles (hadrons)*. There were some schemes for leptons with "weak" isospins and strangeness and various kinds of *lepton numbers*, but these schemes were again quite loosely and artifically related to the systematics of hadrons.

It was then quite natural to assume that every interaction must have its own symmetry and classification scheme where only the particles characteristic of a particular interaction are included. Such an idea was proposed in 1961 by an Estonian theoretician Harry Õiglane [204]. In this paper a unified model was proposed on the basis of a 6-dimensional internal space with the symmetry group $SO(6) \cong SU(4)$ unifying the *symmetry of hadrons* (a $SO(4)$-subgroup), *electromagnetic interactions* (an another $SO(4)$-subgroup), and *weak interactions* (a $U(1) \times SO(4)$-subgroup) by *intersecting* subgroups. Every particular systematics (fixed particular internal symmetry) breaks the systematics of other interactions through the noncommutativity of the corresponding symmetry operators.

The same idea was proposed by Baker and Glashow [9], where the symmetry groups of all three interactions were isomorphic (to the group $U(1) \times SU(2)$), but they were included into the unifying group $SU(3)$ in different ways as intersecting subgroups (where $SU(2)$-subgroups of different interactions may be regarded as the I-(isospin), U- and V-spin groups [13]) so that the operators of "good" quantum numbers were noncommuting for different interactions. In this way each interaction disturbes the symmetries (and systematics) of the other two interactions.

An analogous scheme was proposed by S.P.Rosen [288]. These schemes were the early predecessors of *Grand Unifying Theories* (GUT). Unfortunately, the symmetry groups in these schemes were chosen in quite a nonadequate manner, also the importance of the gauge principle was not yet fully understood and explored.

In the second half of 1960ties the unifying theory for electromagnetic and weak interactions (the theory of *electroweak* interaction or *flavordynamics*) was developed, the most simple model of which has the group $U(1) \times SU(2)$ and is known as the *Glashow-Salam-Weinberg* (GSW-) model [105, 373, 303]. This model already explores the ideas of local (gauged) symmetry and spontaneous symmetry breaking (realized through the *Higgs mechanism*). The gauging of the $U(1) \times SU(2)$-symmetry gives four vector gauge mesons, through the spontaneous symmetry breaking mechanism three of which acquire masses and only one (the photon) remains massless. The triumph of this model was the discovery of massive vector mesons, the *intermediate* $W^{\pm}$-, Z^0-bosons in CERN in 1982-1983.

On the other hand, it appeared that the "pure" strong interaction between quarks has an exact "color" $SU_c(3)$-symmetry, in the gauged form it is the basis of the fundamental theory of strong interactions called *quantum* *chromodynamics* (QCD).

The theories of electroweak interactions and quantum chromodynamics form the so-called *Standard Model* with the minimal symmetry group $[SU(2)\times U(1)]\times SU_c(3)$. Already the form of this symmetry group indicates that the unification of three interactions in these frames is quite incomplete.

Later (since 1974) GUT models were proposed which pretended to the unification of weak, electromagnetic and strong interactions. GUTs automatically unify quarks and leptons, symmetry transformations transform quarks into leptons and vice versa, or even more simply – in GUTs quarks and leptons are symmetrical. Mathematically it means that the minimal group of the Standard Model should be included into some larger simple Lie group with the rank ≥ 4.

The first model of GUTs was proposed in 1974 by G. Georgi and Sh. L. Glashow [101], the group of this model was $SU(5)$. The gauging of this symmetry gives 24 Yang-Mills vector bosons, some of which, the leptoquark bosons (or X-bosons), mediate the transitions between quarks and leptons, leading particularly to the decay of the proton (not observed insofar). These bosons must have very big masses, something about $10^{14}-10^{15}$ GeV. This energy is oftenly called the GUT energy.

In reality such a symmetry may have existed only in the extreme conditions at superhigh temperatures in the very early period (10^{-43} sec) after the Big Bang. Here in the models of Grand Unification high-energy physics for the first time got into contact with the cosmology of early Universe.

The proton lifetime according to GUT must lie in the interval of $10^{26}-10^{32}$ years (depending on the model). Unfortunately this problem must be regarded as an unsettled one because in specially arranged underground experiments no such decay has been registered insofar.

In the GUT models very many groups of rank ≥ 4 have been explored, among them the exceptional groups E_6, E_7, and E_8. The most popular and successful groups have been $SU(5)$, $SO(10)$, and E_6. Here also the supersymmetrically extended "flipped" $SU(5)$ model of John Hagelin et al. [5] should especially be noted. *Supersymmetry* ([284], see also some remarks in the next subsection) is the most profound type of unification in Nature, it appears that it is supersymmetry that allows the solving of some principal problems of unification of fundamental interactions.

And finally ... according to John Hagelin [135]: *the unifying field for all interactions in Nature is the Consciousness!*

1.6 Supersymmetry and supergravity

Unlike the majority of applications where nonassociative algebras have been introduced as an initial principle (except of M_3^8-algebra in the algebraic analysis of QM) octonions and groups of the E-series emerged quite unexpectedly in supergravity.

1.6.1 The principle of supersymmetry and Lie superalgebras

Insofar our symmetry considerations have allowed us to collect into a multiplet (i.e. into a symmetrical figure) particles with different masses and charges and even with different spins, but in the latter case the spins should belong to one and the same statistics. Commonly symmetry transformations cannot connect bosons and fermions.

In 1971 Ramond [283] and Neveu and Schwarz [244], attempting to generalize the Veneziano (dual string) model to include fermions, introduced a new algebraic structure which turned out to be a *graded Lie algebra.* In other papers [112, 374, 369], the graded version of the Poincaré algebra was introduced and a field theory with interaction invariant under this *supersymmetry* algebra was built [369]. Then gravity was generalized through the gauged supersymmetry to a theory called *supergravity* [365, 62] (see the next subsection).

Graded Lie algebras (GLA) generalize the common Lie algebra structure

$$[I_i, I_j] = c_{ijk} I_k\,; \quad i,j,k = 1,2,\ldots,r\,, \tag{1.11}$$

$$\sum_{cycl} [I_i, [I_j, I_k]] = 0 \quad \text{(Jacobi identity)} \tag{1.12}$$

to include other "spinorial" generators Q_α, which under mutual *anticommutation* relations close into the Lie algebra:

$$[I_i, Q_\alpha] = f_{i\alpha\beta} Q_\beta\,, \tag{1.13}$$

$$\{Q_\alpha, Q_\beta\} = d_{i\alpha\beta} I_i\,, \tag{1.14}$$

$$i,j,k = 1,2,\ldots,r\,; \quad \alpha,\beta = 1,2,\ldots,s$$

In the Kac's classification [170] simple finite-dimensional GLAs are divided into two classes:

1) *classical* GLA, the Lie part of which is a sum of simple Lie algebras, modulo abelian generators;

2) *hyperclassical* GLA, the Lie part of which consists of a simple Lie algebra together with some of its representations, c.f. also [284].

Local (gauged) Poincaré supersymmetry gives *simple supergravity* (SSG). The gauging of supersymmetry demands the introduction of fields with spin 2 (*gravitons*) and 3/2 (*gravitinos*) [365, 62].

Introduction of internal symmetries for the spinor generators gives us *extended supergravities* (ESG), here only (1.14) is essentially changed:

$$\{Q^a_\alpha, Q^b_\beta\} = 2\delta_{ab}(\gamma_c)^{\alpha\beta}D^c + \text{additional terms}\,, \tag{1.15}$$

$$a, b, c = 1, 2, \ldots, N$$

Here the quantity N may take values 1,2,...,8 only (it is the consequence of equations of motion and of the requirement of nonexistence of spin>2 particles). The case $N = 1$ is SSG, the case $N = 8$ is called *maximally extended supergravity* (MESG).

1.6.2 Algebraic structure of supergravity

The internal symmetry group of ESG, i.e. the group connecting particles with equal spins, is the group $SO(N)$. Here we are interested mainly in the MESG with the internal symmetry group $SO(8)$. The only irreducible multiplet of MESG contains the following massless particles:

spin-chirality	± 2	$\pm\frac{3}{2}$	± 1	$\pm\frac{1}{2}$	0
$SO(8)-$multiplet	1	8	28	56	70

For a realistic theory the group $SO(8)$ must be extended because it does not contain the group $(SU(2) \times U_1) \times SU_c(3)$ of the Standard Model. In fact the symmetry of MESG is not confined to the group $SO(8)$, but it has an additional $SU(8)$-symmetry. Both these groups may be gauged but their realizations are different. The group $SO(8)$ is considered as a Yang-Mills group with fundamental vector fields (and with the corresponding kinematical term in the Lagrangian). The corresponding $SU(8)$-fields are regarded as dependent (composite) fields, because in the fundamental MESG-multiplet there is no place for 63 fundamental gauge vector bosons. Besides, the coupling constant for $SO(8)$ is arbitrary, but for $SU(8)$ its is fixed.

The most consistent approach to these problems is provided by the method of *dimensional reduction* [47, 46]. In this reduction process the "superfluous" coordinates of the initial high-dimensional space-time are compactified and turned into the topological product of the 4-space-time and a compact space, the latter depending on the process of reduction. For example, proceeding from the SSG with 11-dimensional space-time, the compactified part is the 7-torus.

In the dimensional reduction process the hidden symmetry of the resulting MESG ($d = 4$), the global group $E_{7,7}$ and the local group $SU(8)$ become apparent. The first of them is a noncompact real form of the exceptional simple Lie group E_7 which is the symmetry group of the equations of motion, the second is the maximal compact subgroup in $E_{7,7}$ and the symmetry group of the Lagrangian.

The particles from the above-mentioned supermultiplet have the following transformation properties: the graviton is an $E_{7,7}$- and $SU(8)$-singlet, fermions transform under the representations 8, 56 of the $SU(8)$-group, vector particles belong to the 56-dimensional irrep of $E_{7,7}$, being also gauge particles for $SO(8)$. Scalars live in the 70–dimensional factorspace $E_{7,7}/SU(8)$.

It is interesting that all MESG ($d = 3, \ldots, 9$) resulting from the 11-dimensional theory have invariance groups from the exceptional "E-series" (see [46]).

From all SG theories the MESG ($N = 8, d = 4$) and SSG ($N = 1, d = 11$) have maximal symmetries and a unique character, where multiplet structure and interactions are fully determined by the requirement of local supersymmetry.

In simple *dimensional reduction* procedure fields in the limit are regarded as independent of superfluous (internal) coordinates, but the latter may also taken more seriously (cf., e.g., [61]).

The structure of the final space after the *spontaneous compactification* of the 11-dimensional theory is determined by the vacuum states which are solutions of the equations of motion. In general, the final space has the form $M^4 \times M^7$, where M^4 is the space-time manifold, and M^7 some internal space with properties depending on the vacuum state chosen.

In[93], for example, we have a solution (in the boson sector) in the form

$$eF_{\mu\nu\rho\sigma} = \pm 3m\varepsilon_{\mu\nu\rho\sigma} , \quad F_{mnpq} = 0 , \tag{1.16}$$

where $F_{....}$ is the field tensor, e – coupling constant of the $SO(8)$ gauge theory; m – a real constant; the indices μ, ν, ρ, σ are related to M^4, the indices m, n, p, q to M^7. Then the particular choice (1.16) leads to the definite spaces M^4, M^7:

$$M^4 \simeq AdS \equiv SO(3,2)/SO(3,1) , \quad M^7 \simeq SO(8)/SO(7) \equiv S^7 , \tag{1.17}$$

where AdS is the anti-de Sitter space and S^7, the 7-*sphere* with the usual Riemann metrics and with the radius $R = m^{-1}$.

There is yet another compactification, $M^{3,1} \times T^7$ [46] ($M^{3,1}$ – the Minkowskian space-time, T^7 – 7-torus). It is proved [22] that these two are the only possible solutions allowing MESG.

Here we have S^7, an exceptional geometric figure intimately related to the octonion algebra. The 7-sphere has interesting geometrical properties, which are unique and exceptional, the *absolute parallelism* among them. This is the property of Lie group manifolds, but the only non-group manifold with this property is the 7-sphere. The octonionic treatment of parallelization of the 7-sphere is given in [133, 51, 216]. In the first paper, there is a very deep analysis of octonion nonassociativity in the Englert's compactification, [69]. In particular, starting from octonions, the Cartan-Schouten equations are derived and the spontaneous symmetry breaking is related to the associator of octonions.

There have been some attempts to relate the four *division algebras* **R, C, H, O** with supersymmetric field theories in space-time dimensions 3,4,6, and 10, respectively. Only for these dimensions there exist classical covariant Green-Schwartz superstrings (see refs. in Sec. 1.10.4).

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About 7-sphere (mathematics and higher-dimensional schemes): [225, 38, 216, 215, 70, 71, 72, 104] [144, 320, 40, 41].

1.7 About algebras in modern mathematics

In 1976 there appeared a paper "Was sind und was sollen Algebren?' [179] ("What are algebras and what should they be?") by Max Koecher, its title was paraphrased from the title of an analogous paper by Richard Dedekind "Was sind und was sollen Zahlen?"(What are numbers and what should they be?") which appeared in 1887. In the latter Dedekind wrote: *"Numbers are free product of human mind and they serve for more easy and subtle differentiation of things"*. Koecher writes about algebras in modern mathematics: *"Algebras as the other algebraic systems must serve in mathematics for the simplification of theoretical constructions"*.

These simple sentences have a profound meaning indeed. We do not even imagine how inconvenient it would be to formulate and describe some relations, facts and situations in mathematics and physics (in QM for example) without group theory or without (hyper-)complex numbers. Let us try to write in components even the most simple expressions of octonion algebra ...

This role of algebras seems to be retained also in future mathematics. There has always been a general trend to abstractions, but nevertheless in the theory of algebras some particular classes of algebras have remained important. So in 1986, 44 years after an analogous passage by Albert (cited in Sec. 1.2), Marshall Osborn [268] writes: *"Without associativity, rings and algebras are not in general well enough behaved to have much of a structure theory. For this reason, the nonassociative algebraist normally studies the class of rings which satisfy some particular identity or set of identities. There are three classes of nonassociative algebras which we regard as more important than the others. First is the class of Lie algebras ... The second important class is the class of Jordan algebras ... The third is the class of alternative algebras"*. And as the most important example of alternative algebras octonion algebra is considered.

There are naturally yet some other natural and more general identities (conditions) which define quite general classes of algebras including the above-mentioned classes of good algebras.

One such condition is *power-associativity* (called also *monoassociativity*). In a power–associative algebra every element generates an associative subalgebra. Foundational work about these algebras belong to Albert [3]. An algebra $\mathcal{A}$ over a field of zero characteristics is power associative iff

$$a^2a = aa^2\,, \quad (a^2a) = a^2a^2\,; \quad \forall a \in \mathcal{A}. \tag{1.18}$$

An algebra $\mathcal{A}$ is called *flexible* [3] if the following identity is satisfied:

$$(ab)a = a(ba)\,; \quad \forall a, b \in \mathcal{A}. \tag{1.19}$$

As we have already seen, quite interesting new classes of algebras may be constructed by simple redefinitions of the multiplication of the algebra under consideration, e.g., passing over to *commutator* or *anticommutator* algebras. An algebra $\mathcal{A}$, for which the commutator algebra $\mathcal{A}^-$ is a Lie algebra, is called *Lie-admissible* [3], about the applications of these algebras see Sec. 1.9.2 and 1.11. Analogously, if the commutator algebra is a Mal'tsev algebra, the initial algebra is called *Mal'tsev-admissible* [234]. Algebras for which (half-)anticommutator algebras are Jordan algebras are called *Jordan-admissible* [3].

Even more evolved redefinitions of the multiplication operation may be considered. For an algebra $\mathcal{A}$ (over a field$F$) the (*left*) (r,s)-*mutation* $\mathcal{A}^{\pm}(r,s)$ [308] may be defined with a new (mutated) multiplication

$$x * y = (xr)y \pm (ys)x\,; \quad r, s \in \mathcal{A},\, , \forall x, y \in \mathcal{A}. \tag{1.20}$$

In [304] (λ, μ)-mutations $\mathcal{A}(\lambda, \mu)$ were introduced with a new multiplication rule

$$x * y = \lambda xy + \mu yx\,; \quad \lambda, \mu \in F\,, \forall x, y \in \mathcal{A}, \tag{1.21}$$

this class contains so-called *quasiassociative* algebras or $(\lambda, 1-\lambda)$-*mutations* [2, 3] with multiplication

$$x * y = \lambda xy + (1-\lambda)yx, \quad \lambda \in F. \tag{1.22}$$

Some classes of well-known good algebras have common properties allowing to include them into some more general "unifying" classes of algebras. For example, Albert [3] defined a class of *standard algebras* (SA), including all associative algebras and (commutative) Jordan algebras (CJA). Each simple algebra in this class is associative or CJA. In [318] Richard Schafer defined an even broader class of *generalized standard algebras* (GSA) including all alternative algebras and SA. Every simple GSA is alternative or CJA. In

[7] there was defineda a class of *more generalized standard algebras* (MGSA) including Lie algebras and GSA. Commutative MGSA are generalized *Lie triple systems.* The latter are *ternary algebras* of some particular type (see [153, 240]). The theory of ternary algebras is not very well-developed (see, e.g., [201, 347]), nevertheless there are few papers concerning physical applications. In connection with octonions and the Cayley-Dickson procedure a particular *ternary sedenion algebra* should be mentioned ([331]; see Sec. 3.4).

Evidently, the most elaborated is the theory of *Lie algebras.* There are also very important recent results in the theory of *Jordan algebras* (see the definition in the next section).

In 1984 Kevin McCrimmon published a paper [223] under the title "Russian revolution in Jordan algebras". Here we bring the abstract of this paper, which is perhaps the best way to introduce it.

"During the last 6 years, Russian mathematicians have made a series of startling advances which essentially complete the general structure theory for Jordan algebras. Most of this work has come from the school of the late A.I.Shirshov at Novosibirsk, especially the young mathematician E.I.Zelmanov. News about these advances seeped slowly to the West, through rumor and terse announcements at international scientific meetings, and only recently have the details begun to appear in print. They give us an incisive description of the structure of infinite-dimensional Jordan algebras, an insight into how this structure comes about. We know that the only simple Jordan algebras are those already discovered in finite dimensions by Jordan, von Neumann, and Wigner at the inception of Jordan theory in 1934: the special algebras either come from quadratic forms or are symmetric elements in an associative algebra with involution, and there is one exceptional algebra of dimension 27. At that time physicists still hoped that an exceptional setting for quantum mechanics might be found in infinite dimensions: "It may well happen that new types of algebra will arise with the removal of (the finiteness) restrictions ... in ordinary quantum mechanics many important features first appear in finite algebras." We now know that this hope is doomed to failure: the only simple exceptional Jordan algebra allotted to mortals is the 27-dimensional Albert algebra."

And of course there is the whole context of recently elaborated Lie-admissible algebras and Lie-isotopies (see a more extended treatment in Sec. 1.9.2 and 1.11 with Refs. therein).

For modern mathematics thick intertwining of very many directions and subdisciplines is typical. So the algebraic structures, nonassociative algebras among them, are percolating other branches of mathematics accomodating to special demands and purposes, by acquiring new features and properties to serve *"for the simplification of theoretical constructions"*.

1.8 Infinite-dimensional algebras of loop groups

Insofar (and also in what follows) we have consciously avoided infinite-dimensional systems. Here we shall swerve aside to make a short excursion into the infinite-dimensional world because our main topics, *octonions* and related problems, are somewhere in the deeper level connected with ∞-*dimensional algebras* of some special types.

These ∞-dimensional algebras are Lie algebras of a special class of infinite-parametric Lie groups called *loop groups* [282], more precisely, these are groups $\mathrm{Map}(X, G)$ of mappings $X \mapsto G$ from a compact space X into a group G. The term itself originates from the simplest case $X \equiv S^1$.

In the quantum field theory (QFT) groups $\mathrm{Map}(X, G)$ originate in two ways: 1) as *gauge groups*, or 2) as *groups of currents*. In these cases the space X a "compactified" physical space. A proper loop group itself arises when the space X is 1-dimensional and compactified into the 1-sphere (loop) S^1, such a situation occurs in the case of *closed superstrings*.

Groups $\mathrm{Map}(S^1, G)$ are comparatively well studied and there is some amount of literature about them (see [282]). Their infinite-dimensional algebras belong to the class of *Kac-Moody (KM-) algebras* [169, 227]. For a brief characterization of these algebras let us recall that in the classical theory of Lie algebras the construction of finite-dimensional semi-simple algebras explores positively-determined Cartan matrices. The rejection of the positive definiteness leads to the KM-algebras. If we confine ourselves to the nonnegative case, then we are lead to the class of *affine Kac-Moody (AKM-) algebras*, and it is these algebras that are the Lie algebras of the loop groups $\mathrm{Map}(S^1, G)$.

For a Lie group with a finite-dimensional Lie-algebra, the general form of commutation relations (CR) for generators is

$$[I_i, I_j] = c_{ij}^k I_k \, ; \quad i, j, k = 1, 2, \ldots, r \, . \tag{1.23}$$

Mappings $S^1 \mapsto G$ are defined by replacing the parameters θ^i ($i = 1, 2, \ldots, r$) of the group G by the functions $\theta^i(z)$, $z \in S^1$; then, considering the firts approximation and expanding $\theta^i(z)$ into the Loran series, the coefficients θ^i_n ($i = 1, 2, \ldots, r$; $n \in \mathbf{Z}$ form an infinite set of parameters for the group $\mathrm{Map}(S^1, G)$ under consideration. Taking now $I^{(n)} = I_i z^n$ as the new generators, we get CR for an ∞-dimensional Lie algebra

$$[I_i^{(m)}, I_j^{(n)}] = c_{ij}^k I_k^{(m+n)}\,, \tag{1.24}$$

where $i, j, k = 1, 2, \ldots, r$; $m, n \in \mathbf{Z}$.

The groups $\mathrm{Map}(S^1, S^1) \equiv \mathrm{Diff}S^1$ and $\mathrm{Map}(S^1, U(1))$ should be distinguished. The first group is the group of diffeomorphisms of the circle S^1, i.e. the group of "general coordinate transformations", or the *reparametrization group*, its algebra being the *Virasoro algebra*. The second group is the gauge group on the circle S^1.

The generators of the Virasoro algebra are (in commonly used notations)

$$L_n = -z^{n+1}\frac{d}{dz}\,; \quad n \in \mathbf{Z} \tag{1.25}$$

with CR

$$[L_m, L_n] = (m-n)L_{m+n}\,. \tag{1.26}$$

The Lie algebras of the groups $\mathrm{Map}(S^1, G)$ and $\mathrm{Diff}S^1$ have nontrivial *central extensions* with CR

$$[I_i^{(m)}, I_j^{(n)}] = c_{ij}^k I_k^{(m+n)} + km\delta_{m,-n}\delta_{ij}\,, \tag{1.27}$$

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n}\,. \tag{1.28}$$

The *Virasoro algebra* is known from CR (1.28), determining the Lie algebra of the *conformal group* in 2 dimensions. This algebra appeared in physics for the first time in the discussion of the dual-resonance model of hadrons, where the operators L_m connect arbitrary states with the states without scalar excitations (ghosts) [368].

There is now an enormous amount of literature about KM and Virasoro algebras, we cite here only some reviews, [241, 107, 266, 326]. The algebras of Kac-Moody type occur in many contexts in mathematics and physics. Let us list here some of them (according to [107]): sporadic finite simple groups, theory of modular forms, completely integrable dynamical systems, gauge fields, theory of superstrings, conformally-invariant field theories, theory of critical phenomena in 2-dimensional statistical systems, etc. The general property of each link in this chain is the natural occurrence of some ∞-dimensional Lie algebra of the type considered. Here *octonions* are in a very natural way connected with the beginning of the chain through *loops* (and through the *Moufang loops* particularly), the latter meaning here algebraic systems (quasigroups) of a special kind, the corresponding perfect codes, triality property, etc. This exotic and exceptional region deserves serious attention, because there are crossing many paths and directions, making the intersection region highly unique. There are already some far-reaching studies of vertex operators by means of representations of KM- and Virasoro algebras and the algebras connected with the *Griess Monster* [31, 88].

1.9 Nonassociative systems in physical applications. Three main approaches

There are several approaches in theoretical physics exploring the notion of nonassociative system. Here in Sec.1.9.1-3 we present a brief discussion of them in the chronological order. Naturally, as always, such a strict and "linear" subdivision of the topics is not completely adequate, because there are many smaller branches, overlappings and intersections.

Here at once some works must be mentioned having more general character and not confined to any particular approach mentioned above.

At first there is a paper by Yoichiro Nambu [237] (originating the *Nambu mechanics*) where some perspectives of nonassociative algebras for generalizing the Hamiltonian mechanics were pointed out. Taking a Liouville theorem (which states that the volume of phase space occupied by an ensemble of systems is conserved) as a guiding principle, a possible generalization of classical Hamiltonian dynamics to a 3-dimensional phase space is considered. The equation of motion involves two Hamiltonians and three canonical variables (instead of classical two, p and q). For any observable F then the Hamilton's time evolution equation

$$\frac{dF}{dt} = \frac{\partial(F, H, G)}{\partial(x, y, z)} = \vec{\nabla} F \cdot (\vec{\nabla} H \times \vec{\nabla} G) := [F, H, G] \tag{1.29}$$

follows, where the right hand side may be regarded as a *generalized Poisson bracket*. As here three observables are present, the associator can be built and some types nonassociative (alternative, Jordan) algebras may be considered.

Then there is also an interesting cycle of papers by Arthur Sagle [295, 302, 301, 296, 297, 298, 300] [299, 139] about nonassociative connection structures on homogeneous spaces with applications in Lagrangian mechanics (the cycle has natural connections with Lie-admissible algebras [296] and analytical loops [139]). For every invariant connection on a reductive homogeneous space G/H the operation of covariant differentiation gives a structure of nonassociative algebra.Geometrical properties of the homogeneous space (geodesics, holonomy, etc.) may be expressed in algebraic properties of the corresponding connection algebra (simplicity, idempotents, etc.). If G/H is interpreted as the configuration space of a mechanical system, many geometric properties of the system (e.g. conservation laws) may be also expressed in algebraic terms. For example, Cayley and Mal'tsev algebras are considered describing the Lagrangian mechanics on 7-sphere [299].

1.9.1 Octonionic approach

The octonionic approach to problems of the fundamental physics consists of the following main stages:

– the discovery of the exceptional Jordan algebra in the algebraic analysis of foundations of QM (Jordan, von Neumann, Wigner, [165]);

– some sporadic applications to the particle classification schemes in the "pre-unitary" period;

– quark confinement and the octonionic Hilbert space (Günaydin, Gürsey [123]), and some further development of nonassociative QM;

– octonions and the related structures (E_8-group, etc.) in the superstring theory.

Investigations in Tartu are summarized in the chart "Octonion approach in Tartu" and presented in Chs. 3,4,6.

We add here only the following very brief comments. A broader treatment of the problems listed above is presented in Sec. 1.10.

For physical applications of nonassociative systems one had to wait for almost 100 years, until in 1934 three great masters [165] undertook an algebraic analysis of foundations of QM and discovered the exceptional Jordan algebra M_3^8 consisting of 3x3-matrices with octonionic elements. This algebra has played an important role in the theory of exceptional Lie groups, and from time to time it has also originated some physical considerations, and recently, vertex operator has been constructed by means of it in the superstring theory. This was the first case when octonions appeared in the physical context, but the most important legacy of the paper [165] (and of course of related papers [159, 161, 160] by Jordan) was the originating of the *Jordan-algebraic approach* to QM with many fruitful connections with projective geometry, lattice theory, etc. Meanwhile this approach was nearly forgotten, but it was revitalized again by the appearance of paper [123]. [For some more detailed discussion of Jordan QM and related problems, see Sec. 1.10.1.]

During the particle classification boom at the end of 1950ies, few papers appeared on the systematics of baryons and mesons by means of octonions (the most direct reasoning here was founded on the magic of the number *eight* – the number of known metastable baryons, and on the connection between the octonions and the groups $SU(3), G_2$, and $SO(8)$). There were also some interesting mathematical papers about exceptional groups and their connections with octonions (and the algebra M_3^8) by J. Tits, H. Freudenthal, and B.A. Rosenfeld [350, 90, 89, 91, 92, 289, 290].

There has been also an everlasting but subdued activity of applying octonions to the formulation of relativistic wave equations. Most of the relativistic invariant equations can be formulated in terms of quaternions. The use of quaternions is principally equivalent with the use of a certain system of matrices. Therefore these formulations do not lead to any essentially new result except more elegant form of equations. As octonions form the direct nonassociative generalization of quaternions (e.g. through the well-known Cayley-Dickson process), there is a hope to get some new knowledge from the octonionic formulation of relativistic-invariant equations. [About symmetry groups and particle equations in connection with octonions, see also Sec. 1.10.2 and Refs. therein].

The interest in octonions was essentially renewed in 1973 by the paper by Feza Gürsey and Murat Günaydin [123], where the idea of the *octonionic confinement* was formulated in terms of the *octonionic Hilbert space* (the last notion was introduced by Goldstine and Horwitz [111, 110]). This paper was followed by a number of publications by these and other authors, and it induced a further development of nonassociative (octonionic) QM along the lines of the Jordan-algebraic and propositional calculus ([157, 278]) formulations (cf. reviews [126, 291]). [See also Sec.1.10.1, 1.10.3].

About earlier activities and the renewal in the 1970ies we refer to review [330], see also Sec. 1.10.

The interest in octonions received a new essential support by the appearance of E_8, the last and the largest exceptional group in the superstring theory. There have also been some attempts to formulate the superstring theory in terms of octonions. Nevertheless, it seems that these have been limited to technical formulations and the foundational role of octonions (and nonassociativity in general) in the superstring theory is not yet clarified. [See also Sec. 1.10.4].

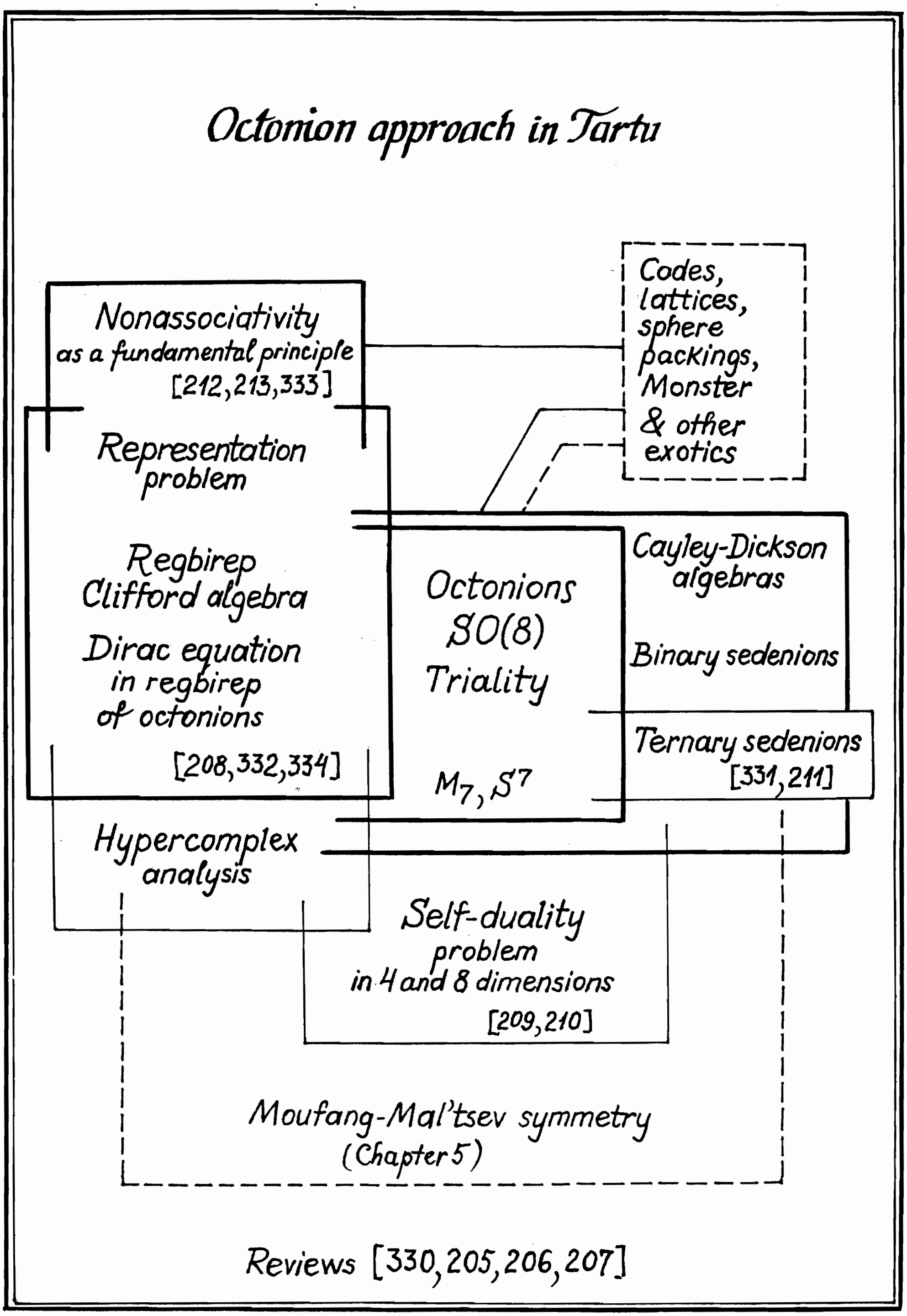
A more recent renewed interest in octonions has been provided by the isotopic generalization of Dirac equation based on *isooctonions*, having some intriguing possibilities for applications (see Sec. 1.11).

In conclusion it may be said that the last 15-20 years of success of the Standard Model, Grand Unification, supersymmetry & supergravity, Kaluza-Klein type theories, and the superstring theory and iso- and genotopic theories have lead us to new higher-dimensional exotic and strange algebraic and geometric structures with new perspectives for new dynamical theories in some generalizations of space-time where hypercomplex numbers, first of all octonions, will play a central role.

The most remarkable properties of octonions and related structures are their uniqueness, exceptionality and finite-dimensionality. The algebra of octonions is unique as the last and the largest, noncommutative and nonassociative alternative division algebra over the reals, it is also the last and largest composition algebra (over the field of $char = 0$). The projective plane coordinatized by octonions is two-dimensional. The chain of exceptional Lie groups (algebras) connected with octonions through the exceptional Jordan algebra M_3^8 contains only five groups. There exists a finite set of exceptional geometries of the octonion plane corresponding to these groups. The 7-dimensional commutator algebra of octonions, the Mal'tsev algebra M_7 is (almost) a unique simple algebra among the non-Lie Mal'tsev algebras.

Octonions have deep relations with sporadic simple finite groups, particularly with the Griess Monster (the Friendly Giant), with perfect self-correcting codes, with exceptional sphere packings and with exceptional lattices in the number theory (about all this see [44]).

In the chart of finite-dimensional linear algebras (Sec. 1.2) the octonion


Octonion approach in Tartu
Codes, lattices, sphere packings, Monster & other exotics
Nonassociativity as a fundamental principle [212,213,333]
Representation problem
Regbirep Clifford algebra
Dirac equation in regbirep of octonions
[208,332,334]
Octonions
SO(8)
Triality
M7, S7
Cayley-Dickson algebras
Binary sedenions
Ternary sedenions [331,211]
Hypercomplex analysis
Self-duality problem in 4 and 8 dimensions [209,210]
Moufang-Mal'tsev symmetry (Chapter 5)
Reviews [330,205,206,207]

algebra occupies the central position on the border and the overlapping of associative, alternative, and Lie-Mal'tsev algebras. The octonion algebra was the second hypercomplex system discovered just after the quaternion algebra. In these times the notion of a hypercomplex system was not yet established. It is quite interesting that an example of a nonassociative system was known even before the matrix calculus was developed by Cayley.

The uniqueness and exceptional properties of octonions also seem to have very important consequences for the physical world. We believe that this highly exceptional mathematical system, where many different structures of mathematics overlap, serves as the foundation of highly-determined, perhaps the only possible physical world.

In Tartu the ideas about nonassociativity started long ago, in about 1950ies, when the attention of one of the authors (LS) was called to octonions by our late colleague Raimund Preem (1918-1988). We have investigated various aspects of octonionic formulations and some general aspects of nonassociativity [330, 206, 207, 205, 212], some mathematical problems (birepresentation, SO(8) group, triality [330, 206, 207, 208, 332], *ternary sedenions* [331, 211] and some particular applications including an octonion formulation of the Dirac equation [208, 332, 334] with color and confinement,and particle spectrum; self-duality in 4 and 8 dimensions [209, 210], and associator quantization [333, 213] (see the chart "Octonionic approach in Tartu").

1.9.2 Lie-admissibility and Lie-isotopy

This approach originates from the papers by R.M. Santilli [304, 305] and rests on the notion of *Lie-admissible algebra* (see Sec.1.3). According to Santilli's suggestion generalizations of physical theories must preserve their initial Lie-algebraic content through the Lie-admissibility, i.e. in the commutator algebra of the generalized algebra. A generalized theory is characterized by a Lie-admissible algebra which under some suitable conditions transfers to the Lie formalism after the redefinition of operation (passing over to the commutator algebra).

The Lie-admissible approach has been applied to numerous problems – analytic mechanics in general, the problems of hadronic interactions, dissipative systems, various problems of QM, field theory, symmetry breaking, etc. (Refs. to earlier papers may be found in [306], see also [330, 206, 207]).

The essential features of this approach can be characteristically illustrated on the example of classical (analytic) mechanics. From the very beginning the Hamiltonian mechanics has been dealing with point-like particles, moving in the homogeneous and isotropic media (vacuum) under conservative forces derivable from potential. However, such *autonomous* systems are rather exceptional in the real world. In general mechanical systems are nonconservative (dissipative), their total energy is not conserved, there are nonpotential forces in action, etc. The Hamiltonian equations with *external terms* (expressing the nonautonomous character of the system) induce a Lie-admissible time evolution law (Lie-admissible brackets). So the Lie-admissibility permits the comprising of nonconservative systems (see [310]).

Santilli has also founded a second group of methods now called *Lie-isotopic formulations.* It is an alternative approach in which the external terms are replaced by a *generalized unit* of the theory leading to the structural generalization of Lie algebras, symplectic geometry, and Hamiltonian mechanics into Lie-isotopic algebras, symplectic-isotopic geometry, and Birkhoff-Santilli mechanics, [307, 171] (see also Sec. 1.11.4).

One of the originating impulses for the Lie-isotopic (or *Lie-Santilli*) theory was the apparent disparity between the *special* and the *general* relativity, the former being reducible to the Poincaré symmetry, while the latter is not believed to admit any reduction to a primitive and universal symmetry. This disparity appears even more forcefully in the transition from the *exterior dynamical problem* (point-like paricles moving in the homogeneous and isotropic media) to the more general *interior dynamical problem* (with extended, deformable particles moving in inhomogeneous and anisotropic media). This implies that theories are *nonlinear, nonlocal* and *noncanonical.* Such structures emerge in varieties of physical conditions, ranging from the traditional Newtonian interior systems (e.g. satellite during the re-entry into the planet's atmosphere) to astrophysics (e.g. protons in the core of a collapsing star) or superconductivity (e.g. the interactions of electron pairing), etc. Due to the nonlinear, nonlocal and noncanonical character of these situations the conventional theory based on Lie symmetries is not applicable.

Santilli's approach consists in the generalization of the current formulation of Lie's theory, originally submitted under the name of *Lie-isotopic theory* [309] and today known as the Lie-Santilli theory (see monographs [171, 6, 337].

Santilli's central idea is the generalization of the fundamental unit of the theory from its trivial n-dimensional form I = diag(1, 1, ...) to an n-dimensional matrix $\hat{I}$ with the general dependence of all essential variables

$$I = \mathrm{diag}(1, 1, \ldots) \Rightarrow \hat{I} = \hat{I}(s, x, \dot{x}, \ddot{x}, \psi, \partial\psi, \partial\partial\psi, \mu, \tau, \ldots) \tag{1.30}$$

under the condition of preserving the original axioms of the unit (nondegeneracy, hermiticity, and positive-definiteness). The "lifting" $I \Rightarrow \hat{I}$ requires, naturally, for necessary compatibility, a generalization of the conventional associative multiplication xy into the so-called *isomultiplication*

$$xy \Rightarrow x * y := xTy\,, \; T = \text{fixed}\,, \tag{1.31}$$

where the quantity T is called the *isotopic element.* Then $\hat{I} = T^{-1}$ is a correct left and right unit element of the theory with respect the new multiplication $*$ and it is called the *isounit.* Proceeding from this idea the entire mathematical structure of physical theories may be reconstructed at all levels, including number fields, vector spaces, transformation theory, functional analysis, algebras, groups, representations, etc.

The name *isotopic* was suggested by Santilli [309] from the Greek $\iota\sigma o\sigma$ $\tau o\pi o\sigma$ (preserving the place or configuration), meaning in this context *axiom-preserving.* In fact, under the assumed topological conditions on T (and therefore on $\hat{I} = T^{-1}$) all novel structures coincide with their original form at an abstract, realization-free level. The advancement of Santilli's theory can be illustrated by the unification of the special and general relativities. In fact all their geometric distinctions cease to exist when the conventional Riemannian spaces are reinterpreted as *isominkowskian spaces* where the isounit is included in the metric tensor.

The notion of Lie-isotopy has appeared to be very fruitful, allowing the reformulation of almost everything with great unifying and heuristic power, it has also lead to some very interesting accompanying concepts, such as *isoduality,* etc. The literature on isotopies, including the isotopic lifting of all various aspects of Lie's theory, of classical mechanics and of quantum mechanics, has by now exceeded ten thousand pages of published research. It is understandable then, that here we have restricted ourselves to only a very brief outline, to the presentation of very basic ideas and very few references on basic reviews and monographs. For the first introduction to the subject we should recommend the short reviev by Kadeisvili [173], a very brief review is also given in Sec. 1.11.

The ideas of Lie-admissibility and Lie-isotopy have also initiated wide and interesting mathematical studies of Lie-admissible algebras, isotopies and related topics by S. Okubo, H.-C. Myung, G. Benkart, J.M. Osborn, M.L. Tomber, A.K. Aringazin, A. Jannussis, D.F. Lopez, M. Nishioka, B. Veljanoski et al. Here we refer to only some reviews and books [15, 355, 16, 233, 234, 171, 337].

In connection with octonion approach a cycle of papers by Susumu Okubo should be specially mentioned: • Generalized (Lie-admissible) composition algebras (including Lie-admissible *pseudooctonions*, etc.) [252, 250, 251, 253, 263, 255, 256, 258] [259] • Flexible Lie-Jordan-admissible algebras: [257] • Construction of nonassociative algebras from representations of Lie algebras: [260, 261] • Nonassociative differential geometry: [262] • Flexible Lie-admissible QM: [254, 256].

1.9.3 The quasigroup approach

This approach is essentially based on new nonassociative algebraic methods in differential geometry where the local properties of some global continuous structures (different from Lie groups) as *quasigroups, loops*, etc., have been studied. Insofar as we restrict ourselves to the Lie groups, we have the apparatus lying at the very basis of *geometry* (manifested by the Erlangen program, [177]) and the classical method of *symmetry and invariance* (see e.g. Wigner [378, 377]). However, the recent development has demonstrated that also various nonassociative systems as quasigroups, loops, odules, etc. play important roles in geometry and also in physical applications.

In physics the main stimulation of the quasigroup approach is provided by modern gauge theories, quantum gravity and by some attempts of the extension or generalization of the classical method of symmetry and invariance. Here we shall give some brief comments about these directions.

• Nowadays gauge theories based on continuous (Lie) groups have come an essential part of modern theoretical physics, providing a unified treatment of fundamental forces of Nature through the localization (gauging) of the global group symmetry. Recently this approach has been generalized for quasigroups by Batalin, Vilkovisky [11, 12], Nesterov [242, 243], and Waldron, Joshi [371].

• Recently a purely algebraic formulation of differential geometry, the *nonlinear geometric algebra*, has been elaborated by L. Sabinin [292, 293, 294]

Methods of nonassociative algebra in physics
according to Nesterov [242]

Quasigroups and quasigroups of transformations	$\Longleftrightarrow$	Gauge theory of defects, geometrical and topological description of nonideal structures
Quasigroup analysis of differential equations	$\Longleftrightarrow$	Amorphous materials and alloys, metallic alloys nonordered magnetics, liquid crystals, nonlinear
Generalized functional self-modelity – renormalization quasigroup, "soft" scaling, bifurcation of invariant charge, Berry phase, etc.	$\Longleftrightarrow$	(gauge) theory, theory of elasticity, materials in strong external fields, plastical deformation of nonhomogeneous materials
Structure theory of smooth quasigroups and quasigroups of transfromations; representations of quasigroups almost periodic functions	$\Longleftrightarrow$	Generalized (soft) symmetries: nonideal structures, life forms
Quasigroup fibrations – generalized gauge theories	$\Longleftrightarrow$	Conservation laws in GR, quasicrystallic structures, phase transitions in nonhomogeneous and nonordered systems, dynamical systems
Cocycles on groups and the problem of topological anomalies	$\Longleftrightarrow$	Irreversibility ?

and M. Akivis [1]. In this approach nonassociativity appears as the algebraic equivalent of the geometrical notion of *curvature*. This geometric algebra provides a new algebraic approach to the theory of gravity, where the space-time is considered as an algebraic system with a (geodesic) multiplication of points in a certain way. The curvature of space-time is then expressed by the nonassociativity of this multiplication. In this line one more attack to one of the most intriguing unsolved problems of contemporary theoretical physics – the *quantization of gravity*, has been undertaken in a series of papers by Piret Kuusk, Eugen Paal and Jüri Õrd [194, 195, 267, 196, 193].

• The *Moufang-Mal'tsev (MM-) symmetry*, a generalization of the classical group-theoretical symmetry treatment through the enlargement (extension) of the basic continuous system of symmetry transformations from Lie groups to Moufang loops (from Lie algebras to Mal'tsev algebras infinitesimally), will be discussed in Ch. 5 (see also Refs. therein).

1.10 Octonion approach (a more detailed discussion)

1.10.1 Jordan formulation of QM, the algebra M_3^8, and projective geometry

In 1932-34 P. Jordan [159, 161, 160] proposed a version of QM (now called *Jordan formulation* of QM or simply Jordan QM) in terms of commutative but nonassociative operator algebra, derivable from an associative noncommutative algebra through the modification of the product

$$\hat{A}\hat{B} \to \hat{A} * \hat{B} := \frac{1}{2}(\hat{A}\hat{B} + \hat{B}\hat{A}), \tag{1.32}$$

which was called *quasimultiplication* by Jordan himself (now the term *Jordan product* is widely used).

In the ordinary QM the product of two noncommuting Hermitian operators is no longer Hermitian. Jordan product preserves hermiticity.

To the *state vector* $|\alpha\rangle$ (or more precisely, to the *ray* $\lambda|\alpha\rangle$, $\lambda\bar{\lambda} = 1$, $\lambda \in \mathbf{C}$) of a Hilbert space in the Jordan formulation, corresponds a *projection operator* P_α, to an orthonormal system $\{|\alpha_i\rangle\}$, $\langle\alpha_i|\alpha_j\rangle = \delta_{ij}$ $(i, j = 1, 2, \ldots)$, the system of projection operators $\{P_{\alpha_i}\}$ satisfying $\mathrm{Tr}(P_{\alpha_i} * P_{\alpha_j}) = \delta_{ij}$. To the transition probability $|\langle\alpha|\beta\rangle|^2$, there corresponds the expression $\mathrm{Tr}(P_\alpha * P_\beta)$. The operators of observables are Hermitian. The mean value of the observable A in the state $|\alpha\rangle$ in Jordan's formulation is given by the expression $\mathrm{Tr}(P_\alpha * \hat{A})$.

The introduction of quasimultiplication (1.32) gives nothing new for the common QM, therefore the problem was posed by Jordan in a more general form. The algebraic properties of the quasimultiplication are manifested by the following equations (we omit hats):

$$A * B = B * A\,, \tag{1.33}$$

$$(A^2 * B) * A = A^2 * (B * A)\,, \tag{1.34}$$

also, all usual axioms of vector space and the distributive law are satisfied. Then the following problem may be stated: is every algebra satisfying conditions (1.33), (1.34) representable by matrices through formula (1.32)? If it is *always* possible, Jordan QM is completely equivalent with the common one. In [165] it has been proved that every algebra with multiplication satisfying (1.33), (1.34) may be obtained by redefinition (1.32) from some matrix algebra, except for an exceptional case of the algebra M_3^8 of Hermitian 3x3-matrices with octonion elements, the *exceptional Jordan algebra* (see also [4], after this work M_3^8 is sometimes called also the *Albert algebra*).

The algebras introduced in [159, 161, 160, 165] satisfying conditions (1.33), (1.34) are now called *commutative Jordan algebras* (CJA). If such an algebra can be obtained from some associative (matrix) algebra by means of redefinition (1.32) of multiplication, it is called *special* CJA. By now the mathematical theory of CJA is well developed with many interesting connections with other branches of mathematics, such as projective geometry, exceptional Lie algebras, etc. (for more information see review [222]). In physics there was a considerable rise of interest in Jordan's QM after the first papers by Günaydin and Gürsey about octonionic Hilbert space (beginning with [123]).

In Jordan's formulation it is very convenient to observe some interesting and important connections with projective geometry and lattice theory lying on the very foundations of QM.

In 1936 a classical paper [25] by Birkhoff and von Neumann appeared, where the logical foundations of QM were investigated and the first axiomatics of QM proposed.

Birkhoff and von Neumann developed a *propositional calculus* of QM which is "*...formally indistinguishable from the calculus of linear subspaces* *with respect to* set products, linear sums *and* orthogonal complements – *and resembles the usual calculus of propositions with respect to* and, or, *and* not". This approach was further elaborated by Jauch [157], Varadarayan [366], and Piron [278] (see also [279]), and its relation to the Jordan algebras was discussed by Emch [68]. [For the axiomatics and algebra of quantum mechanics we refer to the papers by Gleason [106] and Gunson [128]].

An investigation of formal properties of QM propositions leads to a special kind of partially ordered sets, a *lattice* [25, 24], the elements of which are propositions. More precisely, the set of QMcal propositions forms an *orthocomplemented atomic complete weak modular lattice* [157, 366, 278]. In the Hilbert space language we can say that all the closed subspaces of a Hilbert space form a lattice of the type mentioned above.

It has been realised for a long time that the essential structural properties of a projective geometry may also be described in terms of intersections and unions of the fundamental geometrical elements (points, lines, planes), in fact a projective geometry is an *atomic modular lattice* [25, 157]. The connection between QM and projective geometry seems to have been established by now, but the *modularity* property of projective geometries is physically untenable [157]. The true interplay of these concepts was clarified by further investigations [157, 366, 278], from which it appears that the orthocomplementarity and modularity imply weak modularity, a property of quantum-mechanical proposition systems. If a projective geometry is orthocomplementable, it has the same lattice-theoretical structure as QM proposition systems.

For a given quantum-mechanical proposition system, the problem of its equivalence to some projective geometry has been resumed as follows [157]: *"Every proposition system is a unique direct union of irreducible proposition systems. Every irreducible proposition system is imbedded in a canonical way into a projective geometry".*

Hence the propositional calculus of QM has the same structure as an abstract projective geometry. This conclusion has been obtained purely by analyzing internal properties of the propositional calculus, whereas a Hilbert space is involved here only indirectly.

There are three different formulations of QM: the Hilbert space (Dirac) formulation, the Jordan formulation in terms of anticommutators, and the propositional calculus approach of Birkhoff and von Neumann and their later followers (the Geneva school).

We just sketched the relations of the first and the third formulation to projective geometries. The Jordan algebraic approach exhibits also expressive projective properties. In the *Hilbert space formulation* the state of a physical system is represented by a ray in a Hilbert space. In the Jordan approach the rays $\lambda|\alpha\rangle$ may be interpreted as the points P_α in the projective space (one dimension lower) of projection operators, $P_\alpha = |\alpha\rangle\langle\alpha|$, $P_\alpha^2 = P_\alpha$. A superposition of two orthogonal states $|\alpha\rangle$ and $|\beta\rangle$ is represented by a point on the line joining the points P_α and P_β. The superposition principle means that if $|\alpha\rangle$ and $|\beta\rangle$ represent physical states, then all points on the line through P_α and P_β correspond to physical states. Projection operators have only eigenvalues 1 and 0, which is related to yes-no experiments of the third formulation.

In the projective geometry there are some general results about the *coordinatization* of projective spaces and *configuration theorems* satisfied in these spaces (see, e.g., [137]). Projective spaces of the dimension $n > 2$ can be realized in vector spaces over (skew) fields. In these geometries the Desarques' theorem follows directly from the axioms of projective geometry, these are the cases corresponding to special Jordan algebras.

In the case of the projective plane ($n = 2$) the situation is more complicated. Here non-Desarquian geometries over alternative skew-fields are posssible, the first example constructed by Moufang [231] as a geometry on the plane coordinatized by octonions. Jordan [162] constructed this plane in terms of M_3^8-algebra. In the latter the points are represented by idempotents of this algebra, or the projection operators in terms of QM. Günaydin, Piron, and Ruegg [126] showed that this projective geometry can be orthocomplemented and hence interpreted as QM. Since 1951 projective planes over octonions have been studied in [91, 289, 75, 363, 60].

After the QM of quarks in the octonion Hilbert space encountered some definite difficulties, some investigators (Biedenharn, Truini et. al.) returned to the exceptional Jordan algebra formulating QM on the complex octonion plane.

The octonion QM [162] in M_3^8-formalism corresponds to the non-Desarquian projective Moufang plane [231, 232], coordinatized by octonions (it is the factor space $F_4/SO(9)$). In this geometry points are idempotents and straight lines, 2-dimensional projection operators determined by two points.

In connection with the Günaydin-Gürsey theory this exceptional QM has been revised and reformulated [126, 291] and its accordance with the postulates of the axiomatics of QM has been established.

The use of the algebra M_3^8 as a charge space demands its *complexification*, as done in [131]. A more thorough justification of the necessity of complexification has been demonstrated by Truini and Biedenharn [356], they also developed a QM on the complex octonion plane [357]. In this connection some new mathematical concepts have been explored, which have never occurred before in a physical context, such as *quadratic Jordan algebras* [221], *inner ideals* [222], *structural group* [154, 178], and *Jordan pairs* [214].

1.10.2 Groups of particle symmetries related to octonions

Some very moderate and episodic interest to octonions arose just in the beginning of the *particle classification* boom (see Sec. 1.5.2), when various *octonion formulations* of internal symmetries were discussed by Souriau [335], Souriau and Kastler [336], Pais [269, 270], Tiomno [349], Gamba [95, 96] (in the Albert algebra context), Penney [276] and Vollendorf [370]. In connection with the particle equations see also Bourret [32], Penney [275], and Buoncristiani [37].

As the approximate $SU(3)$-symmetry for hadrons was soon established, the obvious sequences of the inclusions $SU(3)/Z_3 \subset SO(8)$ and $SU(3) \subset G_2 \subset SO(7) \subset SO(8)$ intuitively lead to a conclusion about the intimate connection between $SU(3)$-symmetry and octonions. The interest in octonions even strengthened when the papers dealing with the $SO(8)$-symmetry were concerned (see numerous Refs. in [330]).

An essential advancement took place after the formation of two important fundamental principles, the *quark confinement*, and the *unification of interactions.*

Confinement of quarks expose an essentially new physical feature of the subhadronic world. In the mathematical language of algebras the *nonobservability* can be interpreted as a manifestation of *nonassociativity.* According to the *proposition calculus* of QM [25] (see also Sec. 1.10.1) observable states are described as those in Hilbert spaces over associative composition algebras **R**, **C**, and **H**. This approach has been developed in the papers by Gürsey and Günaydin [129, 130, 131], where octonion formalism is elaborated for describing the nonobservability of colored quarks with an exact *color symmetry* group $SU_c(3)$), and for observable "white" (non-colored) states as $SU_c(3)$-

singlets.

Gürsey [130] was the first to use exceptional groups of the types F and E for the "superunifying" groups. It was quite natural from the mathematical point of view, because all these groups are intimately connected with octonions (most obviously through the exceptional Jordan (Albert) algebra M_3^8). Physically it means an essential extension (enlargement) of the charge space, quite desirable for the inclusion of new fundamental particles, *quarks* and *leptons* on an equal (symmetrical) footing. Nowadays such groups are regarded as GUT groups with definite chains of maximal subgroups of symmetries conserved in *spontaneous symmetry breaking.* These groups include the *flavor*- and *color*-type degrees of freedom:

$$G \supset G^{\text{flavor}} \times G^{\text{color}} \supset [SU(2) \times U(1)]_{\text{GSW}} \times SU_c(3)\,,$$

the last group in this chain being the group of the Standard Model (see Sec.1.5.5). Exceptional groups appear also in supergravity and Kaluza-Klein-type theories, but in a very drastic manner in the *superstring theory* (the group E_8).

There is an interesting set of groups of the *Freudenthal-Tits magic square* [350, 90, 89, 91, 92] consisting of automorphism groups of exceptional Jordan algebras $M_3(\mathcal{H}(i,j))$ over *Rosenfeld algebras* [289] $\mathcal{H}(i,j) = \mathbf{H}_i \otimes \mathbf{H}_j$, where $i,j = 1,2,4,8$, and $\mathbf{H}_1 \simeq \mathbf{R}$, $\mathbf{H}_2 \simeq \mathbf{C}$, $\mathbf{H}_4 \simeq \mathbf{H}$, $\mathbf{H}_8 \simeq \mathbf{O}$. The Albert algebra in these notations is $M_3^8 \equiv M_3(\mathcal{H}(1,8))$.

Table 2. **Tits-Freudenthal magic square**

$i\backslash j$	1	2	4	8
1	$SU(2)$	$SU(3)$	$Sp(6)$	F_4
2	$SU(3)$	$SU(3) \times SU(3)$	$SU(6)$	E_6
4	$Sp(6)$	$SU(6)$	$SO(12)$	E_7
8	F_4	E_6	E_7	E_8

The magic square includes, besides the exceptional groups G_2, F_4, E_6, E_7, and E_8, also the groups $SU(2)$, $SU(3)$, $SU(3) \times SU(3)$, $SU(6)$, $Sp(6)$, and $SO(12)$. All these groups have been involved in various particle symmetry schemes. On the other hand, the most popular GUT groups, $SU(5)$ and $SO(10)$, may be regarded as exceptional because the schemes of simple roots (for their Lie algebras A_4, D_5) resemble those of groups of type E (and may be denoted as E_4, E_5, respectively).

In conclusion it must be emphasized that almost all the groups which have participated in particle symmetry schemes are somehow connected with octonions.

BIBLIOGRAPHY: • Octonions in particle systematics (and some related papers): [335, 336, 269, 270, 349, 273, 95, 96, 276] [370, 228, 230, 229, 323, 324, 166, 53, 54, 55, 56] [57, 58, 114, 334]. • Octonions and relativistically-invariant wave equations: [32, 275, 276, 36, 37, 181, 64, 65, 66, 264, 277, 26] [27, 28, 27, 146, 147, 198, 380, 208, 332, 334, 247, 346] [280, 151, 152]. • Octonions and Yang-Mills equations: [36, 235, 248, 117, 327, 280, 371, 151]. • Octonionic field theory: [185, 186, 187, 188, 189, 190] • Division algebras (mathematics): [281, 48, 263, 18, 17] [343, 43, 58]. • Division algebras (physical applications): [245, 184, 343, 43, 132, 14, 73, 249, 85, 86, 359, 360] [361, 56, 57, 219, 58].

1.10.3 Quantum theory of quarks in the octonionic Hilbert space

The basic mathematical concept of this approach, the *Hilbert space with nonassociative (octonionic) scalars* was introduced and investigated by Goldstine, Horwitz, and Biedenharn [111, 110, 142] about 10 years before the papers of Günaydin and Gürsey [123, 130, 122, 124] (see also reviews [330, 207], where the basic formulations of this approach were published). These papers marked also the beginning of the rebirth of Jordan QM and octonion approach as a whole.

According to the theory of the observable states developed by Birkhoff and von Neumann [25], we can have observable states only in Hilbert spaces over associative composition algebras (see also [157]). The orthodox QM explores the Hilbert space over complex numbers. The quaternion formalism of QM (with quaternionic Hilbert space) has been developed by a number of investigators [77, 78, 80, 79, 67]. The general hope was to introduce additional internal degrees of freedom (isospin) by such an enlargening of the quantum-mechanical Hilbert space, but finally it was proved [158] that quaternionic representations of the Poincaré group did not generate new physical states.

So the abandonment of associativity here means the appearance of nonobservable states. It is quite an agreeable feature if the latest ideas and convictions about *nonobservability* of quarks (*confinement* or imprisonment) are to be taken seriously.

Also, in the algebraic formulation of quantum mechanics in terms of Jordan algebras, octonion-valued observables may be treated only in a nonobservable Hilbert space (except the case of three degrees of freedom). A field theory may be developed where octonion field equations would imply dynamical relations in the observable Hilbert subspace.

To begin with Gürsey-Günaidin theory, in [123] the split octonion algebra is studied by means of a formalism exhibiting an explicit quark structure. The groups $SO(8)$, $SO(7)$, and G_2 are represented by octonions as well as by 8x8-matrices, and the reduction is made through the subgroups $SU(3)$ and $SU(2) \times SU(2)$ of the group G_2, the group of automorphisms of octonions.

In [122], a 1-particle unitary representation of the Poincaré group is constructed in a split octonionic Hilbert space of state vectors with octonionic components but complex-valued scalar products, an internal $SU(3)$ group arising as the automorphism group of this representation.

In [124, 129, 118], an octonion formalism was developed for the description of nonobservable colored quarks with exact color symmetry $SU_c(3)$ and observable noncolored ("white") hadronic states. In [124, 129], this formalism is based on split octonions. In [118], it was demonstrated that the same results are obtainable also in the framework of the corresponding division algebra. Here we can also find the first remark about Mal'tsev algebra as a *color algebra* with the group of automorphisms $SU_c(3)$.

The essence of the Gürsey-Günaydin formalism lies in the use of a special basis where octonion field operators split into *longitudinal* and *transversal* components. All algebraic formulation of confinement are reduced to the principle of the observability of longitudinal ($SU_c(3)$-singlet) states and the nonobservability of the transversal (color triplet) states. A more detailed analysis [278] (see also [207]) demonstrates that this principle is not fully realisable.

Octonion QM of quarks was analyzed and criticized in papers [278, 182] by Kosinski and Rembielinski. An inconsistency in the separation of observable states was pointed out, also it was asserted that the color confinement cannot have an algebraic origin in the framework used.

We should finally mention that *nonassociative* Lie-admissible algebras play a fundamental role in Santilli's [313] model of *nonobservability* of quarks via the isotopies of $SU(3)$-group (see Sec. 1.11.5).

BIBLIOGRAPHY about octonionic Hilbert space and exceptional QM in connection with the Gürsey-Günaydin theory: [111, 110, 122, 123, 124, 129, 130, 131, 118, 119, 126, 291, 182, 285, 382] [20, 21, 199].

1.10.4 Octonion formalism in the theory of superstrings

Papers about the octonion formalism in the superstring theory may be divided into two main groups: 1) those elaborating and investigating the 10-dimensional formalism of the superstring theory in terms of octonions, [184, 343, 319, 49, 83, 74, 42, 82, 132, 73, 84, 14, 249, 85, 86, 148, 344, 345, 192] and 2) papers on vertex operators and M_3^8-algebra [108, 109, 45, 125, 121, 76] (see also [120, 88].

In the superstring theory, which may be correctly formulated only in the framework of 10-dimensional space-time (see e.g. [116]), the octonion formalism arises in a very natural manner as a particular case of a more general construction involving four division algebras.

If the simple supersymmetric Yang–Mills theories are considered, where every gauge particle has only one superpartner, then all possible dimensions may be determined as those for which the numbers of vector and spinor degrees of freedom are equal. A more detailed analysis [184, 343] shows that the only possible space-time dimensions are $d = 3, 4, 6, 10$, and the number of physical degrees of freedom for the corresponding spinors of Maiorana, Maiorana-Dirac, Weyl, and Maiorana-Weyl type equals 1,2,4,8, respectively.

In the spaces of these dimensions the properties of spinors follow from local isomorphisms, allowing one to treat the corresponding Lorentz and conformal groups in terms of matrices over the corresponding division algebras **R, C, H, O**, respectively:

$$\begin{array}{llcll}
d=3 & SO(2,1) & \simeq & SL(2,\mathbf{R})\,, & \mathbf{R} \\
 & SO(3,2) & \simeq & Sp(4,\mathbf{R})\,; & \\
d=4 & SO(3,1) & \simeq & SL(2,\mathbf{C})\,, & \mathbf{C} \\
 & SO(4,2) & \simeq & SU(2,2\mathbf{C})\,; &
\end{array}$$

$$
\begin{array}{llcll}
d=6 & SO(5,1) & \simeq & SL(2,\mathbf{H})\,, & \mathbf{H} \\
 & SO(6,2) & \simeq & SO^*(8)\simeq SO(4,\mathbf{H})\,; & \\
d=10 & SO(9,1) & \simeq & SL(2,\mathbf{O})\,, & \mathbf{O}\,, \\
 & SO(10,2) & \simeq & Sp(4,\mathbf{O})\,. &
\end{array}
$$

To this chain of dimensions, i.e. to the division algebras **R, C, H, O** corresponds also a quadruple of Jordan algebras M_3^i $(i = 1,2,4,8)$ over **R, C, H, O**, respectively, which are connected with the construction of *vertex operators* [108, 109, 45, 125, 121, 76].

In [108], a vertex operator is constructed for a certain type of representations of affine Kac-Moody algebra by using free boson and interacting fermion fields. This construction involves as particular cases all four division algebras and the *Freudenthal-Tits magic square.* In [45], a connection is demonstrated between the exceptional Jordan algebra M_3^8 and the vertex operators built on three nonequivalent 8-dimensional irreps of the group $SO(8)$.

In [125], the algebra M_3^8 is constructed by means of the Fubini–Virasoro vertex operator [94], the latter being an important algebraic tool in the superstring theory.

In [121], a general method for the "vertex" construction of nonassociative algebras, ternary systems and their superanalogs is presented.

1.11 Lie-admissible and Lie-isotopic approach (main aspects)

1.11.1 Isonumbers and genonumbers

Further generalizations of the concept of number were introduced by the theoretical physicist Ruggero Maria Santilli [309, 314] for specific physical needs under the names of *isonumbers* and *genonumbers*, jointly with a new antiautomorphic conjugation, *isoduality*, yielding additional new numbers called *isodual isonumbers* and *isodual genonumbers.* We here give brief outline of these new number systems.

In essence, all preceding number systems, including octonions, are defined with respect to the simplest possible unit 1 related to the simplest possible multiplication, say, $a \times b$ under which 1 is the correct right and left unit, $1 \times a = a \times 1 \equiv a$, $\forall a \in S$ (S denoting the number system under consideration). Santilli generalized the unit 1 into the most general possible integro-differential quantity $\hat{1}$ (1.30) depending on time t, coordinates r, momenta p, accelerations $\dot{p}$, wavefunctions Ψ and their derivatives $\partial\Psi$, density μ of the medium in which motion occurs, its temperature τ, etc., so $\hat{1} = \hat{1}(t, r, p, \dot{p}, \psi, \partial\psi, \mu, \tau, \ldots)$. Santilli classified the possible *liftings* into two primary classes, those with Hermitean $\hat{1}$, $\hat{1} = \hat{1}^\dagger$, called *isonumbers*, and those with nonhermitean $\hat{1}$, $\hat{1} \neq \hat{1}^\dagger$, called *genonumbers* [314].

The lifting of the unit

$$1 \rightarrow \hat{1}\,, \quad \hat{1} = \hat{1}^\dagger\,, \tag{1.35}$$

(i.e. when $\hat{1}$ is Hermitean!) was called an *isotopy* [309] from the Greek meaning of the word expressing the capability of $\hat{1}$ to preserve all axioms of 1. In this case Santilli introduced a corresponding generalization of the conventional multiplication $a \times b$ into a form $a \hat{\times} b$, which he called *isomultiplication*, such that $\hat{1}$ is the correct left and right unit of the theory. The simplest possible solution is given by

$$a \times b \rightarrow a \hat{\times} b := a \times T \times b\,, \quad \hat{1} = T^{-1}\,, \tag{1.36}$$

in which case (only) $\hat{1}$ is called the *isounit* because it preserves the original left and right unit axioms: $\hat{1} \hat{\times} a = a \hat{\times} \hat{1} \equiv a$, $\forall a \in S$, and T is called the *isotopic element.* Note that the lifting of the multiplication $a \times b \rightarrow a \hat{\times} b = a \times T \times b$, T fixed and invertible, preserves all original axioms of $a \times b$ (associativity, commutativity, alternativity, etc.).

Consider now the natural numbers $a, b, \ldots$, etc., with trivial unit 1. Since $\hat{1}$ is outside the original set by conception, closure of the image of the set under lifting $1 \rightarrow \hat{1}$ with respect to the right and left distributive laws reguired Santilli (see [314] for details) to generalize the very concept of number into the *isonumbers* which are given by $\hat{a} := a \times \hat{1}$. This implies that conventional numbers, such as the integer number 2, is lifted into the new quantity $\hat{2}$ with an *integral* structure.

The generalization of the structure of the unit permitted (Santilli [314]) also to introduce a new antiautomorphic *isoduality* conjugation $\hat{1} \rightarrow \hat{1}^d := -\hat{1}$ with consequential *isodual isomultiplication* $a \hat{\times}^d b = a \times T^d \times b = -a \times T \times b$, $T^d = -T$, under which $\hat{1}^d$ is again the correct right and left unit of the theory. Closure then require the lifting of the number a into the *isodual isonumbers* $\hat{a}^d := a \times \hat{1}^d = -a \times \hat{1}$. The most salient difference between isonumber and its isoduals is that, while the former admit a positive-definite *isonorm* $|\hat{a}|! = |a|\hat{1} > 0$, where $|a|$ is the conventional norm, isodual isonumbers can only admit a *negative-definite isonorm* $\uparrow \hat{a}^d \uparrow = |a|\hat{1}^d < 0$ with profound physical implications reviewed later on.

It permitted Santilli [314] to identify *two* classes of structurally generalized numbers, the *isoreals, isocomplexes, isoquaternions* and *isooctonions* and their isoduals. In correspondence Santilli introduced [314] *two* classes of generalized fields, the *isofields* and their isoduals.

The covering case of nonhermitean $\hat{1}$ is algebraically even more intriguing. It lead to the discovery of a "hidden ordering" in the ordinary multiplication of numbers *which emerges only under the nonhermitean generalization of the unit*. Consider two integer numbers, e.g. 2 and 3. It is evident that the multiplication of 2 by 3 to the right, denoted by Santilli $2 > 3$ is identical to the multiplication of 3 time 2 to the left, indicated $2 < 3$, i.e. *under the condition that the trivial value 1 is unit* $2 > 3 \equiv 2 < 3$. [We use here the terminology and notations used by R.M. Santilli, which is somewhat different from ours.] Suppose now that the unit is generalized into a *Hermitean* form $\hat{1}$. Then it is easy to see that the isomultiplication to the right and to the left again coincide, $a > b = a \times T \times b \equiv a < b = a \times T \times b$, $T = T^\dagger$. Suppose now that the generalized unit $\hat{1}$ is no longer Hermitean. Now it is evident that the multiplication to the right is no longer necessarily equal to that of the left. In fact, one can introduce the generalized unit for multiplication to the right $\hat{1}^> = (T^>)^{-1}$ and its conjugate ${}^<\hat{1} = ({}^<T)^{-1} = (\hat{1}^>)^\dagger$ for multiplication to the left. In this case $2 > 3 = 2 \times T^> \times 3 \neq 2 < 3 = 2 \times {}^<T \times 3$. Note that *each* of the new multiplications $a > b$ and $a < b$ preserves all original axioms of $a \times b$ (associativity, commutativity, alternativity, etc.).

When the generalized unit is no longer Hermitean, Santilli called the liftings of the unit and multiplication

$$1 \rightarrow \{{}^<\hat{1}, \hat{1}^>\} = \{{}^<\hat{1} = ({}^<T)^{-1},\ \hat{1}^> = (T^>)^{-1}\}, \tag{1.37}$$

$$a \times b \rightarrow \{a < b = a \times {}^<T \times b,\ a > b = a \times T^> b\} \tag{1.38}$$

a *genotopy* [309] from the Greek meaning expressing the fact that the new units *do not* preserve the original axioms of 1, but induce *covering axioms.* Closure under distributive laws per each ordering of the product then requested the lifting of numbers into two classes of *genonumbers* $\hat{a}^> := a \times \hat{1}^>$

and ${}^{<}\hat{a} := {}^{<}\hat{1} \times a$. Genonumbers themselves can be subjected to isoduality, yielding the isodual genonumbers $\hat{a}^{>d} := -a \times \hat{1}^{>}$ and ${}^{<}\hat{a}^{d} := -{}^{<}\hat{1} \times a$. This permitted to introduce [6] the following four additional generalizations – *genoreals, genocomplexes, genoquaternions* and *genooctonions* (with corresponding *genofields*, etc.).

The Georgian mathematician J.V.Kadeisvili [172] classified isounits into five topologically different classes defined with respect to the Hermitean part of the isounit: **Classes I** (for the Hermitean part of $\hat{1}$ positive-definite, yielding *isonumbers* and *genonumbers*), **II** (for the Hermitean part of $\hat{1}$ negative-definite yielding the *isodual isonumbers* and *isodual genonumbers*), **III** (the union of **I** and **II**), **IV** ($\hat{1}$ singular) and **V** ($\hat{1}$ general, e.g. discontinuous or discrete). This introduces additional possible number concepts, such as *numbers with a singular or discrete unit*, etc.

Numbers being the fundamental concept of mathematics and also of physics, these novel generalizations discussed above give also interesting and far reaching possibilities developing essentially new mathematical and physical theories. The totality of contemporary mathematics with all its fundamental concepts including vector spaces, algebras, groups, functional analysis, etc., is based on *conventional* number fields. The lifting of these fields into isofields and their isoduals leads to a natural generalization of conventional mathematics under the name of *isotopic formulations* [309], while the further generalizations based on genofields and their isoduals are developed under the term of *genotopic formulations* [313]. We here merely mention that *isonumbers* are at the foundations of the Lie-isotopic generalization of Lie's theory [6, 171, 310, 311, 337], while genofields are at the foundation of the yet more general Lie-admissible generalization of the Lie-isotopic theory [311, 313]. For instance, angles, Euclidean spaces, rotational symmetry, Fourier transform, etc., must be suitably reformulated under the lifting, see e.g. [313] for the *functional isoanalysis.*

There are interesting and far reaching physical applications of the theory discussed above, which will be briefly discussed in subsections 1.11.4, 1.11.5.

1.11.2 Isolinear and genolinear algebras

The isotopic generalization of the unit and of the multiplication discussed in the previous subsection and expressed by (1.35), (1.36) is axiom-preserving by conception, thus implying the condition of *isolinearity*

$$A\hat{\times}(a\hat{\times}x + b\hat{\times}y) = a\hat{\times}(A\hat{\times}x) + b\hat{\times}(A\hat{\times}y\ , \tag{1.39}$$

which (for Kadeisvili Class I) coincides with the conventional linearity at the abstract, realization-free level.

The genotopic liftings of the unit (1.37) and corresponding genotopies of the multiplication (1.38) imply the corresponding *genolinearity to the right* [313]

$$A > (a > x + b > y) = a > (A > x) + b > (A > y) \tag{1.40}$$

and the *genolinearity to the left*

$$A < (a < x + b < y) = a < (A < x) + b < (A < y)\,. \tag{1.41}$$

Iso-and genotopic liftings have important implications. As shown by Santilli [314, 307, 313], *any nonlinear theory, e.g., of the type $A(x,\ldots) \times x$ can always be reformulated into an identical isolinear form $A(x,\ldots) \times x \equiv \hat{A}\hat{\times}x$,* $\hat{A}T = A$, where $\hat{A}$ is independent of x. As a result, *nonlinearity is not an axiomatic property because it can be made to disappear at the abstract, realization-free level.*

A similar occurrence holds for nonlocality which can be embedded into the notion of *isolocality*, that is, with all integral terms embedded in the isotopic element T. This allowed Santilli to introduce a new *isolinear* and *isolocal topology* which is everywhere linear and local-differential *except at the isounit* (see also the topological studies by Sourlas and Tsagas [337]).

For the better understanding of further constructions let us recall the appearance of Lie algebras g from their universal associative enveloping algebras $\xi(g)$ and associated group G. Consider an one-dimensional Lie group of unitary transformations of an operator A on a Hilbert space with conventional inner product $\langle , \rangle$ over the field of complex numbers $\mathbf{C}$ with (Hermitean) generator H and parameter t:

$$A(t) = e^{iHt} \times A(0) \times e^{-iHt}\,. \tag{1.42}$$

The Lie algebra emerges in the familiar infinitesimal law for $t \approx 0$

$$i\frac{dA}{dt} = [A, H] = A \times H - H \times A\,, \tag{1.43}$$

where $[A, H]$ is the familiar Lie product.

Lie-isotopic and Lie-admissible theories consists of two sequential, step-by-step generalizations of *all* various aspects of the above structures. The first is characterized by the isotopies represented by liftings (1.35), (1.36), which imply an isotopic generalization [313] of the field $\mathbf{C} \to \hat{C}$, of the Hilbert space into the *isohilbert space* $\mathcal{H} \to \hat{\mathcal{H}}$ with a *isoinner product* $\langle \hat{,} \rangle \in \hat{C}$ (under which the original Hermitean operators remain Hermitean, and thus observable). Also the underlying carrier space and all the quantities and relations in the Poincaré-Birkhoff-Witt theorem (discussed already in the original proposal [309]) undergo to the liftings. The notion of exponentiation is lifted into the notion of *isoexponentiation*

$$\hat{e}^{iHt} = \hat{1} \times e^{itTH} \equiv e^{iHTt} \times \hat{1} \tag{1.44}$$

with the corresponding lifting of the formula (1.42), where *Lie-isotopic group* acts as

$$\begin{aligned} A(t) = \hat{e}^{iHt} \hat{\times} A(0) \hat{\times} \hat{e}^{-iHt} &= \hat{1} \times e^{itTH} \hat{\times} A(0) \hat{\times} e^{-iHTt} \times \hat{1} = \\ &= e^{iHTt} \times A(0) \times e^{itTH} , \end{aligned} \tag{1.45}$$

and the corresponding infinitesimal law

$$i \frac{dA}{dt} = [A \hat{,} H] := A \hat{\times} H - H \hat{\times} A = A \times T \times H - H \times T \times A , \tag{1.46}$$

where T is the isotopic element from (1.36) with all its "integro-differential" content, and $[A \hat{,} H]$ is a Lie-isotopic analogue of the Lie product, which may be referred to as the *Lie-Santilli product* [6, 171, 337].

One can therefore see the mathematical and physical nontriviality of the *Lie-Santilli theory*, which, mathematically, implies the lifting from the conventional group structure into the corresponding isotopic form *with a completely unrestricted integro-differential operator T in the exponent.* All conventional linear and local-differential results of Lie's theory are then structurally generalized into a covering nonlinear and integro-differential form. Yet, owing to the lack of axiomatic character of nonlinearity and nonlocality, the *Lie and Lie-Santilli theory (of Kadeisvili Class I) coincide at the abstract level by construction.*

Physically, an inspection of the Lie-Santilli time evolution (1.46) indicates the possibility of representing local-potential interactions via the Hamiltonian H and all nonlocal-nonpotential interactions via the isotopic element T.

Now let us stop for a moment on the introduction of *genotopies*. The terms $A \times H$ and $H \times A$ in (1.43) have a *well defined ordering of the multiplication*. In fact, an inspection of law (1.42) clearly establishes that the term $A \times H$ originates from the action of the enveloping associative algebra *to the left*, while the term $H \times A$ originates from the action *to the right*. Such an ordering is completely inessential under Hermitian units, as it is the case for Lie theory and Lie-Santilli theory. However, the moment the isounit is lifted into a nonhermitean form, the identification of such ordering becomes crucial for consistency.

For this reason Santilli [309, 311, 313] introduced two additional genotopic liftings of the entire formalism of the Lie-isotopic theory, including units, fields, Hilbert spaces, carrier spaces, enveloping associative algebra, etc., one for multiplication to the right and one to the left. For instance, we have the two *genoexponentiations*

$$\hat{e}^{>iHt} = e^{iH \times T^{>} \times t} \times \hat{1}^{>}\,, \quad {}^{-itH\,<}\hat{e} = {}^{<}\hat{1} \times {}^{-it \times {}^{<}T \times H}e\,. \tag{1.47}$$

This yields a Lie-admissible generalization of Lie groups:

$$A(t) = \hat{e}^{>iHt} > A(0) <^{itH<} \hat{e} = e^{itT^{>}H} \times A(0) \times e^{-iH^{<}Tt} \tag{1.48}$$

and related infinitesimal form

$$i\frac{dA}{dt} = (A, H) := A < H - H > A = A \times {}^{<}T \times H - H \times T^{>} \times A\,, \tag{1.49}$$

where the product (A, H) is manifestly non-Lie, it is *Santilli's fundamental Lie-admissible product* [309, 314, 310, 311, 307, 313], where the notion of *Lie-admissibility* is used to indicate that the attached antisymmetric product *is not Lie, but Lie-isotopic*, $(A, H) - (H, A) = [A\hat{,}H] = A \times T \times H - H \times T \times A\,,\ T = {}^{<}T + T^{>}$.

There are interesting mathematical and physical implications. The local structure of (1.48) is not a Lie algebra, but *Lie-admissible algebra*. Moreover, the genotopy implies the deformation of the product in the amount here unified for both orderings ${}^{<}T^{>}$, *while jointly the underlying unit is lifted of the inverse amount* ${}^{<}1^{>} = ({}^{<}T^{>})^{-1}$. This implies that all essential aspects of Lie's theory, such as orbits, geodesics, etc., coincide with the corresponding images in the covering Lie-admissible theory when represented in their appropriate genospaces over the appropriate genofields (see [313] for details).

By recalling the fundamental role of Lie's theory for most of mathematics, the fundamental role of Santilli's sequential generalizations of Lie-isotopic and Lie-admissible types is evident.

Finally let us consider the physical motivations of above *dual* generalizations. Recall that Lie's product $[A,H] = A \times H - H \times A$ is totally antisymmetric, and as such, it is particularly suited to represent the conservation of the total energy H, $i\dot{H} = [H,H] = H \times H - H \times H \equiv 0$.

The transition to the covering Lie-isotopic theory permits the representation of nonhamiltonian interactions via the isotopic element T. The fundamental physical point is that the Lie-Santilli product $[A\hat{,}H]$ remains fully antisymmetric. This illustrates the statement of the preceding section to the effect that the isotopic formulations represent systems with conserved total energy H while admitting nonhamiltonian internal forces. In fact, we have the law $i\dot{H} = [H\hat{,}H] = H \times T \times H - H \times T \times H \equiv 0$.

However, closed-isolated systems represent a conceptual abstraction because, strictly speaking, there exist no closed-isolated system in the physical reality (expressed, e.g., by *Mach principle* of general relativity). In order to achieve the direct universality recalled earlier, Santilli was therefore forced to study a second level of generalizations of his own Lie-isotopic theory with the fundamental brackets which, by central condition, are neither totally antisymmetric nor totally symmetric. The necessary condition of admitting the Lie-isotopic and Lie theories as particular cases led in a unique way to the Lie-admissible generalization. In fact, time evolution (1.49) represents the *time-rate-of-variation* of the energy H of the system when considering the rest of the Universe as external. We therefore have the chain of physical laws

$$\begin{aligned} &\text{Lie: } i\dot{H} = [H,H] = H \times H - H \times H \equiv 0 \longrightarrow \\ &\text{Lie-isotopic: } i\dot{H} = [H\hat{,}H] = H \times T \times H - H \times T \times H \equiv 0 \longrightarrow \\ &\text{Lie-admissible: } i\dot{H} = (H,H) = H < ({}^{<}T - T^{>}) > H \neq 0 . \end{aligned} \tag{1.50}$$

The term *Lie-Santilli theory* has been referred in the literature [6, 171, 337] to the Lie-isotopic generalization of Lie's theory, no name has been introduced for the more general Lie-admissible theory. We here propose the use of the same term *Lie-Santilli* theory with the prefix *isotopic* when the basic product is antisymmetric and *genotopic* when it is not. The term *Lie-Santilli genotopic theory* would then be suitable for the Lie-admissible generalization.

1.11.3 Isosymmetries and conservation laws. Genosymmetries and nonconservation laws

Physical systems are local-differential-potential when the test bodies can be effectively approximated as being *point-like*, thus implying local-differential topologies, algebras and geometries, as it is the case for the *exterior dynamical problem* (motion in empty space). In general, however, physical systems are nonlocal, integral, nonpotential, as it is the case for *interior dynamical systems* (motion of extended deformable bodies within physical media). It is evident that conventional symmetries, being based on the conventional Lie's theory, symplectic geometry and analytic mechanics, are *inapplicable* to the latter systems. For this reason Santilli introduced the notion of *isosymmetries* as the invariance under a *isotopic group* of the type (1.45) with the following primary features:

1) *Nonlinear, nonlocal, noncanonical transformations.* All Lie symmetries of current use in physics, such as rotations, Lorentz transformations, translations, etc., are *linear, local and canonical.* A first implication of the unrestricted integro-differential isotopic element T in the exponent of the isoexponentiation (1.44) is that of producing the most general possible nonlinear, nonlocal, noncanonical transformations. At any rate, this structure is evidently *necessary* to reach symmetries of interior dynamical systems (see below for a simple example).

2) *Conventional conservation laws.* The basis of a vector space remains invariant under isotopies (up to possibel renormalization coefficients) [313]. This property applies also to the basis of a Lie algebra. As a result, *the basis of a conventional Lie symmetry remains unchanged in the transition to the covering isosymmetry.* This is clearly illustrated in the transition from (1.42) (a one-dimensional symmetry with generator H) to (1.45) (a one-dimensional isosymmetry *with the same generator H*). The generators of a symmetry or isosymmetry are the conserved quantities. *Isosymmetries imply conventional total conservation laws.* In different terms, the *structure* of the symmetry is generalized taking into account nonlinear, nonlocal, nonpotential internal forces, but the total physical quantities must evidently remain the same precisely because nonpotential interactions have no effect on the total energy.

Here the concept of a *bound system* is involved. A typical example of such systems represented by a space-time symmetry such as the Galilean symmetry, is the *planetary system*, which is an N-body system moving in vacuum without collisions. Isosymmetries characterize different systems which can be visualized at the classical level with the stucture of Jupiter and at the operator (quantum mechanical) level with the structure of a neutron star, in which the constituents are *extended objects* (as molecules for Jupiter and neutrons for a neutron star) being in continuous internal contact interactions and collisions thus causing the nonlinear, nonlocal and nonpotential internal effects. However, visual observations show that the total physical quantities of Jupiter are fully Galilean despite internal nonpotential effects, and conserved, as it is correctly represented by isosymmetries.

The selection between of a symmetry or an isosymmetry can also be done in a simple yet suggestive way. Conventional symmetries characterize systems with the *Keplerian nucleus*, in classical or quantum mechanics, as it is the case for solar system and the atom. On the contrary, *isosymmetries characterize systems without the Keplerian nucleus*, which is replaced by the so-called *isonucleus* [307, 313] which is any particle heavier or ligther than all remaining constituents. The latter occurence is indeed verified in Nature, as it is the case for Jupiter or a neuton star where the Keplerian nucleus cannot possibly exist due to the compression of all constituents one against the other, thus resulting isonucleus, in fact arbitrary particle to be in the center.

Quantum mechanics while being exact for the atomic structure is expected to be only *approximately valid* in nuclear and hadronic structures, though the approximation is excellent. The isotopic generalization of QM under the name of *hadronic mechanics* (HM) has been proposed [313] to attempt deeper understanding of the structure of nuclei, hadrons and stars.

3) *Isotopic reconstruction of exact symmetries when believed to be broken.* Conventional symmetries are broken under nonlinear, nonlocal and nonpotential interactions, in which case the covering isosymmetries are .exact. However, Santilli has proved that all isosymmetries with a positive-definite isounit (Kadeisvili Class I) are locally isomorphic to the original symmetry. The 3rd meaning of isosymmetries is therefore the reconstruction of exact space-time and internal symmetries when believed to be broken in the framework of orthodox methods.

The first significant example is that of the *rotational isosymmetry* introduced and elaborated by Santilli [312].

The second level of generalizations, the *genosymmetries and nonconservative laws* are perhaps less known and more intriguing because its character of *direct universality*, that is applicability without the restriction that the system considered is isolated from the rest of the Universe (as is necessary to have conervation laws).

The genosymmetries are evidently given by the invariance of nonlinear, nonlocal, nonhamiltonian systems under the *Lie-Santilli genogroups*. As an example, the one-dimensional genogroup (1.48) characterizes a clear invariance of the system considered, trivially, because every system is invariant under its own time evolution. The fundamental novelty is that the genosymmetry cannot characterize conservation laws anymore because the product is no longer antisymmetry. On the contrary, *genosymmetries characterize time rate variations of physical quantities*, as illustrated by the nonconservation of energy (1.50). The covering character of the latter approach over the isosymmetries and symmetries is evidenly established by the fact that *conservation laws are simple particular case of time-rate variations.*

In [313] the general methods for the construction of the *genotopic image* of all the space-time and internal symmetries outlined above. It is merely given by relaxation of the Hermiticity of the isounit and the now familiar assumption of two ordering of the multiplication, one to the right, representing forward motion in time, and one to the left, representing backward motion in time.

As a final remarks, isosymmetries are *stucturally reversible*, that is they verify the time-reversal invariance for a time-symmetric Hamiltonian. As such, they provide a quantitative representation of the *time reversal invariance of the center-of mass trajectory of interior systems* such as Jupiter or a neutron star. We can say that *isosymmetries provide an axiomatization of reversibility under irreversible internal processes.*

On the contrary, *genosymmetries are structurally irreversible*, that is, they are irreversible irrespective of whether the Hamiltonin is reversible. *As such they provide a direct representation of irreversible internal processes*, such as Jupiter's internal vortices with monotonically nonconserved (decreasing or increasing) angular momenta. We can then say that *genosymmetries provide an axiomatization of irreversibility for open nonconservative conditions in such a way to be compatible with the reversibility of the center of mass for the sytems when closed into an isolated form* [307, 313].

1.11.4 Lie-admissible formulation of Hamiltonian mechanics

As well known (see e.g. [309, 313]) the equations originally conceived by Hamilton *are not* those of the contemporary literature, but rather the equations with *external terms*

$$\dot{r}_k = \frac{\partial H(t,r,p)}{\partial p_k}, \quad \dot{p}_k = -\frac{\partial H(t,r,p)}{\partial r_k} + F_k^{\mathrm{NSA}}(t,r,p,\dot{p},\dots), \tag{1.51}$$

where NSA stands for variational *nonselfadjointness*, i.e. the violation of the integrability conditions for the existence of a potential or a Hamiltonian in the local coordinates considered.

As pointed out by Santilli [309], (1.51) do not possess a consistent algebraic structure because their brackets violate the scalar and distributive laws (see[313] for details). For this reason, he reformulated Hamilton's original equations in the *identical* form of *Hamilton-Santilli equations* [6, 171, 337]

$$\dot{a}^\mu = S^{\circ\mu\nu}(t,a,\dot{a},\dots)\frac{\partial H(t,a)}{\partial a^\nu}, \tag{1.52}$$

where

$$a = \{a^\mu\} = \{r,p\}, \quad S^{\circ\mu\nu} = \omega^{\mu\nu} + s^{\mu\nu}, \quad (s^{\mu\nu}) = \mathrm{diag}(0, F^{\mathrm{NSA}}/(\partial H/\partial p)),$$

and $\omega^{\mu\nu}$ is the conventional canonical Lie tensor in the cotangent bundle.

The emerging new brackets

$$(C,D) = \frac{\partial C}{\partial a^\mu} S^{\circ\mu\nu}(t,a,\dot{a},\dots)\frac{\partial D}{\partial a^\nu} \tag{1.53}$$

then characterize a *general Lie-admissible algebra* because the attached antisymmetric brackets define Lie algebra.

The above occurrence is nontrivial. It was believed over a century that the Lie algebras were the only nonassociative algebras appearing in physics. Santilli pointed out that the Lie-admissibility characterize the true Hamilton equations when written in an algebraically consistent way. This eastablished the physical relevance of one of the largest known class of nonassociative algebras in the most fundamental aspect of dynamics, the brackets of the classical time evolution.

The direct universality of the Lie-admissible algebras in Newtonian mechanics is transparent from (1.53). The construction of the explicit representation is also so simple to appear trivial.

The above Lie-admissible formulation of analytic mechanics of 1978 was based on *conventional* numbers, fields, vector spaces, etc. Santilli was subsequently *forced* to reformulate it in 1993 [313] in terms of genonumbers, genofields, genospaces, etc., for numerous independent methodological reasons, such as the lack of existence of a variational principle, with consequental lack of Hamilton-Jacobi equations, lack of unambiquous quantization, etc.

We finally note that comprehensive and systematic applications of Santilli's Lie admissible formulations have been done by the Greek physicist A. Jannussis and his school at the University of Patras (Greece) [6, 155, 156].

1.11.5 The problem of hadronic structure

It seems that the most intricate and perspective applications of ideas of Lie-isotopy and Lie-admissibility lie in the realm of subnuclear particle world. An interesting novel approach to the problem of hadronic structure has been proposed in [311, 307, 313], quite different from orthodox views and models. We shall review here some essential points as expressed by the author himself, not dwelving into methodological analysis or even criticism. We only hope we are adequately presenting the essence of main ideas. And from the beginning we regret we cannot here on some pages to represent the whole rich spectrum of ideas of the wide topics which may be called *hadronic mechanics*, an interested reader must turn to the monographs cited above.

The main objective of the hadronic mechanics is a systematic study of the nonlocality of the strong interactions, and of the structure of hadrons in particular due to mutual overlapping of the wave packets of hadrons in a way preserving causality, measurement theory, and the basic features of QM.

The main working hypothesis of the hadronic mechanics is *the generalization of Planck constant into an integro-differential operator* through the isotopic lifting of the unit described in Sec. 1.11.1: $\hbar = 1 \to \hat{\hbar} = \hbar\hat{1}$.

The need for the generalization of the unit, and of the corresponding associative product, originates from the fact that the nonlocal interactions due to wave-overlappings, whether in electron pairing in superconductivity, or in deep inelastic scattering, or in other events, are of "contact" type; that is, of a type which does not admit potential energy. Conventional Hamiltonians $H = K + V$ can therefore represent the kinetic energy K and all possible action-at-a-distance interactions with potential V. However, the contact interactions due to mutual wave-penetration, by conception, cannot be represented with the Hamiltonian H and, in this sense, they are called "nonhamiltonian". In [313] these interactions are represented via the generalized unit of the theory.

The physical relevance of isotopic and genotopic methods is well-established and permits quantitative studies of the transition from the *exterior dynamical problem*, characterized by motion of point-like particles within the homogeneous and isotropic vacuum, to the *interior dynamical problem*, characterized by motion of extended and therefore deformable particles within inhomogeneous and anisotropic physical media, resulting in the most general known dynamical equations.

The *isotopies* are used when interior structural problems are studied as a whole with conserved conventional total quantities under a generalized interior structure. The genotopies are instead used to characterize one individual constituent while considering the rest of the system as external, thus resulting in the nonconservation of its physical quantities in a way compatible with total conservation laws.

The *classical isotopies and genotopies* realize the isotopies and genotopies of contemporary algebras, geometries, mechanics, symmetries and relativities. In hadronic mechanics the corresponding *operator isotopies and genotopies*, that is, the *axiom-preserving isotopies* and *axiom-inducing genotopies* of QM are constructed. This topic is known under various terms as *hadronic generalization of QM*, *isotopic completion of QM*, *isotopic realism*, and perhaps the most as *hadronic mechanics* or HM in short.

As quarks seem to have some deep connections with nonassociativity it is interesting to give some remarks about quarks in Santilli's theory of hadronic structure. According to the general philosophy *quarks cannot be even defined in Santilli's structure model of hadrons.* Compatibility with modern views is however achieved considering a *family of hadrons* rather than individual ones. In Santilli's HM quarks are built inside hadrons as suitable collections of massive stable particles. Also, quarks achieve a *strict confinement* under isotopies, that is, an identically zero transition probability to exist in free state even in the absence of a potential barrier. It is well-known that the conventional QM does not permit a convergent perturbation theory for quark interaction. Santilli's model achieves convergent isoperturbation expansions in a way which is so simple as to be trivial: via realizations of the isotopic element T with modulus smaller than 1, $|T| \ll 1$.

The quantitative isotopic treatment of quarks is based on the isosymmetries $S\hat{U}(3)$ characterizing *isoquarks* which not only are locally isomorphic to $SU(3)$ but preserve all its quantum numbers in their standard irreps. This means that there is no possibility to discern experimentally between quarks and isoquarks. The best realization seems the one via the operator form of Nambu's mechanics [237] for triplets with realization of the isotopic element

$$T = H_1^{-1} + H_2^{-1}, \tag{1.54}$$

where H_1 and H_2 are Nambu's Hamiltonians. Since for strong interactions $H_1, H_2 > 1$, it then follows that $|T| \ll 1$. Isoquarks are then automatically confined in the strict sense indicated above and trivially admit convergent isoperturbative expansions (see [313] for details).

In summary, conventional QM unitary models are assumed as providing the final classification of hadrons. In regard to the structure, Santilli lifts conventional QM quarks into isoquarks which are strictly confined but composed of ordinary massive particles freely produced in spontaneous decays.

Nonassociative Lie-admissible algebras play a fundamental role in Santilli's model [313] of *nonobservability* of quarks via the isotopies of $SU(3)$. In fact, the basic isoproduct is given by $[A\hat{,}B] = A * B - B * A = ATB - BTA$ but it can be *identically* rewritten as $[A\hat{,}B] = (A, B) - (B, A) = (ARB - BSA) - (BRA - ASB)$ with $T = R - S$ $(T, R, S, R \pm S \neq 0)$, that is, $[A\hat{,}B]$ can be interpreted as being a Lie isotopic algebra attached to a *nonassociative* Lie-admissible algebra with the product $(A, B) = ARB - BSA$. Santilli contends that the **nonobservability of quarks in his model is due to the nonassociativity of** (A, B). This particular use of nonassociativity produces a quark theory with all conventional quantum numbers, multiplets, etc. (because of the local isomorfism $S\hat{U}(3) \sim SU(3)$) and an *exact confinement*.

The current state of HM may be characterized as follows:

1) the mathematical consistency of hadronic mechanics is now established, thus allowing rigorous quantitative treatments of interior particle problems in a form suitable for experimental tests.

2) there are already number of experimental verifications which, even though evidently preliminary, nevertheless confirm the predictions of the HM.

3) In addition, HM suggests a number of *novel* experiments on internal nonlinear-nonlocal-nonhamiltonian effects beyond the descriptive and predictive capacities of conventional theories.

As for experimental tests we refer to [311](vol.3), [313](vol.3), [315, 316].

Perhaps we shall finish this section with last sentences of the foreword to the first volume of [313]: *"Above all, a primary reason for writing these books is to point out for young minds of all ages that hadronic mechanics identifies the apparent existence of a new technology I tentatively called* **hadronic technology**, *because emerging from mechanisms in the structure of individual hadrons, while the current technologies emerge from the mechanisms in the structure of molecules, atoms, and nuclei. The societal implications of these possibilities, e.g., for possible new forms of energy, new approaches to cold fusion, new computer modeling, new medical applications, etc., have warranted this first identification of the state of the art in the conceptual, mathematical, theoretical and experimental foundations of hadronic mechanics."*

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Chapter 2

Limit transitions between algebras: contractions and deformations

There is an interesting, well-determined and non-trivial possibility to introduce new (nonassociative) algebras through *contractions* and *deformations* of some initial ones. These are parameter-dependent continuous transitions in the algebraic manifold of fixed (n-) dimensional algebras usually restricted by some additional identities leading to some "good" algebras.

This chapter may fall somewhat aside from the main topics of this monograph, especially in some sections (as Sec. 2.4), but the justification to include it is that the deformation procedure is *one of the ways obtaining nonassociativities* (at least we have a simple but non-trivial example in Sec. 2.3). We hope that this chapter is of some heuristic value. The presentation here is extremely elementary, mainly for the interested nonspecialist reader. To the cohomology-specialists this chapter is certainly hopelessly naive.

To be more definite we restrict ourselves with Lie algebras where we describe essential technical points (as in Sec. 2.2). There is quite an enormous body of literature about contractions and deformations, mainly of associative and Lie algebras, listing perhaps some 800 items (without q-deformations, the very recent and popular topics!).

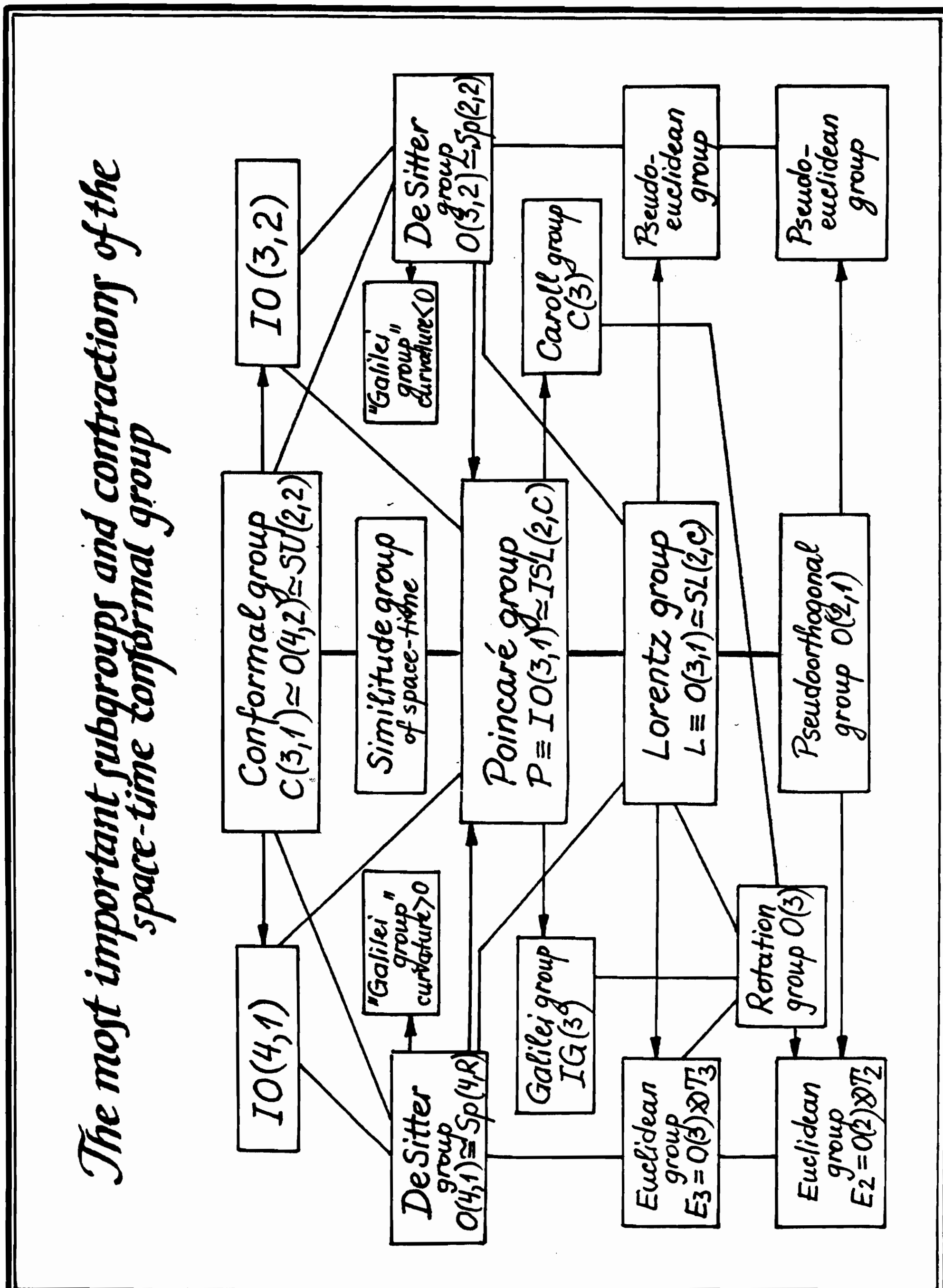
We shall give a general discussion of the problem of limit-type transitions between algebras (contractions and deformations) in Sec. 2.1, then present an elementary introduction to the deformation theory on an example of Lie algebras (Sec. 2.2). Sec. 2.3 is devoted to "generalised" deformations extending beyond the specified class of initial algebras. And finally, contractions and deformations between spectrum algebras of simple dynamical systems will be discussed in Sec. 2.4.

2.1 Historical remarks

It seems that contractions and deformations are the only known parameter-dependent continuous transformations of algebras (groups) which, in general, may give us algebras *nonisomorphic* (but with the same dimension!) to the initial ones. In mathematics (especially in geometry) situations and examples have been known to lead to such limit relations also between related algebraic structures (see e.g. [15]), but the first explicit formulation appeared first for the contractions related to physical problems [19]. It is a classical paper by Inönü and Wigner, which starts with words: "*Classical mechanics is a limiting case of relativistic mechanics. Hence the group of the former, the Galilei group must be in some sense a limiting case of the relativistic mechanics group* (i.e. the Poincaré group). *The representations of the former must be limiting cases of the latter's representations. There are other examples for similar relations between groups. Thus the inhomogeneous Lorentz group (the Poincaré group) must be, in some sense, a limiting case of the de Sitter groups*". The limit process called *contraction* by Inönü and Wigner is a result of the limit of some parameter: the velocity of light becoming infinite ($c \to \infty$) for the contraction *Poincaré group* $\Rightarrow$ *Galilei group*, or the radius of curvature $R \to \infty$ for the contraction *de Sitter group* $\Rightarrow$ *Poincaré group.*

Even some years earlier Segal [31] considered sequences of Lie groups with structure constants converging to the structure constants of some nonisomorphic Lie group.

In lines of historical remarks also the contribution of Russian theoretician Georgii Aleksandrovich Zaitsev (1935-1986) should be specially mentioned. Fatally diseased from the very youth, tangled into wheel chair he did enormous work in mathematics and physics, which is almost unknown in Western scientific community (as for example, he developed independently the $(V - A)$-theory of weak interactions, etc.). Zaitsev also developed independently the theory of limit transitions between Lie groups (algebras), he called them *limit Lie groups (algebras).* The results were reported at 1961 Kiev


The most important subgroups and contractions of the space-time conformal group
Conformal group C(3,1) ≃ O(4,2) ≃ SU(2,2)
IO(3,2)
IO(4,1)
De Sitter group O(3,2) ≃ Sp(2,2)
De Sitter group O(4,1) ≃ Sp(4,R)
"Galilei" group curvature<0
"Galilei" group curvature>0
Similitude group of space-time
Poincaré group P ≡ IO(3,1) ≃ ISL(2,C)
Caroll group C(3)
Galilei group IG(3)
Lorentz group L ≡ O(3,1) ≃ SL(2,C)
Pseudo-euclidean group
Pseudo-euclidean group
Pseudoorthogonal group O(2,1)
Rotation group O(3)
Euclidean group E3 = O(3) ⊗ T3
Euclidean group E2 = O(2) ⊗ T2

Conference on Problems of Geometry by late academician A.Z. Petrov, but have remained unpublished. Zaitsev's idea was to incorporate all physical theories into a whole system on the ground of limit relations between them, algebraically expressed through the apparatus known today as the theory of contractions and deformations. Zaitsev's legacy consists of some hundred papers and an interesting book [34].

By now these ideas are far developed. There is an entire system of limit relations between the space-time groups (see the chart included!), there is a theory of deformations connecting classical mechanics and QM (developed by the Lichnerowicz school [2], and also an interesting theory of *quantum groups* (see, e.g., [9, 33]).

Among the pioneers of the theory of group contractions A.S. Fedenko [8] and B.A. Rosenfeld [29] should be mentioned. In more recent times interesting contributions (also related to quantum groups) have been presented by N.A. Gromov [13]. But these are only some few excerpts from the huge amount of literature about limit groups, or *contractions* and *deformations*, as called nowadays.

2.2 Elementary introduction to the contractions and deformations of Lie algebras

2.2.1 Contraction problem

To clarify the very nature of the limit process in contraction we begin with the simplest example of the contraction of the rotation group $SO(3)$ to the group of motions of 2-plane, the Euclidean group $E(2)$:

$$SO(3) \Rightarrow E(2)\,.$$

In geometric terms this contraction may be interpreted as the "flat space limit of the sphere". In such kind of limit examples the notion of contraction was for a long time hidden in geometry and was first expressed explicitly in terms of Lie algebras (groups) by Fedenko [8], where he discusses symmetric spaces with nonsemisimple fundamental groups as limits of symmetric spaces with semisimple fundamental groups.

To expose the essence of the contraction procedure let us consider in some details the most simple case mentioned above. Here the technical details do not overshadow the principal moments. Let us write CR for the Lie algebra of the rotation group $SO(3)$:

$$[I_1, I_2] = I_3\,, \quad [I_2, I_3] = I_1\,, \quad [I_3, I_1] = I_2\,.$$

Then let us introduce new parameter-dependent generators (i.e. let us perform a parameter-dependent transformation of the Lie algebra basis, we call them *ε-transformations* or *prelimit* transformations):

$$\tilde{I}_1(\varepsilon) = \varepsilon I_1\,, \quad \tilde{I}_2(\varepsilon) = \varepsilon I_2\,, \quad \tilde{I}_3 = I_3\,,$$

the CR for these generators are

$$\begin{aligned} [\tilde{I}_1(\varepsilon), \tilde{I}_2(\varepsilon)] &= \varepsilon \tilde{I}_3(\varepsilon)\,, \\ [\tilde{I}_2(\varepsilon), \tilde{I}_3] &= \tilde{I}_1(\varepsilon)\,, \\ [\tilde{I}_3, \tilde{I}_1(\varepsilon)] &= \tilde{I}_2(\varepsilon)\,. \end{aligned} \tag{2.1}$$

Now for the limit

$$\lim_{\varepsilon \to 0} \tilde{I}_i(\varepsilon) = \tilde{I}_i\,, \tag{2.2}$$

the most important thing to understand is that in the limit $\varepsilon \to 0$ the generators $\tilde{I}(\varepsilon)$ do not disappear (vanish) but they undergo a limit process and become new generators with new CR. In our particular case the resulting CR

$$[\tilde{I}_1, \tilde{I}_2] = 0\,, \quad [\tilde{I}_2, \tilde{I}_3] = \tilde{I}_1\,, \quad [\tilde{I}_3, \tilde{I}_1] = \tilde{I}_2 \tag{2.3}$$

are the CR of the Euclidean group algebra

$$E(2) \simeq ISO(2) \simeq SO(2) \otimes T_2\,,$$

which is a nonsemisimple algebra of a special structure, the algebra with a *commutative radical.* Note that (nontrivial) contractions always lead to nonsemisimple algebras.

Naturally all this would be quite a nonsense if you regard generators as represented by some matrices. Then they indeed will vanish, and for the representations to survive special tricks must be applied [19].

Insofar we have not yet given any rigorous definition for the contraction procedure, it may be stated as follows. Let us have a Lie algebra $\mathcal{L}$ with some basis consisting of generators I_i $(i = 1, 2, \ldots, r)$ and let us allow the transformation A of the basis depend on some continuous parameter ε, i.e. $A(\varepsilon)$, and become singular for some limit values of the parameter. If the transformed structure constants $c_{ij}^k(\varepsilon)$ approach some well-defined limit at a particular (limit) value of the parameter, then we call this new algebra $\tilde{\mathcal{L}}$ defined by the new structure constants $\tilde{c}_{ij}^k$ a *contraction* of the initial algebra $\mathcal{L}$. Also, the accompanying well-defined process of *group contraction* can be considered, but it involves additional differential-geometric and topological aspects.

In the contraction process the dimension of the algebra remains invariant, but the algebraic stucture changes and the contracted algebra in general is not isomorphic to the initial one.

Now a problem arises which is the suitable structure of the parameter-dependent matrix $A(\varepsilon)$, the ε-transformation? Insofar interesting nontrivial geometrical and physical examples exploit very simple types of ε-transformations. The matrix $A(\varepsilon)$ used by originators was

$$A(\varepsilon) = u + \varepsilon v = \begin{pmatrix} 1+\varepsilon w & 0 \\ 0 & \varepsilon \end{pmatrix}, \tag{2.4}$$

the most simple version of which is

$$A(\varepsilon) = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \varepsilon & & \\ & & & & \ddots & \\ & & & & & \varepsilon \end{pmatrix}. \tag{2.5}$$

We call these contractions *Inönü-Wigner (IW-) contractions.* Let us list here some characteristic features of these simplest contractions.

1. The first s generators form a subalgebra both in initial and contracted algebras. It is necessary and sufficient condition for pure IW-contractions. The algebraic structure of this subalgebra is not altered by contraction.

2. "Contracted" generators $\tilde{I}_j^{(2)}$, $(j = s+1, \dots, r)$ form an Abelian subalgebra in the contracted algebra, i.e. there appears the structure of the semidirect sum

$$\tilde{\mathcal{L}} = \mathcal{A} \oplus\!\!\!\!\!\supset \mathcal{H},$$

where $\mathcal{A}$ is the nonaltered (stable) subalgebra and $\mathcal{H}$, the Abelian subalgebra of contracted generators.

3. The contracted algebra does not change if we apply again the same contraction, i.e. with exactly the same ε-transformation.

The ε-part of $A(\varepsilon)$ of the pure diagonal form (2.5) transforms the generators $I_j^{(2)}$ and the corresponding parameters α^j in the following manner:

$$\tilde{I}_j^{(2)}(\varepsilon) = \varepsilon I_j^{(2)}\,, \quad \alpha^j = \varepsilon\tilde{\alpha}^j\,; \quad j = s+1,\ldots,r$$

It is easy to see now that at the fixed values of the parameters $\tilde{\alpha}^j$ at $\varepsilon \to 0$ the parameters α^j become gradually smaller, i.e. the initial group G is contracted to the very close neighbourhood of the subgroup A (corresponding to the subalgebra $\mathcal{A}$), which may also be the pictoral meaning of the term "contraction". We see also that new values for parameters obtain unbounded values, which means the loss of compactness of the group as a whole. The contraction procedure has very nicely been described in a book [11] by R. Gilmore. The infinitesimal aspect of the contraction process has recently been investigated from the point of view of *nonstandard analysis* [12].

The pure form (2.5) of the ε-transformation fits nicely with the *Cartan decomposition* of simple groups, which in terms of Lie algebras means

$$\mathcal{L} = \mathcal{M} + \mathcal{N}\,,$$

where CR have a general form

$$[\mathcal{M},\mathcal{M}] = \mathcal{M}\,, \quad [\mathcal{M},\mathcal{N}] = \mathcal{N}\,, \quad [\mathcal{N},\mathcal{N}] = \mathcal{M}\,. \tag{2.6}$$

Now contracting with respect of the subalgebra $\mathcal{M}$ we have a nonsemisimple algebra

$$\tilde{\mathcal{L}} = \mathcal{M} \mathbin{\ni\!\!\!\!+} \tilde{\mathcal{N}}$$

with characteristic CRs

$$[\mathcal{M},\mathcal{M}] = \mathcal{M}\,, \quad [\mathcal{M},\tilde{\mathcal{N}}] = \tilde{\mathcal{N}}\,, \quad [\tilde{\mathcal{N}},\tilde{\mathcal{N}}] = 0\,. \tag{2.7}$$

There are naturally more involved possibilities, more complicated cases of $A(\varepsilon)$, for example, *Saletan contractions* [30] (for the earlier development of contractions and deformations, up to 1968, see review [24]).

2.2.2 Deformation problem

The *deformation problem* arises as an *inverse problem* for contraction, i.e. as the problem of finding possible initial or *precontracted* algebras (or groups) for some given Lie algebra (or group). If we have a contraction process $\mathcal{L} \to \tilde{\mathcal{L}}$, is there any well-defined procedure to get $\tilde{\mathcal{L}} \Rightarrow \mathcal{L}$ (or more precisely, for a given $\tilde{\mathcal{L}}$ to find all $\mathcal{L}$ such that they are contractible to $\tilde{\mathcal{L}}$)?

The answer is yes: *for a given algebra $\mathcal{L}$ all algebras which are contractible to $\mathcal{L}$ can be located among the deformations of $\mathcal{L}$.*

Let us proceed now to the general definition of this very important concept and to the description some of its properties. Our presentation here follows closely the treatment given in papers [10, 27, 28, 26].

Let V be a vector space of finite dimensionality and $\mathcal{M}$, the set of all possible Lie algebras, i.e. all possible commutation relations (CR) on this vector space. The vector space V then becomes an algebraic manifold with a quite complicated structure. We have a complete description of this manifold only in low dimensions up to $n = 6$, for some more special Lie algebra structures, such as for nilpotent Lie algebras, up to $n = 7$, but this is all what we have. Already in the case of 3-dimensional Lie algebras there appears a family of nonisomorphic algebras depending on a continuous parameter. It is a characteristic feature. In higher dimensions there appear such families depending on several continuous parameters.

Now we can see that the contraction problem and the inverse problem is essentially a problem of "geography" of this algebraic manifold of 3^n dimensions.

Let us specify some Lie algebra $\mathcal{L}$ from this set $\mathcal{M}$, which means the specifying of the Lie algebra operation

$$\mu(x,y) := [x,y]\,; \quad \forall x,y \in \mathcal{L}\,, \tag{2.8}$$

for which there are two characteristic properties, the *antisymmetry*

$$\mu(x,y) = -\mu(y,x)\,, \quad \forall x,y \in \mathcal{L} \tag{2.9}$$

and the *Jacobi identity*

$$\sum_{cycl} \mu(x,\mu(y,z)) = 0\,; \quad \forall x,y,z \in \mathcal{L}\,. \tag{2.10}$$

Now the *deformations of the Lie algebra* $\mathcal{L}$ are defined as one-parameter families of Lie algebras

$$\mu_t(x,y) = \mu(x,y) + tF_1(x,y) + t^2F_2(x,y) + \dots, \tag{2.11}$$

where $\mu \in \mathcal{M}$ and $F_1, F_2, \ldots \in \{f : V \times V \to V\}$, f being bilinear antisymmetric map. For $\mu_t(x,y)$ to be in $\mathcal{M}$ these *deformation functions* $F_1, F_2, \ldots$ must satisfy certain conditions following from (2.9) and (2.10) for μ_t.

In a very general form these conditions, the *deformation equations*, can be written as an infinite system (k=1,2,...)

$$\sum_{cycl} \sum_{\substack{i+j=k,\\ k=1,2,\dots}} F_i(F_j(x,y),z) + F_j(F_i(x,y),z) = 0\,. \tag{2.12}$$

For $k = 0$ we obtain the Jacobi identity for $\mu(x,y) = F_0(x,y)$, i.e. for the initial algebra. $k = 1$ gives us the equation of the *infinitesimal deformation*,

$$\partial F_1(x,y,z) = 0\,, \tag{2.13}$$

where

$$\partial F_1(x,y,z) = \sum_{cycl} \mu(x, F_1(y,z)0 + \sum_{cycl} F_1(x, \mu(y,z))\,. \tag{2.14}$$

As a particular Lie algebra can be represented in very many bases related with the regular transformations from the group $GL(n,V)$, there are *orbits* in the manifold $\mathcal{M}$ representing particular Lie algebras. It is the mutual location of orbits that determines the possible transitions between nonisomorphic algebras (contractions and deformations), which is in essence the problem of the topology of that algebraic manifold. There is a special kind of topology, developed for algebraic manifolds, the *Zariski toplogy* fitting with the algebraic structure of the manifold (in this topology the set of closed subsets coincides with the set of algebraic submanifolds).

And now it appears that the structure of the series of the n-th powers, t^n, and the n-th deformation functions, F_n, i.e. the consequential structure of deformation equations, gives the possibility to apply here a powerful machinery of algebraic topology, the apparatus of *cohomology groups* (well developed for Lie algebras already some times ago [4, 16]).

Let there be a Lie algebra $\mathcal{L}$ and some of its representations W ($\mathcal{L}$-modules), then in terms of algebraic topology the following construction, called *complex*, can be built:

$$C^0 \xrightarrow{\delta_0} C^1 \xrightarrow{\delta_1} C^2 \xrightarrow{\delta_2} \ldots \xrightarrow{\delta_{n-1}} C^n \xrightarrow{\delta_n} C^{n+1} \xrightarrow{\delta_{n+1}} \ldots, \tag{2.15}$$

where the *cochain space* $C^n \equiv C^n(\mathcal{L}, W)$ consists of all antisymmetric n-linear maps of the n-fold direct product $\mathcal{L} \otimes \mathcal{L} \otimes \ldots \otimes \mathcal{L} \to W$, the *cochains.* A *n-cochain* f is a function of n arguments from the Lie algebra with values in some $\mathcal{L}$-module W. The 0-cochain is a constant function.

For any cochain f there can be constructed a $(n+1)$-cochain δf $(\equiv \delta^n f)$, a $(n+1)$-*coboundary* for f:

$$\delta f(x_1, x_2, \ldots, x_{n+1}) = \sum_{q=1}^{n+1} (-1)^{n-q+1} x_q f(x_1, \ldots, \hat{x}_q, \ldots, x_{n+1}) +$$
$$\sum (-1)^{r+q} f(x_1, \ldots, \hat{x}_q, \ldots, \hat{x}_r, \ldots, x_n, \mu(x_q, x_r)), \tag{2.16}$$

where arguments with hats are supposed to be omitted.

The cochain space $C^n(\mathcal{L}, W)$ is mapped into the cochain space $C^{n+1}(\mathcal{L}, W)$ by the *coboundary operator* δ:

$$C^n(\mathcal{L}, W) \xrightarrow{\delta} C^{n+1}(\mathcal{L}, W),$$

which is a *link* of the complex (2.15) written above. A n-cochain is called *n-cocycle* if $\delta f = 0$ and *n-coboundary,* if $f = \delta g$, $g \in C^{n-1}(\mathcal{L}, W)$. Therefore n-cocycles form the nucleus of the coboundary homomorphism δ:

$$Z^n(\mathcal{L}, W) = \mathrm{Ker}\delta_n . \tag{2.17}$$

The image of the $(n-1)$-cochain space $C^{n-1}(\mathcal{L}, W)$ under the coboundary map δ_{n-1} forms a subspace in $C^n(\mathcal{L}, W)$. We denote it $B^n(\mathcal{L}, W)$ and call the n-th *coboundary space*

$$B^n(\mathcal{L}, W) = \mathrm{Im}\delta_{n-1} . \tag{2.18}$$

The general property of the complex is that

$$B^n(\mathcal{L}, W) \subseteq Z^n(\mathcal{L}, W), \tag{2.19}$$

or, equivalently,

$$\mathrm{Im}\delta_{n-1} \subseteq \mathrm{Ker}\delta_n , \tag{2.20}$$

and, consequently,

$$\delta_n \delta_{n-1} = 0 \,, \tag{2.21}$$

or even more briefly,

$$\delta^2 = 0 \,. \tag{2.22}$$

Now we can consider factor spaces

$$H^n(\mathcal{L}, W) = Z^n(\mathcal{L})/B^n(\mathcal{L}, W) \,, \tag{2.23}$$

which are called *n-th groups of cohomology* of the Lie algebra $\mathcal{L}$ with respect to a $\mathcal{L}$-module W.

For deformation theory purposes we may confine ourselves with the cohomology groups $H^n(\mathcal{L}, \mathcal{L})$, i.e. we take for W the adjoint representation $\mathrm{ad}_{\mathcal{L}}x :\ \mathrm{ad}_{\mathcal{L}}x(y) = \mu(x, y)$.

Now some identifications may be arranged with the quantities of the deformation problem. A detailed consideration leads to the following situation. All the infinitesimal deformations determined by $F_1(x, y)$ form the cocycle space $Z^2(\mathcal{L}, \mathcal{L})$ and the trivial infinitesimal deformations belong to $B^2(\mathcal{L}, \mathcal{L})$, so the second group of cohomology $H^2(\mathcal{L}, \mathcal{L})$ characterizes the possible set of nontrivial (infinitesimal) deformations. If $H^2(\mathcal{L}, \mathcal{L}) = 0$, then such deformations are absent (as it is the case for the semisimple Lie algebras). The algebra $\mathcal{L}$ is then called *infinitesimally rigid.*

Under quite general conditions (concerning the underlying field) an infinitesimally rigid algebra is also *rigid* [27, 28].

Some "geographical" picture may be drawn for the location of algebras in the algebraic manifold $\mathcal{M}$. A point $\mu \in \mathcal{M}$ represents a Lie algebra $\mathcal{L}(V, \mu)$ in some fixed basis. When the basis is transformed by $GL(V)$, we have isomorphic algebras, all points μ representing isomorphic algebras from the orbit $GL(\mu)$ in the manifold $\mathcal{M}$, the group $GL(V)$ acting transitively upon orbits (on each orbit separately). The power series μ_t represents an analytic curve, which starts in the point μ with the value $t = 0$. The infinitesimal deformation F_1 is a tangent vector to this analytic curve in the point $t = 0$. The tangent space of all these tangent vectors coincides with the space $Z^2(\mathcal{L}, \mathcal{L})$ of 2-cocycles, [10]. The analytic curve μ_t may belong to the orbit, then tangent vectors belong to the coboundary subspace $B^2(\mathcal{L}, \mathcal{L}) \subseteq Z^2(\mathcal{L}, \mathcal{L})$. The exact picture is expressed by the cohomology group $H^2 = Z^2/B2$.

All this is quite well-behaving when the algebraic manifold $\mathcal{M}$ is sufficiently smooth and the point μ is not a *singular* one, but a simple point of the manifold.

An infinitesimal deformation is *integrable* when it can be embedded into a 1-parameter family μ_t of deformations, i.e. when all the higher deformation equations with higher deformation functions are satisfied. Here a decisive result is expressed by the third cohomology group $H^3(\mathcal{L}, \mathcal{L})$, its elements can be regarded as *obstructions* to the integrability, and when $H^3(\mathcal{L}, \mathcal{L}) = 0$ (trivial), these obstructions are absent.

A result has been proved [28] for a set of all possible deformations: this set is parametrised by the *zeros* of the mapping $H^2(\mathcal{L}, \mathcal{L}) \mapsto H^3(\mathcal{L}, \mathcal{L})$.

Interrelation between *contractions* and *deformations* seems quite straightforward in general, but it must be treated with a great care because it depends upon the positions of orbits and possible singularities. In the contraction process the dimension of the algebra remains invariant, so the contracted algebra also belongs to the same algebraic manifold $\mathcal{M}$. If we proceed from some particular contraction type, then the precontracted CR may always be arranged as the deformation series μ_t and it represents the deformation by which we get back the precontracted algebra from the contracted one. On the other hand there may be deformations for which the initial algebra may not be reproduced by IW-contraction procedure, but singular contractions are needed. Deforming some algebra into a continuous family of nonisomorphic algebras, we do not get automatically the precontracted algebra because the deformation process itself (alone) is not able to choose the right one. In every particular case these problems need a special treatment. And there is yet an interesting problem of treating contractions as deformations. An interested reader may have a start with these problems e.g. from [14].

2.3 More general deformations

Analogously to the deformation of associative and Lie algebra structures [10, 27] the *general deformation of n-dimensional linear algebra* $\mathcal{A}$ may be defined ([22]) as

$$(x, y)_\varepsilon = xy + \varepsilon F_1(x, y) + \varepsilon^2 F_2(x, y) + \dots , \qquad (2.24)$$

where xy is the binary product in the initial algebra $\mathcal{A}$, $(x,y)_\varepsilon$ is the deformed product, ε is the *deformation parameter* and $F_i(x,y)$, $i = 1,2,\dots$ are the *deformation functions*, whereby $F_0(x,y) = xy$ in complete analogy with the deformations defined by (2.11) for Lie algebras.

If we want to preserve strictly the type of algebra, i.e. to deform associative algebras into associative ones, Lie algebras into Lie algebras, etc., we must superimpose the corresponding conditions (properties) for the deformed product (2.24). From these conditions there follows a certain (infinite) system of *deformation equations* for the deformation functions $F_i(x,y)$. In this sense we are speaking about deformations of associative algebras, about deformations of Lie algebras, etc.

Let us now define *generalized deformation* for a linear algebra as a procedure which modifies the operation (product, multiplication) of the initial algebra according to (2.24), but without any strict demand of exact preservation of the initial type of algebra. Naturally, to avoid inconsistency the following trivial condition must hold: *For generalized deformations with a deformed product* (2.24) *the class of initial algebras must form a subclass in the class of deformed algebras.* For example, associative algebras, Lie algebras and CJA can be deformed into Lie-admissible algebras because they all are Lie-admissible. Likewise we can consider alternative deformations of associative algebras. In what follows we shall consider the last case in some more details.

The *associativity* condition for a deformed algebra $\mathcal{A}(\varepsilon)$ (for a deformed product $(xy)_\varepsilon$)

$$(x,y,z)_\varepsilon = ((xy)_\varepsilon z)_\varepsilon - (x(yz)_\varepsilon)_\varepsilon = 0\,, \quad \forall x,y,z \in \mathcal{A}(\varepsilon) \tag{2.25}$$

leads to the following *deformation equations*

$$\sum_{\substack{i+j=k\\ i,j\geq 0,k=0,1,2,\dots}} [F_i(F_j(x,y),z) - F_i(x,F_j(y,z))] = 0\,. \tag{2.26}$$

The alternativity conditions in the form

$$(x,x,y)_\varepsilon = (x,y,y)_\varepsilon = 0\,, \quad \forall x,y \in \mathcal{A}(\varepsilon) \tag{2.27}$$

lead to the corresponding equations for deformation functions

$$\sum_{i+j=k} [F_i(F_j(x,x),y) - F_i(x,F_j(x,y))] = 0\,,$$

$$\sum_{i+j=k} [F_i(F_j(x,y),y) - F_i(x,F_j(y,y))] = 0\,, \tag{2.28}$$

where $i, j \geq 0,\ k = 1, 2, \ldots$.

Detailed equations for the first-order *deformation coefficients* ${}^1f_{ij}^k$ (defined by $F_1(e_i, e_j) = {}^1f_{ij}^k e_k$) follow from (2.26) and (2.28):

$$c_{pk}^l f_{ij}^p + c_{ij}^p f_{pk}^l - c_{ip}^l f_{jk}^p - c_{jk}^p f_{ip}^l = 0\,, \tag{2.29}$$

for associative deformations (from (2.26)) and

$$\begin{aligned}
&c_{ij}^p f_{pk}^l + c_{pk}^l f_{ij}^p + c_{ji}^p f_{pk}^l + c_{pk}^l f_{ji}^p - \\
&c_{ik}^p f_{jp}^l - c_{jp}^l f_{ik}^p - c_{jk}^p f_{ip}^l - c_{ip}^l f_{jk}^p = 0\,, \\
&c_{jk}^p f_{ip}^l + c_{ip}^l f_{jk}^p + c_{kj}^p f_{ip}^l + c_{ip}^l f_{kj}^p - \\
&c_{ij}^p f_{pk}^l - c_{pk}^l f_{ij}^p - c_{ik}^p f_{pj}^l - c_{pj}^l f_{ik}^p = 0\,,
\end{aligned} \tag{2.30}$$

$$i, j, k, l, p = 1, 2, \ldots, n;\ \text{summation over } p$$

for alternative deformations (from (2.28)). The entities c_{ij}^k are the structure constants of the initial algebra; superscript 1 is omitted for ${}^1f_{ij}^k$.

After solving Eqs. (2.29) and (2.30) we superimpose the *condition of finiteness* on the quantities f_{ij}^k and get equations for the *first-order finite deformations*:

$$f_{ij}^p f_{pk}^l - f_{jk}^p f_{ip}^l = 0 \tag{2.31}$$

for the associative deformations and

$$\begin{aligned}
&f_{ij}^p f_{pk}^l + f_{ji}^p f_{pk}^l - f_{ik}^p f_{jp}^l - f_{jk}^p f_{ip}^l = 0\,, \\
&f_{jk}^p f_{ip}^l + f_{kj}^p f_{ip}^l - f_{ij}^p f_{pk}^l - f_{ik}^p f_{pj}^l = 0\,,
\end{aligned} \tag{2.32}$$

$$i, j, k, l, p = 1, 2, \ldots, n\,;\ \text{summation over } p$$

for the alternative deformations.

After solving Eqs. (2.31) or (2.32) we get the structure constants $\tilde{c}_{ij}^k = c_{ij}^k + f_{ij}^k$ for the deformed associative or alternative algebra, containing, however, many superfluous extra parameters leading only to trivial deformations (to changes of the basis) of the initial algebra. As we are interested only in nontrivial (nonisomorphic) deformations, we must get rid of the trivial ones.

Therefore we must compute the effect of the change of basis (by finite transformations!) on the structure constants for every particular algebra under consideration and compare it with the deformed structure constants. Then the superfluous parameters corresponding to trivial deformations may be eliminated, and only the essential nontrivial deformations remain.

In an ordinary deformation procedure one finds a) *infinitesimal deformations* (which can be identified with cocycles in the case with an applicable cohomology theory, i.e. in the cases of associative and Lie algebras), b) *nontrivial infinitesimal deformations* (identifiable with $H^2(A, A)$) and c) *finite nontrivial deformations.*

In some very early papers [21, 22] we have proceeded along a somewhat different line: we have found a) infinitesimal deformations, then b') we have found *finite deformations*, and c') using finite transformations of basis, we have separated *finite nontrivial deformations.* The two last steps b') and c') cannot be interpreted in terms of cohomology theory. For purely technical reasons the step c') is possible only for 2-dimensional algebras and it should be admitted that this procedure is quite exceptional and it was followed only for preliminary studies, to separate nontrivial deformations by finite transformations of the basis. Finitenesss is important for identifying the contractions which correspond to the singularities of the transformation of the basis solely in the finite case.

In such primitive manner some types of deformations of 2-dimensional power-associative algebras were described in [21, 22] (in [22] also some deformations of the Pauli algebra), and in [23] power-associative deformations of the Pauli algebra by M. Kõiv and R.-K. Loide.

2.4 Contractions and deformations between spectrum-generating groups of simple dynamical systems

Now we shall continue with a group-theoretical discussion of symmetry and invariance of simple dynamical systems, started in Sec. 1.5.4, but within the context of the present chapter, i.e. in the context of contractions and deformations. We would like to underline the fact of the systematic occurrence of *triples* of spectrum-generating groups (algebras) mutually connected by these limiting procedures discussed above. Such situations have appeared to take place in the group-theoretical formulation of *simple dynamical systems* [25], and in *strong coupling* models [5] of hadrons. As it became clear recently [17, 18], analogous triples of gauge groups appear also in some *supergravity theories.*

2.4.1 Description of spectrum-generating groups

Usually, a SGG (for algebras SGA, for terms we refer Sec. 5.4) is considered as a *noncompact* group with the DDG as the *maximal compact subgroup.* The generators lying outside the Lie algebra of the DDG do not commute with the Hamiltonian of the dynamical system under consideration. Under the action of these generators the states of one energy level are transformed into the states of some other energy levels (generally the states of different energy levels are mixed). The DDG acts irreducibly on the degenerate states of every energy level.

For a simple dynamical system a unitary infinite-dimensional irrep of the *noncompact* SGG (ncSGG) may be chosen, which contains the whole spectrum of the states of the given dynamical system. The restriction, *branching* with respect to the subgroup DDG, gives an infinite direct sum of (different) irreps D(DDG) acting on the degenerate states of energy levels.

The noncompact SGG were first introduced in [1, 7] for the description of hadron spectrum states.

The *compact* SGG (cSGG) is a *compact real form*, a compact companion of the ncSGG, both of which are the real forms of the same complex semisimple Lie group. Like every compact group, the cSGG has also only finite-(but unlimited) dimensional unitary irreps. It appears that the finite number of states of the lowest energy levels may always be incorporated into some unitary finite-dimensional irrep D(cSGG). Restricting ourselves to the DDG (which is also a subgroup in cSGG) gives a reducible representation, i.e. a direct sum of irreps D(DDG) corresponding to the chosen lower-energy levels. Quite a remarkable fact is that for every choice of these lower levels and for the corresponding finite-dimensional irrep D(cSGG) there exists a unitary infinite-dimensional irrep of the ncSGG which incorporates all the remaining inifinite number of higher states of the whole spectrum (the "tail" of the D(cSGG)).

If the cSGG and ncSGG are real forms of a (semi-) simple classical group, the set of spectrum-generating groups may be very naturally completed into a *triple* of groups, considering also the *inhomogeneous* (nonsemisimple) SGG (ihSGG) related to cSGG and ncSGG through a well-defined limit process, the IW-contraction (see Sec.2.2.1), arranged with respect to the subgroup $M(\equiv \mathrm{DDG})$ in the Cartan decomposition, see (2.6), (2.7) for CRs in precontracted and contracted Lie algebras:

$$\mathrm{cSGG},\ \mathrm{ncSGG} \overset{\text{contraction}}{\longrightarrow} \mathrm{ihSGG}\,. \tag{2.33}$$

Conversely, the ihSGG may be deformed back into the cSGG and nc SGG through the Lie group (algebra) *deformation* procedure after Gerstenhaber [10] (see Sec. 2.2.2):

$$\mathrm{ihSGG} \overset{\text{deformation}}{\Longrightarrow} \mathrm{cSGG},\ \mathrm{ncSGG}\,. \tag{2.34}$$

Such remarkable triples of groups – compact and noncompact real forms with the same maximal compact subgroup, and the corresponding inhomogeneous group obtained from both of them by an IW-contraction with respect to the maximal subgroup – arise naturally in numerous contexts in mathematics and theoretical physics. In connection with group contractions these triples have been mentioned in [29].

Another example of dynamical situations, very similar to simple dynamical systems, where our triples of SGG appear as well, is the old *strong coupling theory* model of baryon-meson scattering (for the group-theoretical treatment, see [5]).

Analogous *triples of gauge groups* have been explored in non-compact gauging of the $N = 8$ supergravity [17, 18]. In [17], an one-parameter family of gaugings is given. For positive values of a real parameter ξ (which may be identified as a limit parameter of IW-contraction or the corresponding deformation parameter) the $SO(8)$ group is recovered, $\xi = 0$ giving the inhomogeneous group $ISO(7)$, while $\xi < 0$ giving the group $SO(7,1)$, i.e. the non-compact gauge group. Also more general gaugings with gauge groups $SO(8)$, $SO(p,q)$, $(p+q=8)$ and $ISO(p)$ are considered.

2.4.2 DDG and SGG for simple dynamical systems

We summarise the results about DDG and SGG of simple dynamical systems, of a strong coupling model, and of the $N = 8$ supergravity gauge groups in the Table 2.

The problem that remains to be investigated is whether the groups ncSGG and ihSGG will give the same qualitative "shape" of the spectrum of the corresponding dynamical system. For that purpose we must have at our disposal detailed descriptions of unitary irreps of all three groups of the "dynamical triple".

We shall demonstrate the positive answer to the problem on an example of H-atom with SGG-triple of groups ncSGG $\simeq SO(4,1)$, cSGG $\simeq SO(5)$, ihSGG $\simeq ISO(4)$, and the DDG $\simeq SO(4)$, i.e. their maximal compact subgroup.

A unitary irrep of the $SO(4,1)$ belongs to one of the three *continuous* series $D^c_{r,\sigma}$ $(r = 1,2,\ldots; \sigma > 0)$, $D^c_{r,\sigma}$ $(r = \frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots; \sigma > \frac{1}{4})$, $D^c_{r\sigma}(r = 0, \sigma > -2)$ (σ a continuous parameter) or of the three *discrete* series $D^{d+}_{r,q}$ $(r = \frac{1}{2},1,\frac{3}{2},2,\ldots; q = r, r-1,\ldots,1)$, $D^{d-}_{r,q}$ $(r = \frac{1}{2},1,\frac{3}{2},2,\ldots; q = r, r-1,\ldots,\frac{1}{2})$, $D^{d0}_{r,q}$ $(r = 1,2,\ldots; q = 0)$ of infinite-dimensional unitary irreps, [6, 32, 20], where $r, \sigma(r,q)$ are the numbers specifying the irrep.

It is known that from the irreps of the DDG of $SO(4)$ in the spectrum of H-atom only $D(k,l)$ with $k = l = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots)$ are realised. The branching rule $r = |k-l|$ indicates that only the irreps of the third, the *degenerate continuous* series, are suitable for the description of the hydrogen spectrum. There is a continuous set of them.

It should be mentioned that the unitary irreps of both degenerate series $D^c_{0,\sigma}$ or $D^{d0}_{r,0}$ may be used for the description of the "tails" of the spectrum left over from the finite-dimensional unitary irreps $D(k,0)$ of the cSGG $SO(5)$.

Table 2. **Triples of SGG and gauge groups**

System	DDG	cSGG	ncSGG	ihSGG
Rotator	$SO(n)$	$SO(n+1)$	$SO(n,1)$	$ISO(n)$
Oscillator	$SU(n)$	$SU(n+1)$	$SU(n,1)$	$ISU(n)$
H-atom	$SO(n+1)$	$SO(n+2)$	$SO(n+1,1)$	$ISO(n+1)$
	$SU(2)$	$SU(2)\times SU(2)$	$SL(2,C)$	$SU(2) \circledS T_3$
Various	$SU(2)\times SU(2)$	$SU(4)$	$SL(4,R)$	$[SU(2)\times SU(2)] \circledS T_9$
models	$SU(2)\times SU(3)$	$SU(6)$	$SL(6,R)$	$[SU(2)\times SU(3)] \circledS T_{24}$
of strong	$SU(4)$	$SU(4)\times SU(4)$	$SL(4,C)$	$SU(4) \circledS T_{15}$
coupling	$SU(6)$	$SU(6)\times SU(6)$	$SL(6,C)$	$SU(6) \circledS T_{35}$
theories	$SU(6)\times SU(6)$	$SU(12)$	$SL(12,R)$	$[SU(6)\times SU(6)] \circledS T_{73}$
	$SU(12)$	$SU(12)\times SU(12)$	$SL(12,C)$	$SU(12) \circledS T_{143}$
$N=8$	$SO(7)$	$SO(8)$	$SO(7,1)$	$ISO(7)$
SG	$SO(p)\times SO(q)$	$SO(8)$	$SO(p,q)$	$ISO(p,q)$

To obtain a unitary infinite-dimensional irrep of the ihSGG $ISO(4)$ with the right "ladder" of the spectrum the contraction process $SO(5) \rightarrow ISO(4)$ must be arranged with the additional condition $\varepsilon k \rightarrow$ const.= quadratic root of the translational invariant of the group $ISO(4)$ (where ε is contraction parameter and k, a natural number specifying the irrep $D(k,0)$ of $SO(5)$). Now in the contraction limit ($\varepsilon \rightarrow 0$, $k \rightarrow \infty$) we obtain an infinite-dimensional irrep $D(p,0)$ of the inhomogeneous group $ISO(4)$ containing all irreps $D(l,l)$ of DDG $\simeq SO(4)$, $l = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$.

The physical unitary irrep $D^c_{0,\sigma}$ of the ncSGG $\simeq SO(4,1)$ may also be contracted to the irrep of $ISO(4)$ with the right $SO(4)$-content.

2.4.3 Physical meaning of the contraction process

A contraction (deformation) procedure may usually be thought as a general formulation of some type of *correspondence* between physical situations where some limits occur (as $c \rightarrow \infty$ from special relativity to classical mechanics, or $h \rightarrow 0$ from quantum to classical mechanics). Here the contractions (deformations) between the groups of SGG-triple must also have a definite (but more particular) physical meaning.

In a strong-coupling model, baryon-meson resonances occurring in the processes of baryon and meson scattering, form an infinite spectrum of hadrons which may be described in terms of unitary irreps of ncSGG or ihSGG. The limit process cSGG→ihSGG may be regarded as the *strong-coupling limit* ($\lambda \rightarrow \infty$) of the coupling constant λ in the Chew-Low expressions of the scattering amplitude (the corresponding contraction parameter is then $\varepsilon = \frac{1}{\lambda^2} \rightarrow 0$) where more and more poles of scattering amplitude are taken into account, which results in the appearance of more and more states in the hadronic spectrum.

In the same way we may also interpret the limit process cSGG→ihSGG for simple dynamical systems — the *whole spectrum limit* where all possible states are included. There is also a contraction ncSGG→ihSGG, so far without any apparent physical significance.

Perhaps the circumstance that the ncSGG may be obtained from cSGG by multiplication of some generators by an imaginary unit, indicates that the contraction process itself may be of no importance.

On the other hand, the geometry of group manifolds and the special functions of ncSGG and ihSGG are quite different. As long as we treat them formally in terms of algebras as spectrum-generating systems, they are indeed SGGs, because they give the same qualitative shapes of spectra. A more detailed discussion indicates that the SGG-triple may describe even slightly different systems [3].

In the gauging of $N = 8$ supersymmetry the contraction parameter ξ is a *field-redefinition parameter* (or one related to it) [17]. This gives a one-parameter family of gauged $N = 8$ supergravity theories. When $\xi = 1$ (or, e-quivalently, $\xi > 0$), one has the initial Nicolai-Wit model with $SU(8) \times SO(8)$ gauge symmetry, at $\xi = 0$ the $SU(8) \times ISO(7)$ gauging is obtained, for $\xi = -1$ ($\xi < 0$) a new, nonequivalent theory with $SU(8) \times SO(7,1)$ gauge symmetry appears. The contraction parameter enters into characteristic entities of these theories, for example, into the expression of the scalar potential added to the Lagrangian to mantain supersymmetry.

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Chapter 3

Cayley-Dickson algebras and their representations

In this chapter a unified treatment of algebras of central interest for this monograph is given. In Sec. 3.1 a very elementary and brief introduction to the *quaternion* and *octonion* algebras is presented. These algebras were the first nontrivial examples of noncommutative and nonassociative hypercomplex systems, and together with real and complex numbers they form an outstanding and exceptional quadruple of composition division algebras related by the classical *Cayley-Dickson procedure* (Sec. 3.2). In this context also some nondefinite and degenerate accompanying modifications appear naturally.

There are some mathematical and physical motivations to extend this finite quadruple of algebras beyond octonions.

Albert [1] has considered *generalized Cayley-Dickson algebras* extending the classical CD procedure without any further modification of the multiplication formula, whereby many good properties of the preceding algebras have got lost. We give a brief sketch of the properties of generalized CD algebras in Sec. 3.3. To be consistent some modification of the CD procedure on the transition $\mathbf{O} \to \mathbf{S}$ from octonions to *sedenions* is needed, where the **nonassociativity** of octonions is to be accounted for. It appears that such a consistent account of nonassociativity demands an introduction of a **ternary operation**, i.e. the next natural algebra in the "principal" chain of CD algebras is the *ternary sedenion algebra* [80, 44] (Sec. 3.4.). So the whole set of algebras originating from the fundamental quadruple **R, C, H,**

O consists of a "principal" chain $\mathbf{R} \to \mathbf{C} \to \mathbf{H} \to \mathbf{O} \to \mathbf{TS} \to ?$, and of "collateral" series originating from each algebra of the principal chain without any further modification of the multiplication law (Sec. 3.4., Table 7).

The real physical world is given us from experiments as arrays of numbers, *results of experiments.* The apparatus mediating the physical reality and the theoretical cognition is the *theory of representations.* We have well-developed theory of representations for associative and Lie algebras (see, e.g. [62, 2, 26]), especially for the semisimple Lie algebras groups and for the Poincaré group and its subgroups. Under the persisiting influence of superstring theory numerous finite and infinite dimensional Lie algebras (as Kas-Moody and Virasoro algebras, W-algebras etc., see reviews [47, 27, 59, 78]) are now under investigation.

In the case of essentially (non-Lie) nonassociative algebras, when no homomorphism into matrix algebra is possible, there exists a general *bimodule representation* or *birepresentation* theory introduced by Eilenberg [16] (see also [71, 31]). In what follows we shall use this type of representation theory but for the sake of simplicity and expediency we are forced to modify it in Ch. 5 when studying Moufang loops and Mal'tsev algebras. In birepresentations nonassociativity appears in the form of matrix relations which will be called the *associative projection.* The representation problems will be discussed in Sec. 3.5 (general theory), Sec. 3.6 (alternative algebras including octonions), and Sec. 3.7 (ternary sedenions). The technical part (mainly in the form of tables) is presented in Appendices 1-3.

3.1 Quaternions and octonions

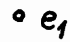
Quaternions were introduced by William Hamilton in 1843 [29], and *octonions* by John Graves and Arthur Cayley in 1845 [28, 7]. These number systems were the very first examples of hypercomplex numbers or algebras and they have had a significant impact on mathematics. Because of their beautiful and unique properties they have attracted many people to study and wonder at them, [45, 81, 9, 84] with an almost Pythagorean belief in their fundamentality. The uniqueness of quaternions and octonions has found a precise expression in the theorems of Frobenius and Hurwitz.

Quaternion (octonion) algebra may be defined as a linear algebra over

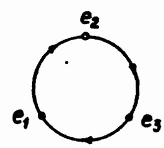
Table 4. **Hypercomplex units as finite projective configurations**

Real numbers
– the corresponding projective geometry consists of empty set of points (dim $=-1$)

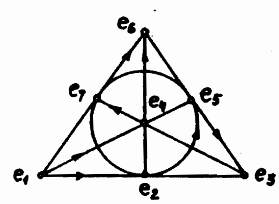
Complex numbers
– the corrresponding projective geometry consists of a single point (dim $= 0$)



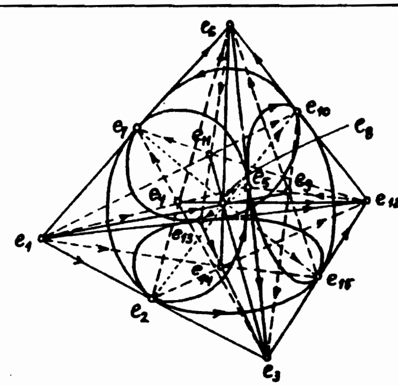
Quaternions
– the corresponding projective geometry consists of three collinear points (dim $= 1$)



Octonions
– the corresponding projective configuration consists of 7 points and 7 lines (3 points on each) (dim $= 2$)



Binary sedenions
– the corresponding projective 3-dimensional configuration consists of 15 points and 35 lines, 7 lines through every point; from purely technical reasons 18 lines are missing on figure



the field of real numbers $\mathbf{R}$ with a general element of the form

$$X = x_0 e_0 + x_i e_i\,, \quad x_0, x_i \in \mathbf{R}\,, \tag{3.1}$$

where $i = 1, 2, 3$ for the *quaternion* algebra $\mathbf{H}$, and $i = 1, 2, ..., 7$ for the *octonion* algebra $\mathbf{O}$; here and afterwards we assume a summation over repeating indices; e_i are quaternion (octonion) units satisfying (as a particular basis) the following multiplication rules:

$$\begin{aligned} e_i e_j &= -\delta_{ij} e_0 + \epsilon_{ijk} e_k\,, \\ e_i e_0 &= e_0 e_i = e_i\,, \\ e_0 e_0 &= e_0\,, \quad e_i^2 = e0\,, \end{aligned} \tag{3.2}$$

where δ_{ij} is the usual Kronecker symbol and ϵ_{ijk} is the Levi-Civita tensor for quaternions and a fully antisymmetric tensor with

$$\epsilon_{ijk} = +1, \quad ijk = 123, 145, 176, 246, 257, 347, 365 \tag{3.3}$$

for octonions.

We often call triples in (3.3) *cycles.* The most profound characterization of octonion units can be found in papers by Freudenthal [24] and van der Blij [85], in fact they form a configuration of points in a finite projective geometry (Table 4).

It is convenient to introduce a uniform terminology for the basic units (elements of the basis). We shall call e_0, e_1, e_2, e_3 ($e_0, e_1, e_2, ..., e_7$) *quaternion (octonion)* units, from these e_0 is the *unit element* of the algebra, the neutral element with respect of multiplication. The elements e_1, e_2, e_3 ($e_1, e_2, ..., e_7$) will be called *pure* quaternion (octonion) units, the elements e_2, e_3 (e_4, e_5, e_6, e_7), the *proper* quaternion (octonion) units. The element $e_2(e_4)$ will be called *essential* quaternion (octonion) unit, and the elements $e_1, e_2(e_1, e_2, e_4)$, the *generic* units for quaternions (octonions). This terminology may be straightforwardly generalized also for *sedenion* units (Sec. 3.3). Further in this chapter we shall deal mostly with octonions and the values 1,2,...,7 are assumed for all indices, $i, j, k, ...$, if not mentioned otherwise. In this section we do not specify the values of indices and formulas may be adjusted for both quaternions and octonions.

Due to linearity the multiplication of basic units determines also multiplication of general elements.

For convenience let us write the multiplication table explicitly.

Table 5. **Multiplication of octonion units**

$e_i \backslash e_j$	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

The multiplication table above relies upon a particular set of structure constants in (3.2) used by Hamilton and Cayley in their original papers. These particular definitions are also used in modern algebra textbooks with some slight modifications. Under linear transformations of the basis the structure constants transform according to the well-known tensor law and multiplication rules may be changed beyond recognition. In this monograph for quaternions, octonions and sedenions we use *multiplicative bases* where every product of any two basic units is zero or equals to some basic unit (but not to their linear combination). Only for these bases we can write simple multiplication tables (as Table 5). There exist 480 multiplicative bases of octonions which may be obtained through renumeration. From all multiplicative bases of octonions we have selected a *fundamental system* consisting of 16 bases which may be obtained from the original Cayley's one by means of reflections (see Appendix 1). This fundamental system of bases allows one to carry through some simple and beautiful constructions.

From two quaternion (octonion) units, e_i, e_j, their *commutator* $[e_i, e_j]$ and *anticommutator* $\{e_i, e_j\}$ can be formed:

$$[e_i, e_j] = 2\epsilon_{ijk} e_k \,, \tag{3.4}$$

$$\{e_i, e_j\} = -2\delta_{ij} e_0 \,. \tag{3.5}$$

From three octonion units, e_i, e_j, e_k, their *associator* (e_i, e_j, e_k) and *antiasso-*

ciator $\{e_i, e_j, e_k\}$ can be formed:

$$(e_i, e_j, e_k) = (e_i e_j)e_k - e_i(e_j e_k) = -\delta_{ij}e_k + \delta_{jk}e_i + (\epsilon_{ijp}\epsilon_{pkr} - \epsilon_{jkq}\epsilon_{iqr}) \,, \quad (3.6)$$

$$\{e_i, e_j, e_k\} = (e_i e_j)e_k + e_i(e_j e_k) = -\delta_{ij}e_k - \delta_{jk}e_i + (\epsilon_{ijp}\epsilon_{pkr} + \epsilon_{jkq}\epsilon_{iqr}) \,. \quad (3.7)$$

The pure quaternion units e_1, e_2, e_3 are anticommutative and associative, the pure octonion units $e_1, e_2, ..., e_7$ are anticommutative and antiassociative, the latter property meaning that the antiassociator vanishes for the basic triples from (3.3).

Because of linearity we can calculate all these quantities for arbitrary quaternions (octonions) of a general form (3.1).

It is easy to see from (3.2),(3.3) (or from Table 5) that the quaternion algebra is a subalgebra in the octonion algebra.

For an arbitrary quaternion (octonion) X the corresponding *conjugate* quaternion (octonion) $\bar{X}$ can be defined as

$$\bar{X} = x_0 e_0 - x_i e_i \,, \quad (3.8)$$

the conjugate mapping $X \to \bar{X}$ being an *involution*, i.e. $\bar{\bar{X}} = X, \overline{XY} = \bar{Y}\bar{X}$.

Then the *norm*

$$N(X) = X\bar{X} = \bar{X}X = (x_0^2 + x_i x_i)e_0 \quad (3.9)$$

and the *inverse* element (for $X \neq 0$)

$$X^{-1} = \frac{\bar{X}}{N(X)} \,, \quad X^{-1}X = XX^{-1} = e_0 \,. \quad (3.10)$$

can be defined for an arbitrary quaternion (octonion) X.

Norm (3.9) is nondegenerate and positively definite for both quaternions and octonions (over the field $\mathbf{R}$ of real numbers) and therefore every element $X \neq 0$ has the unique inverse element X^{-1}. It means that quaternion and octonion algebras defined insofar are *division algebras* (i.e. algebras with the unique division, without *zero divisors*), but as we shall see in the next Section, they have nonisomorphic companions with degenerate and nondefinite norm forms.

Speaking of the division in a noncommutative algebra left and right quotients must be considered. Let us have two quaternions or octonions, A, B;

$B \neq 0$. By the definition, the *left quotient* $B\backslash A$ of A divided by B "from the left" is the solution of the equation

$$BX = A\,,$$

and the *right quotient* A/B of A divided by B "from the right" is the solution of the equation

$$YB = A\,.$$

It is easy to verify that

$$X = B\backslash A = \frac{\bar{B}A}{N(B)}\,, \tag{3.11}$$

$$Y = A/B = \frac{A\bar{B}}{N(B)}\,. \tag{3.12}$$

In quaternion and octonion algebras the norm $N(X)$ satisfies the *composition law* well known for complex number algebra

$$N(XY) = N(X)N(Y)\,. \tag{3.13}$$

The algebras with such property are called *composition algebras*. The composition properties of the sums of four and eight squares were known long before the discoveries of quaternions and octonions (Euler, 1748, ref. in [87]; Degen 1822 [8], respectively).

In conclusion let us give some general characterization of quaternion and octonion algebras.

The quaternion algebra **H** is a simple associative noncommutative division algebra with involution. The octonion algebra **O** is a simple alternative nonassociative division algebra with involution. Both algebras are the normed composition ones. All alternative algebras are power-associative and so is the octonion algebra.

All these facts can be easily inferred from our previous treatment in Ch. 1. Further we restrict ourselves to real numbers as the basic underlying field of scalars, i.e. we shall deal only with the algebras over **R**. Such a restriction simplifies greatly the whole picture, because, as has said Hermann Weyl, [96], really bewildering things occur not in algebras themselves but in their underlying fields.

Finally let us give a brief sketch of uniqueness properties of quaternion and octonion algebras. As we already know, among the infinite set of all linear algebras there are some extraordinary and unique ones – the algebra of real numbers ($\mathbf{R}$), the algebra of complex numbers ($\mathbf{C}$), the algebra of quaternions ($\mathbf{H}$), and the octonion algebra ($\mathbf{O}$, will also be referred to as the *Cayley algebra*). Their uniqueness lies in some important properties shared by them and lacking in other algebras, these properties being summarized in the following theorems.

Frobenius' theorem [25], see also [40]:

Each associative division algebra is isomorphic either to the algebra of real numbers, to the algebra of complex numbers or to the quaternion algebra.

Generalized Frobenius' theorem [40]:

Each alternative division algebra is isomorphic either to the algebra of real numbers, to the algebra of complex numbers, to the quaternion algebra or to the octonion algebra.

Dimensions 1,2,4,8 are obligatory also for nonalternative finite-dimensional real division algebras (Bott, Milnor [3]).

Hurwitz' theorem [30]:

Each normed composition algebra with a unit element is isomorphic to one of the following algebras: the algebra of real numbers, the algebra of complex numbers, the quaternion algebra or the octonion algebra.

For elementary proofs of these theorems we refer to the books [34, 40]. Here the modern formulations of the Frobenius' and Hurwitz' theorems are given. In the original papers [25, 30] the problem was tackled in terms of quadratic forms. In 1938 Linnik [200] proved a theorem (in terms of algebras) from which both the Frobenius' and Hurwitz' theorems followed. In terms of algebras the Hurwitz' theorem was also proved by Albert [1] and Dubish [12] (see also refs. in [71], p. 73, and [84]). The algebras $\mathbf{R},\mathbf{C},\mathbf{H},\mathbf{O}$ are sometimes also called the *Hurwitz algebras* and denoted as $\mathbf{H}_i$, $i = 1, 2, 4, 8$.

BIBLIOGRAPHY (resumed). • Original papers: [29, 28, 7] • Bibliographies, reviews: [45, 81, 9, 43, 84, 79, 10] • About octonion units: [24, 85, 79] [37, 39, 36, 35, 38, 64] • Uniqueness theorems: [25, 30, 42, 1, 12, 40, 3] (see also [84, 71, 34]) • Composition properties: [8, 30, 42, 1, 12] • Automorphism group G_2: [99] (original paper) • Vector-matrix representation (and the alternative algebra context) [97, 98] • Bimodule representations: [24, 63, 79] • Relation with exceptional groups: [24, 83, 22, 21, 66, 23, 86, 19, 11, 20, 67]

• Construction of nonassociative algebras (including octonions) from representations of Lie algebras: [56, 57] • Related algebras and modifications; ternary composition algebras: [72, 73, 74, 75, 77, 76], Lie-admissible composition algebras (pseudo-octonions, etc.): [50, 48] [49, 51, 58, 52, 53, 54, 55], some other relations: [88, 89, 90, 91, 92, 93, 94, 95].

3.2 Cayley-Dickson procedure and Cayley-Dickson algebras

Let A be an n-dimensional linear algebra over a field F and with an identity element e_0 and *involution* $p \to \bar{p}$, (the latter meaning that $\forall p \in A, \exists \bar{p} \in A$: $p + \bar{p} = t(p)\,, p\bar{p} = \bar{p}p = n(p) \in F$; $\bar{\bar{p}} = p$; $\overline{pq} = \bar{q}\bar{p}$, $\forall p, q \in A$). Then we can construct the "doubled" algebras $A(\mu)$ of a dimension $2n$ (over F), consisting of the pairs $(p, P) : p, P \in A$ with the operations

$$
\begin{aligned}
(p,P)+(q,Q) &= (p+q, P+Q)\,, && (3.14)\\
\lambda(p,P) &= (\lambda p, \lambda P)\,, && (3.15)\\
(p,P)(q,Q) &= (pq-\mu\bar{Q}P, Qp+P\bar{q})\,, && (3.16)
\end{aligned}
$$

and with the identity element $\tilde{e_0} = (e_0, 0)$ and a new hypercomplex unit $e = (0, e_0)$ (with $e^2 = -\mu\tilde{e_0}$). The pairs may be represented as

$$(p,P) = (p,0) + (0,P) = p + Pe \tag{3.17}$$

with the multiplication rule (following from (3.16))

$$(p+Pe)(q+Qe) = pq - \mu\bar{Q}P + (Qp + P\bar{q})e \tag{3.18}$$

and involution

$$p + Pe \to \overline{p+Pe} = \bar{p} - Pe\,. \tag{3.19}$$

The elements $(p, 0)$, $p \in A$ form a subalgebra $A' \subset A(\mu)$ isomorphic to the initial algebra A.

The process described above is usually called the *Cayley-Dickson procedure*, [9] (cf. also [71]).

Proceeding from the real numbers $\mathbf{R}$, we have at the first step the algebras $\mathbf{R}(\mu_1)$ with the general elements

$$c = x_0e_0 + x_1e_1\,, \quad e_1^2 = -\mu_1 e_0\,, x_0, x_1; \mu_1 \in \mathbf{R}\,, \tag{3.20}$$

with the norm

$$N(c) = c\bar{c} = \bar{c}c = x_0^2 + \mu_1 x_1^2\,. \tag{3.21}$$

Taking the values $\mu_1 = \pm 1, 0$, we have after the first step three number algebras: *complex numbers* $\mathbf{C}$ ($\mu_1 = 1$), *double (split complex) numbers* ($\mu_1 = -1$) and *dual numbers* ($\mu_1 = 0$). These algebras may be denoted straightforwardly as $\mathbf{R}(1)$, $\mathbf{R}(-1)$, $\mathbf{R}(0)$, respectively (in Rosenfeld's [65] denotations $R(i), R(e), R(\varepsilon)$, respectively, where i is the proper *imaginary* unit ($i^2 = -1$), e is the *double number* unit ($e^2 = 1$), and ε the *dual* unit ($\varepsilon^2 = 0$)).

From (3.21) we can see that only the complex number algebra is the normed division algebra.

For double and dual numbers and some other higher-dimensional modifications as semiquaternions, etc., we refer to an earlier exposition of hypercomplex numbers by Cartan and Study in the *Encyclopédie des Sciences Mathématiques* [81] (see also the bibliography [45]) and to the book [65] by Rosenfeld.

After the second step we have the algebras $\mathbf{R}(\mu_1, \mu_2)$. Here we give general elements and norm forms for the algebras $\mathbf{R}(1, \mu_2)$:

$$\begin{aligned}
q &= c_1 + c_2 e_2\,; \quad e_2^2 = -\mu_2 e_0\,, \quad c_1, c_2 \in \mathbf{C}\,, && (3.22)\\
c_1 &= x_0 e_0 + x_1 e_1\,, \quad c_2 = x_2 e_0 + x_3 e_1 \quad ; x_0, x_1, x_2, x_3 \in \mathbf{R}, &&\\
q &= x_0 e_0 + x_1 e_1 + x_2 e_2 + x_3 e_3\,, \quad e_3 = e_1 e_2\,; &&\\
N(q) &= q\bar{q} = \bar{q}q = x_0^2 + x_1^2 + \mu_2(x_2^2 + x_3^2)\,. && (3.23)
\end{aligned}$$

There are Hamilton's *quaternions* $\mathbf{H}$ ($\mu_1 = \mu_2 = 1$) , *split quaternions* or *antiquaternions* ($\mu_1 = 1, \mu_2 = -1$), and *semiquaternions* ($\mu_1 = 1, \mu_2 = 0$)here. These algebras can be denoted as $\mathbf{R}(1{,}1)$, $\mathbf{R}(1{,}-1)$, $\mathbf{R}(1{,}0)$, respectively (in Rosenfeld's denotations $R(i,j), R(i,e), R(i,\varepsilon)$). In addition to these three algebras there are two others among the algebras $\mathbf{R}(\mu_1, \mu_2)$ – the *antisemiquaternions* $\mathbf{R}(-1{,}0)$, and "zero" quaternions $\mathbf{R}(0{,}0)$. The following isomorphisms may be arranged: $\mathbf{R}(1,-1) \simeq \mathbf{R}(-1,1)$, $\mathbf{R}(-1,-1) \simeq \mathbf{R}(1,-1)$, $\mathbf{R}(0,1) \simeq \mathbf{R}(1,0)$, $\mathbf{R}(0,-1) \simeq \mathbf{R}(-1,0)$.

At the third step, starting with quaternions, we are lead to the algebras $\mathbf{R}(1, 1, \mu_3)$ with general elements and norms

$$\begin{aligned}
X &= q_1 + q_2 e_4\,; \quad q_1, q_2 \in \mathbf{H}\,, \quad e_4^2 = -\mu_3 e_0\,, && (3.24)\\
q_1 &= x_0 e_0 + x_1 e_1 + x_2 e_2 + x_3 e_3\,,\ q_2 = x_4 e_0 + x_5 e_1 + x_6 e_2 + x_7 e_3\,, &&
\end{aligned}$$

$$\begin{aligned} X &= x_0e_0 + x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 + x_5e_5 + x_6e_6 + x_7e_7\,, \\ e_5 &= e_1e_4\,,\ e_6 = e_2e_4\,,\ e_7 = e_3e_4\,, \\ N(X) &= X\bar{X} = \bar{X}X = q_1\bar{q}_1 + \mu_3 q_2\bar{q}_2 = \\ &= x_0^2 + x_1^2 + x_2^2 + x_3^2 + \mu_3(x4^2 + x_5^2 + x_6^2 + x_7^2)\,; \end{aligned} \tag{3.25}$$

there are Cayley's *octonions* **O** ($\mu_3 = 1$), *split octonions* or *antioctonions* ($\mu_3 = -1$), and *semioctonions* ($\mu_3 = 0$) here. We denote these algebras **R**(1,1,1), **R**(1,1,−1), **R**(1,1,0) respectively (Rosenfeld's denotations: $R(i,j,k), R(i,j,e), R(i,j,\varepsilon)$). Among the algebras $\mathbf{R}(\mu_1,\mu_2,\mu_3)$ there are four more new algebras **R**(1,−1,0), **R**(1,0,0), **R**(−1,0,0), **R**(0,0,0). Isomorphisms between the algebras $\mathbf{R}(\mu_1,\mu_2,\mu_3)$ have been arranged in [61](see also [65]).

In what follows, under the *Cayley-Dickson algebras* (CD-*algebras*) we mean the algebras obtained by successive CD-processes from a field **F** (*char* $\neq$ 2). (In fact we consider only the algebras over **R**, but to formulate some facts about higher CD-algebras we must have at hand a general field).

For each first three steps we have a division algebra ($\mu = 1$), a split algebra ($\mu = -1$) and an algebra with a nilideal ($\mu = 0$) at each step.

With these first three steps binary alternative algebras are exhausted. The CD process gives alternative algebras only if the initial algebra is associative. Further there are no division algebras over **R**, [1] (for sedenions it was noticed by Schafer [70]), but there is always a suitable initial field F, which gives a division algebra for the m-th step, [4].

BIBLIOGRAPHY (resumed). • Original paper: [52] • About modifications (as dual and double numbers, etc.): [45, 81, 65, 61, 71] • Generalized CD-algebras (Sec.3.3): [1, 4, 17, 70, 46]

3.3 Generalized Cayley-Dickson algebras. Binary sedenions

Sedenion algebras and further algebras obtained in the steps $r \geq 4$ are called *generalized Cayley-Dickson algebras* [1, 70].

All CD-algebras and generalized CD-algebras (steps $r = 1, 2, \ldots$) share the following properties [70]:

- centrality
- simplicity
- flexibility
- power-associativity
- Jordan-admissibility
- degree two: $x^2 - t(x)x + n(x) = 0$, $\forall x \in A$
- derivation algebra G_2 for all A, $r \geq 3$
- squares of basic units equal $-\tilde{e}_0$ (for all $\mu_i = 1$)

Generalized CD-algebras are nonalternative, also division algebras (over **R**) are lacking (Albert, [1]; Schafer, [70]). But there always exists such a ground field **F** on starting with which we get (after n steps) a generalized CD-algebra which is a division algebra [4].

In [17] some *weakened versions* of alternativity and composition properties are considered. In the algebras $A_m = A_3(\mu_4, ..., \mu_m)$, $m > 4$ for all $x \in A_m$, $y \in A_3$ the following properties of *weakened alternativity* are satisfied: $xy^2 = (xy)y, y^2x = y(yx)$. For the algebras A_m also some *partial composition* properties are present: there exists a subalgebra $S \subset A_m$ and a nondegenerate quadratic form q such that $q(xs) = q(sx) = q(x)q(s)$, $\forall s \in S$, $\forall x \in A_m$.

Table 6. **Multiplication of sedenion units**

i\j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	-0	3	-2	5	-4	-7	6	9	-8	-11	10	-13	12	15	-14
2	2	-3	-0	1	6	7	-4	-5	10	11	-8	-9	-14	-15	12	13
3	3	2	-1	-0	7	-6	5	-4	11	-10	9	-8	-15	14	-13	12
4	4	-5	-6	-7	-0	1	2	3	12	13	14	15	-8	-9	-10	-11
5	5	4	-7	6	-1	-0	-3	2	13	-12	15	-14	9	-8	11	-10
6	6	7	4	-5	-2	3	-0	-1	14	-15	-12	13	10	-11	-8	9
7	7	-6	5	4	-3	-2	1	-0	15	14	-13	-12	11	10	-9	-8
8	8	-9	-10	-11	-12	-13	-14	-15	-0	1	2	3	4	5	6	7
9	9	8	-11	10	-13	12	15	-14	-1	-0	-3	2	-5	4	7	-6
10	10	11	8	-9	-14	-15	12	13	-2	3	-0	-1	-6	-7	4	5
11	11	-10	9	8	-15	14	-13	12	-3	-2	1	-0	-7	6	-5	4
12	12	13	14	15	8	-9	-10	-11	-4	5	6	7	-0	-1	-2	-3
13	13	-12	15	-14	9	8	11	-10	-5	-4	7	-6	1	-0	3	-2
14	14	-15	-12	13	10	-11	8	9	-6	-7	-4	5	2	-3	-0	1
15	15	14	-13	-12	11	10	-9	8	-7	6	-5	-4	3	2	-1	-0

Taking the Cayley's original multiplication table for the octonion units (Table 5) we get through the doubling CD procedure a particular multiplication table for the binary sedenion units $\tilde{e}_0, \tilde{e}_i (i = 1, 2, ..., 15)$ (Table 6, where only the numbers of the indices k are indicated for the nonzero structure constants $c_{ij}^k = \pm 1$ with the corresponding signs; here $e_i e_j = c_{ij}^k e_k$, $i, j, k = 0, 1, 2, ..., 15$; in this table also multiplication tables of quaternions and octonions are contained).

Naturally, this algebra is obtainable from the real number algebra $\mathbf{R}$ after four steps and may be denoted as $\mathbf{R}(1,1,1,1)$. In what follows we use also the denotation $B\mathbf{S}$. The term *sedenion* appeared first in the title of Sylvester's paper [82], 1884.

Some survived properties (compared with octonions) may be listed:

- the ones satisfied for all generalized CD algebras;
- anticommutativity of basic units;
- multiplication of basic units satisfies "nonlinear" alternative laws $(xx)y = x(xy)$, $(xy)y = x(yy)$ (but not for general elements!);

There are also properties which are violated, e.g.

- antiassociativity for the noncyclic triplets of octonion units: $(e_i e_j)e_k = -e_i(e_j e_k)$, $ijk \neq 123, 145, 176, 246, 257, 347, 365$;
- "linear" alternativity (3.41) for basic units and general elements;
- the *Moufang identity* is modified by an additional associator term:

$$(e_i e_j)(e_k e_i) = e_i(e_j e_k)e_i + (e_i, e_j, e_k e_i)\,. \qquad (3.26)$$

3.4 Ternary sedenions

Cayley-Dickson multiplication formula (3.16) was combined by Dickson [9] to get 8-dimensional alternative algebras. We can see that in every step from real numbers to octonions this formula internally allows for a certain new property for the "doubled" algebra (see Table 7):

1) the *square of thenew unit* equals minus identity (for $\mu = 1$);

2) the presence of *involution* as a complex conjugation in the second step and the involutive conjugation for quaternions and further algebras;

3) *noncommutativity* of quaternions (and further algebras).

Table 7. **Algebras related to CD-procedure**

Real numbers			
1 ▽ **Complex numbers** as pairs of real numbers	2 ▷ **Quaternions** as pairs of complex numbers	3 ▷ **Octonions** as pairs of quaternions	4 ▷ 5 ▷? **Ternary sedenions** as pairs of octonions
▽ 2^n-dim.($n \geq 1$) *complex commutative algebras* with binary multiplication rule $(p, P)(q, Q) =$ $(pq - PQ,$ $pQ + Pq)$	▽ 2^n-dim.($n \geq 2$) *Clifford-like algebras* with binary multiplication rule $(p, P)(q, Q) =$ $(pq - P\bar{Q},$ $pQ + P\bar{q})$	▽ 2^n-dim.($n \geq 3$) *generalized CD-algebras* with binary multiplication rule $(p, P)(q, Q) =$ $(pq - \bar{Q}P,$ $Qp + P\bar{q})$	▽ 2^n-dim.($n \geq 4$) *ternary algebras* with ternary multiplication rule (3.27)
$n = 1$: complex numbers	$n = 2$: quaternions	$n = 3$: octonions	$n = 4$: ternary sedenions
$n = 2$: bicomplex numbers ...	$n = 3, 4$: algebras resembling biquaternions and Dirac algebra ...	$n = 4$: binary sedenions ...	$n = 5$: ternary algebra of sedenion pairs ...

For the fourth step (from octonions to sedenions) the Dickson formula gives no further property to allow for, in fact it disregards the nonassociativity of octonions. The extension of the CD-procedure into the region of generalized CD-algebras seems somewhat formal.

We can proceed with an unchanged multiplication rule at every step, then we get "collateral" series of new algebras originating "sideways" from each algebra of the main sequence (see Table 7).

Let us try now to generalize the Dickson formula for the fourth step (from octonions to sedenions) so that the nonassociativity of octonions would be accounted for. It is clear that nonassociativity may appear only when there are three elements of the algebra combined, therefore the binary multiplication rule cannot account for nonassociativity at all and we must introduce some kind of *ternary operation*, the new meaningfully generalized algebra will be a *ternary algebra*, [80, 44].

Let us define the corresponding ternary $*$-*associator product* for arbitrary three sedenions $A = x + Xe$, $B = y + Ye$, $C = z + Ze$; $x, y, z, X, Y, Z \in \mathbf{O}$:

$$*(A,B,C) = *(AB)C - A(BC)* =$$

$$\begin{aligned} *[(x+Xe)(y+Ye)](z+Ze) - (x+Xe)[(y+Ye)(z+Ze)]* = \\ = (xy)z - \underline{\bar{Y}(Xz)} - \bar{Z}(Yx) - \bar{Z}(X\bar{y}) - \\ -x(yz) + \underline{(x\bar{Z})Y} + (\bar{y}\bar{Z})X + (z\bar{Y})X + \\ [\underline{(Zx)y} - Z(\bar{Y}X) + (Yx)\bar{z} + (X\bar{y})\bar{z} - \\ -(Zy)x - (Y\bar{z})x - \underline{(X\bar{z})\bar{y}} + X(\bar{Y}Z)]e\,. \end{aligned} \qquad (3.27)$$

The difference between the $*$-product $*(A, B, C)$ and the associator (A, B, C) computed in the binary sedenion algebra (i.e. according to the (3.16)) lies in the position of brackets in the underlined terms (see also Fig. 1). We call the 16-dimensional (sedenion) algebra with the ternary $*$-product (3.27) *ternary sedenion algebra* and its elements, *ternary sedenions.*

$*$-associator product has the common linearized alternativity property

$$*\,(A,B,C) = (-1)^{\sigma} *\,(PA,PB,PC)\,, \qquad (3.28)$$

where P represents some permutation and σ is the parity of the permutation. So it may be said that the alternativity property is restored, and so is the

Multiplication structure of ternary sedenions

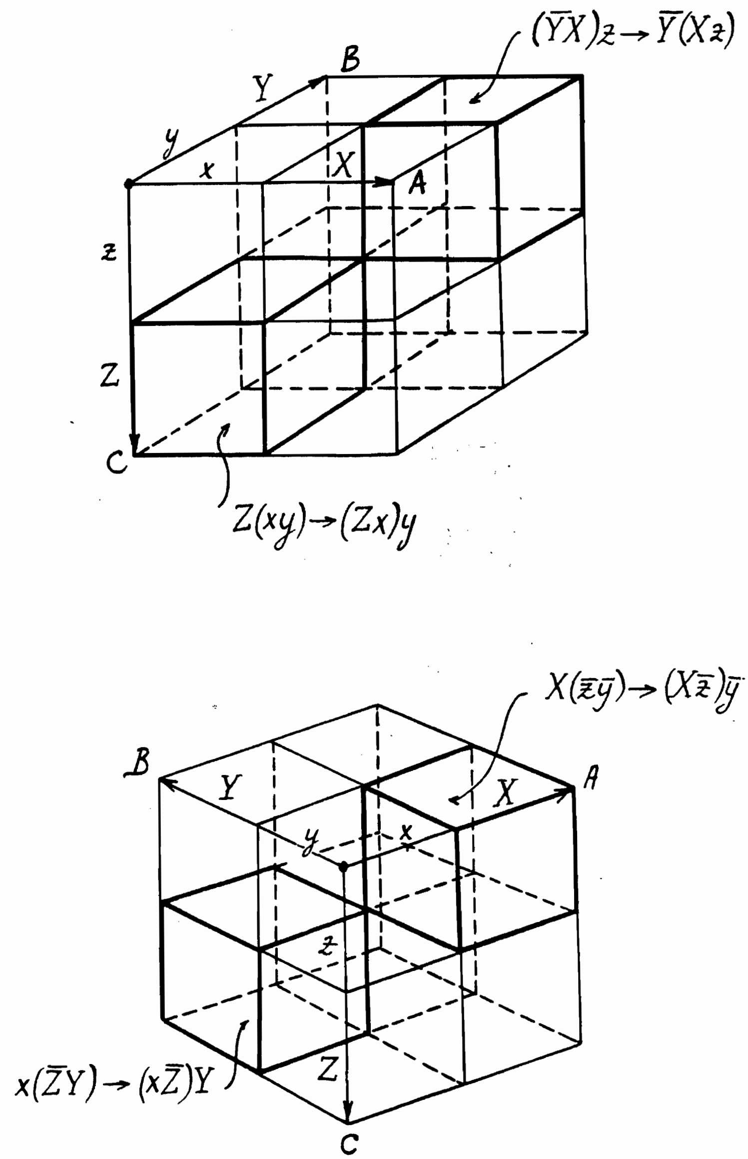


Fig. 1.

antiassociativity property for the basic units (now in terms of $*$-product). We shall return to the comparison of binary and ternary sedenion algebras when we discuss the problems of hypercomplex analysis (in Ch. 4).

There is a connection with the binary operation through the *half-ternary* products $*(AB)C$ and $*A(BC)$ which may be traced from (3.27)

$$*\,(AB)e_0 = A(Be_0)* = AB, \tag{3.29}$$

where AB is the product of A and B in the binary sedenion algebra.

However, it must be said that these properties of our algebra are not purely ternary, because this modification has been derived from binary algebra. The wievpoint of pure ternary operation demands the consideration of associativity relations involving five elements, etc. We shall not discuss this problem here.

3.5 Bimodule representations of nonassociative algebras

Let A be a finite-dimensional linear algebra (over a field $\mathbf{F}$) with a generally nonassociative binary multiplication.

Definition. A linear (vector) space V is called *A-bimodule* (over $\mathbf{F}$) if the following linear maps are defined

$$L: \qquad A \times V \to V, \quad L(a,v) \equiv av \ \in V\,, \tag{3.30}$$

$$R: \qquad V \times A \to V, \quad R(v,a) \equiv va \ \in V\,. \tag{3.31}$$

The vector space V is called the *left* or the *right A-module* if only the map L or R is defined, respectively.

A natural example of an A-bimodule is the algebra A itself if the operators of the *left* and the *right multiplication* are defined:

$$L(a,x) \;=\; L_a x = ax\,, \tag{3.32}$$

$$R(x,a) \;=\; R_a x = xa; \quad a, x \in A\,. \tag{3.33}$$

This bimodule is called the ***regular*** or the *natural A-bimodule.*

In a fixed basis the operators L_a, R_a are the matrices acting upon the elements of the algebra A as vectors of the linear space of the algebra. They form the *associative multiplication algebra* $\mathcal{M}(A)$ of the algebra A. This algebra may be considered as the simplest, *regular bimodule representation* or *regular birepresentation* (or even *regbirep* in a short argot mode). For our purposes in what follows it suffices to deal with this particular type of birepresentation.

Definition. Let us have an A-bimodule V over a field F. We can write mappings (3.30),(3.31) in the form

$$v \mapsto S_a v = L(a,v) \equiv av\,, \tag{3.34}$$
$$v \mapsto T_a v = R(v,a) \equiv va\,. \tag{3.35}$$

A pair of linear mappings from the algebra A into the space $L(V)$ of A-bimodule

$$S: \quad a \mapsto S_a \in L(V)\,, \tag{3.36}$$
$$T: \quad a \mapsto T_a \in L(V)\,, \tag{3.37}$$

is called *bimodule representation* or *birepresentation* (S,T) of the algebra A.

Birepresentation is completely determined by the bimodule and *vice versa*, these are equivalent concepts. Every algebra A has at least one birepresentation corresponding to the regular bimodule of the algebra.

In birepresentations of nonassociative algebras one of the principal properties of common representations of associative algebras, the *homomorphism* property gets lost. In the regular birepresentation it may be demonstrated very easily:

$$(a,b,x) = (L_aL_b - L_{ab})x\,, \quad (x,a,b) = (R_bR_a - R_{ab})x\,,$$

In the regular birepresentation, the **nonassociativity of the algebra** A **is measured by the noncommutativity of its** L- and R-**matrices:**

$$(a,x,b) = (ax)b - a(bx) = [R_b, L_a]x\,; \quad \forall a,x,b \in A\,. \tag{3.38}$$

In the case of associative algebra the birepresentation concept reduces to the concept of common representation by homomorphism.

From the defining identities of algebra relations for the operators of the bimodule (S,T) follow, which may be obtained, by substituting $(S,T) \leftrightarrow (L,R)$. Below we give some types of identities in terms of L-,R-operators.

The identities containing equal elements must be linearized. In general, if we have some identity $(x_1, x_2, ..., x_n) = 0$ for (different) elements of the algebra A, then the linear space V is an A-bimodule if the following conditions,

$$(v, x_2, ..., x_n) = (x_1, v, ..., x_n) = ... = (x_1, x_2, ..., v) = 0 , \tag{3.39}$$

are satisfied for all $x_1, x_2, ..., x_n \in A$, $v \in V$.

Birepresentations have been studied for alternative algebras [68, 69, 33, 5], Jordan algebras [33, 5, 32], Mal'tsev algebras [41, 6] and Moufang loops (for the two latter see Refs. in Ch. 5).

Here we describe briefly the regular birepresentation defined by Eqs. (3.32),(3.33). Let us formulate some essential properties of the operators L_a, R_a of left-right multiplications.

• The operators L_a, R_a are linear operators of the vector space $V(A)$ of the algebra A, therefore there exists the matrix form $ax = L_a x$, $xa = R_a x$ of the action of these operators.

• The sets $L_A = \{L_a;\ a \in A\}$, $R_A = \{R_a;\ a \in A\}$ of operators are linear subspaces in the linear space $L(V(A))$ of all linear operators of the linear space $V(A)$ of the algebra A.

Definition. The algebra generated by the spaces of operators L_A, R_A is called an *associative multiplication algebra* $\mathcal{M}(A)$ of the algebra A.

• To an ideal $L \subset A$ there corresponds an invariant subspace in the multiplication algebra $\mathcal{M}(A)$.

• The algebra A is simple iff $\mathcal{M}(A)$ is an irreducible set of operators.

• If the algebra A is a division algebra, then the operators (matrices) are regular for each $a \in A$, and

$$L_{a^{-1}} = L_a^{-1}, \quad R_{a^{-1}} = R_a^{-1}. \tag{3.40}$$

A following paraphrased definition may be given: A pair of linear maps $a \to L_a, R_a$ from the algebra A into the multiplication algebra $\mathcal{M}(A)$ is called *regular birepresentation* or *L-,R-representation* of the algebra under consideration.

Defining identities (or relations) between the elements of the algebra can be represented through the operators L_a, R_a:

• Commutativity: $L_a = R_a$

• Anticommutativity: $L_a = -R_a$

- Lie algebras (anticommutativity + Jacobi identity):

$$a(bc) + c(ab) + b(ca) = 0 \ \Rightarrow\ [L_a, L_b] = L_{ab}$$

- Associative algebras:

$$L_{ab} = L_a L_b\,, \quad R_{ab} = R_b R_a\,, \quad R_b L_a = L_a R_b$$

- Alternative algebras (see the next Section).

3.6 Birepresentations of alternative algebras. Octonions and $SO(8)$-group

3.6.1 Birepresentations of alternative algebras

Alternative algebras have been defined as algebras with an alternating associator

$$(x, y, z) = (-1)^P (Px, Py, Pz). \tag{3.41}$$

This is a linearized form of the left and the right alternativity conditions (1.1) not suitable for the representation theory.

From (3.41) we get the relations between the operators of the regular birepresentation

$$L_{ab} = L_a L_b + [L_a, R_b]\,, \tag{3.42}$$

$$R_{ba} = R_a R_b + [R_a, L_b]\,, \tag{3.43}$$

$$[L_a, R_b] = [R_a, L_b] \quad \forall a, b \in A\,. \tag{3.44}$$

According to the correspondence $(L, R) \leftrightarrow (S, T)$ we have also the same relations between the operators of a general birepresentation:

$$[S_a, T_b] = S_{ab} - S_a S_b = T_{ba} - T_a T_b = [T_a, S_b]\,. \tag{3.45}$$

For alternative algebras we have also a concept of *Casimir operator* [69, 5] well known from the theory of Lie algebras. Let us brief here some main points.

Let A be an alternative algebra with some fixed basis $\{e_1, e_2, ..., e_n\}$. For every birepresentation $(S, T) : x \mapsto S_x, T_x$ there are defined invariant symmetric bilinear *trace forms* $\mathrm{Tr}(S_x S_y)$, $\mathrm{Tr}(T_x T_y)$.

The algebra A is called *nondegenerate* if at least one of the trace forms is nondegenerate. Such an algebra is decomposable into the direct sum of simple ideals.

Let us choose another basis $\{e'_1, e'_2, ..., e'_n\}$ in A such that $\mathrm{Tr}(R_i R'_j) = \delta_{ij}$, where R_i, R'_j are R-operators of the regular birepresentation of the algebra, corresponding to the elements of the bases e_i, e'_j, respectively. Now we can define the *Casimir operators of the birepresentation* (S, T):

$$\Gamma_S = \sum_{i=1}^{n} S_i S'_i \,, \quad \Gamma_T = \sum_{i=1}^{n} T_i T'_i \,, \tag{3.46}$$

where $S_i = S_{e_i}$, $T_i = T_{e_i}$, $S'_i = S_{e'_i}$, $T'_i = T_{e'_i}$.

For a nondegenerate alternative algebra A the Casimir operators Γ_S, Γ_T of the birepresentation (S, T) commute with all operators S_x, T_x, $\forall x \in A$, moreover, $\Gamma_S = S_e$, $\Gamma_T = T_e$, where e is the identity element of the algebra A. Casimir operators for the regbirep are identity operators.

It can be said that the most interesting alternative algebra is the algebra of octonions, therefore we shall continue with some elaboration of the representation aspect for the octonion algebra. In the next chapter we shall apply this apparatus to the Dirac equation and some other problems.

3.6.2 Regular birepresentation of octonions

The *regular birepresentation* (regbirep) of the octonion algebra is determined when L-,R-operators for the octonion units e_i, $i = 0, 1, ..., 7$ are given. The matrices of these operators consist essentially of the structure constants of the algebra:

$$\begin{aligned} e_i \to L_i \equiv L_{e_i} : \quad & e_i x = L_i x \,, \quad L_i = (c_i) \,; \\ e_i \to R_i \equiv R_{e_i} : \quad & x e_i = R_i x \,, \quad R_i = (\tilde{c}_i) \,; \\ (c_i)^k_j = c^k_{ij},\ (\tilde{c}_i)^k_j = c^k_{ji} = -c^k_{ij}; \quad & i, j, k = 0, 1, ..., 7, \end{aligned} \tag{3.47}$$

where the upper index (k) indicates the row number, the lower index (j), the column number, and the index i inside parentheses is the matrix number. One can write straightforwardly (from (3.42-3.44)):

$$c^k_{ij} L_k = L_i L_j + [L_i, R_j] \,,$$

$$\tilde{c}^k_{ij} R_k = R_i R_j + [R_i, L_j] \, , \qquad (3.48)$$
$$[L_i, R_j] = [R_i, L_j] \, .$$

In Appendix 2 the matrices L_i, R_i are given for all fundamental modifications of the octonionic basis. Due to linearity we can now compute the regbirep matrices L_x, R_x for all general elements x of the octonion algebra.

According to our guiding principles, if nonassociativity has to play some role in physics, it must be exhibit itself through some kind of *associative projection*, for which we have an ideal apparatus — nontrivial bimodule representations with two noncommuting sets of the matrices S_i, T_i.

Let us give here for convenience and completeness some details about the L-,R-matrices which will prove useful in the next chapter.

Let us write Eqs.(3.48) for general octonions x, y

$$xy \to L_{xy} = L_x L_y + [L_x, R_y] \equiv L_x * L_y \, , \qquad (3.49)$$
$$xy \to R_{xy} = R_y R_x + [R_x, L_y] \equiv R_x * R_y \, , \qquad (3.50)$$

where the asterisk denotes a new kind of operation defined by the equations above.

From the *Moufang identity*

$$(db)(ad) = d((ba)d) \, , \quad \forall a, b, d \in \mathbf{O} \qquad (3.51)$$

the $*$-operation for R-matrices can be expressed as follows:

$$R_a * R_b = L_b R_b R_a L_b = R_{ab} \, . \qquad (3.52)$$

Let us stop here for a while on the R_i-matrices of the octonion regbirep which will be used for the expression of Dirac γ-matrices.

R_i-matrices satisfy the following equations:

$$\{R_i, R_j\} = R_i R_j + R_j R_i = -2\delta_{ij} \mathrm{I} \, . \qquad (3.53)$$

They are anticommuting antisymmetric matrices forming the 64-dimensional Clifford algebra C_6 with 6 generic elements ($R_1, R_2, R_3, R_4, R_5, R_6$ for example, as $R_1 R_2 R_3 R_4 R_5 R_6 = R_7$).

For R_i-matrices $*$-operation (3.50) may be represented in a "projective" form:

$$R_i * R_j = \Gamma_{(ijk)} R_j R_i = R_{ij} \, , \qquad (3.54)$$

where (ijk) denotes the positive cycle of the octonion units containing the indices i, j. The products $\Gamma_{(ijk)} = R_i R_j R_k$ with ijk positive cycles (3.3) are diagonal matrices of a *reflection type* (i.e. with ± 1 on the main diagonal).

The R_i-matrices corresponding to different multiplication tables are different. The transition between two tables may be carried out by L_i-matrices. From Moufang identity (3.51) it follows directly that

$$- L_i R_{ji} L_i = R_i R_j \tag{3.55}$$

and the transition formula looks like

$$- L_i^{(0)} R_j^{(0)} L_i^{(0)} = R_j^{(i)} , \tag{3.56}$$

where the upper indices in parentheses denote the numbers of table modification.

For nonassociative algebras, and for octonions among them, there is no such well-developed representation theory as for semisimple Lie algebras. But the octonion algebra is insofar the only algebra for which a whole series of nontrivial birepresentations different from the regbirep can be indicated without any special studies. It is possible due to a specific relation between octonions and the rotation group in 8 dimensions, the group $SO(8)$.

3.6.3 Birepresentations of octonions and the group $SO(8)$

The group $SO(8)$, the invariance group of the octonion norm, is the largest group which can be related to the octonionic space with one dimension. The exceptional groups F_4, E_6, E_7, E_8 are groups of various geometries of the octonionic plane. Here we confine ourselves only to a very brief consideration of the group $SO(8)$ and some of its interesting subgroups.

The group $SO(8)$ has an interesting spectrum of irreducible representations. In dimension 8 there exist 3 nonequivalent irreducible representations: $\bar{8}^{s_1}, \bar{8}^{s_2}$ (so-called *halfspinors*) and $\bar{8}^{v}$ (vector) (in what follows we shall use for the corresponding Lie algebra D_4 the same notations without bars!). Analogous triples of irreps appear also in certain higher dimensions 35, 56, 112, ... (see [18]). Irreps in the triple may be transformed into each other by the *outer automorphisms* of the group, these are in fact the permutations S_3 of the scheme of simple roots of the corresponding Lie algebra. In fact it is

Table 8. **Low-dimensional IRs of $SO(8)$**

IR $D^{\dim}(\ell_1,\ell_2,\ell_3,\ell_4)$	Highest weight components $\lambda_1,\lambda_2,\lambda_3,\lambda_4$	Short design.	Branching rules → $SO(7)\equiv B_3$	G_2	$SU(3)\equiv A_2$
$D^1(0,0,0,0)$ scalar	0,0,0,0	1	1	1	1
$D^8(1,0,0,0)$ vector	1,0,0,0	8^v	1,7	1,7	8
$D^8(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ semispinor of the 1st kind	0,1,0,0	8^{s1}	8	1,7	8
$D^8(\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2})$ semispinor of the 2nd kind	0,0,1,0	8^{s2}	8	1,7	8
$D^{28}(1,1,0,0)$ regular	0,0,0,1	28	7,21	7,7,14	$8,10,\overline{10}$
$D^{35}(2,0,0,0)$	2,0,0,0	35	1,7,27	1,7,27	8,27
$D^{35}(1,1,1,1)$	0,2,0,0	35′	35	— " —	— " —
$D^{35}(1,1,1,-1)$	0,0,2,0	35″	35	— " —	— " —
$D^{56}(1,1,1,0)$	0,1,1,0	56	21,35	1,7,7,14,27	$1,8,10,\overline{10},27$
$D^{56}(\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	1,1,0,0	56′	8,48	— " —	— " —
$D^{56}(\frac{3}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2})$	1,0,1,0	56″	8,48	— " —	— " —

The numbers $\ell_1,\ell_2,\ell_3,\ell_4$ are simultaneously integers or half-integers, $\ell_1 \geqslant \ell_2 \geqslant \ell_3 \geqslant 0$, ℓ_4 may be positive or negative (or 0); $\lambda_1=\ell_1-\ell_2$, $\lambda_2=\ell_3+\ell_4$, $\lambda_3=\ell_3-\ell_4$, $\lambda_4=\ell_2-\ell_3$.

the origin of the appearance of such triples of irreps. This situation has a consequence called *triality*, [24]. In the Lie algebra D_4

$$AX \cdot Y + X \cdot AY = C(X \cdot Y)\,; \quad X, Y \in \mathbf{O}\,, \tag{3.57}$$

where $A, B, C \in 8^{s_1}, 8^{s_2}, 8^v$ respectively, and the dot denotes the octonionic multiplication. In terms of group irreps

$$\varphi_1(X) \cdot \varphi_2(Y) = \varphi(X \cdot Y)\,; \quad X, Y \in \mathbf{O}\,, \tag{3.58}$$

$\varphi_1, \varphi_2, \varphi \in \bar{8}^{s_1}, \bar{8}^{s_2}, \bar{8}^v$ respectively. In the reduction (branching) $SO(8) \to G_2$ three nonequivalent 8-dimensional irreps reduce similarly into the direct sum $1 \oplus 7$. In the reduction $SO(8) \to SO(7)$ both halfspinors reduce to the 8-dimensional spinor irrep of the group $SO(7)$, and the vector is subject to the canonical reduction $1 \oplus 7$. In the reduction $SO8) \to SU(3)/Z_3$ all 8-dimensional irreps reduce to the octet irrep of the Eightfold Way group.

The generators for the 8-dimensional irreps of the algebra D_4 in terms of regbirep matrices of octonions have been given by Freudenthal [24]. For vector representation 8^v the generators can be written as follows:

$$G_{ij} = [L_i + R_i, L_j + R_j]\,; \quad i, j = 1, 2, ..., 7\,; \tag{3.59}$$

$$G_{ij}e_j = e_i\,, \quad G_{ij}e_i = -e_j\,, \quad G_{ij}e_k = 0\,; \quad i \neq j \neq k\,, \tag{3.60}$$

satisfying the common commutation relations for the generators of the rotation group

$$[G_{ij}, G_{kl}] = (\delta_{ik}\delta_{jm}\delta_{ln} - \delta_{jk}\delta_{im}\delta_{ln} + \delta_{jl}\delta_{im}\delta_{kn} - \delta_{il}\delta_{jm}\delta_{kn})G_{mn}\,. \tag{3.61}$$

The generators of halfspinor irreps may be represented as follows:

$$\{L_{ij}\}: \quad L_{i0}x = L_i x = e_i x\,, \quad L_{ij}x = L_j L_i x = e_j(e_i x), \tag{3.62}$$

$$\{R_{ij}\}: \quad R_{i0}x = R_i x = x e_i\,, \quad R_{ij}x = R_j R_i x = (x e_i)e_j. \tag{3.63}$$

The subgroup $SO(7)$ does not alter the unit element e_0 ($8^v \to 1 \oplus 7$), automorphisms form a still narrower subgroup $G_2 \subset SO(7)$. In the transition to the subgroup $SO(7)$ from generators of $SO(8)$ the generators $L_{0i}(R_{0i})$ separate, forming the Mal'tsev algebra M_7 with the multiplication

$$[L_{0i} * L_{j0}] = L_{i0} * L_{j0} - L_{j0} * L_{i0}\,, \tag{3.64}$$

where the $*$-multiplication is defined by (3.49).

BIBLIOGRAPHY (resumed). • About representations: [18] • In connection with octonions: [24, 13, 14, 13], and also with Mal'tsev algebras [60].

3.7 Representation of binary and ternary sedenions

For the binary sedenion algebra regular birepresentation (regbirep) $\mathcal{L}, \mathcal{R}$-operators can be constructed easily. The operators representing the binary sedenion units e_i, $i = 0, 1, ..., 15$

$$e_i \to \mathcal{L}_i, \mathcal{R}_i : \quad \mathcal{L}_i x = e_i x \,, \; \mathcal{R}_i x = x e_i \,, \quad x \in B\mathbf{S} \tag{3.65}$$

are given in Appendix 3. If $\mathcal{L}_i, \mathcal{R}_i$ are to be thought as 16x16-matrices, then the elements x, $e_i x$, $x e_i$ must be 16-columns (vectors).

A general element $A = x + Xe$ of the binary sedenion algebra is then represented by 16x16-matrices

$$\mathcal{L}_A = \begin{pmatrix} L_x & -\bar{R}_X \\ \bar{L}_X & R_x \end{pmatrix}, \quad \mathcal{R}_A = \begin{pmatrix} R_x & -L_{\bar{X}} \\ L_X & R_{\bar{x}} \end{pmatrix}, \tag{3.66}$$

where $L, R, \bar{L}, \bar{R}$ are the corresponding regbirep 8x8-matrices and their conjugates for octonions.

We can introduce a *mixed representation* where one sedenion in the product is represented by a 16x16-matrix and the other one, by a 16-column. For the binary algebra this representation type may seem artifical but for the ternary algebra it gives a nontrivial possibility to construct a representation — we call it also *mixed representation for ternary sedenions.* In this representation two sedenions, A, C, in the ternary "half-products" $*(AB)C, A(BC)*$ in (3.27) are represented by $\mathcal{L}$-,$\mathcal{R}$-type 16x16-matrices and the third one, B, and the halfproducts themselves, by 16-columns (underlined):

$$\underline{*(AB)C} = *\mathcal{R}_C \mathcal{L}_A \underline{B} \,, \quad \underline{A(BC)*} = *\mathcal{L}_A \mathcal{R}_C \underline{B} \,, \tag{3.67}$$

where $*\mathcal{R}_C\mathcal{L}_A, *\mathcal{L}_A\mathcal{R}_C$ are special nonassociative products of $\mathcal{L}$-,$\mathcal{R}$-matrices calculated by means of ternary product (3.27):

$$* \, \mathcal{R}_C \mathcal{L}_A = \begin{pmatrix} R_z L_x - L_{\bar{Z}} L_X & -\underline{\bar{R}_{X_z}} - L_{\bar{Z}} R_x \\ \underline{L_{Zx}} + R_{\bar{z}} \bar{L}_X & -L_Z \bar{R}_X + R_{\bar{z}} R_x \end{pmatrix}, \tag{3.68}$$

$$* \, \mathcal{L}_A \mathcal{R}_C = \begin{pmatrix} L_x R_z - \bar{R}_X L_Z & -\underline{L_{x\bar{Z}}} - \bar{R}_x R_{\bar{z}} \\ \underline{\bar{L}_{X\bar{z}}} + R_x L_Z & -\bar{L}_X L_{\bar{Z}} + R_x R_{\bar{z}} \end{pmatrix}. \tag{3.69}$$

Here we make the following replacements (corresponding to underlined terms in (3.27))

$$
\begin{aligned}
R_z\bar{R}_X \to \bar{R}_{Xz}\,, \quad & L_Z L_x \to L_{Zx}\,,\\
L_x L_{\bar{Z}} \to L_{x\bar{Z}}\,, \quad & \bar{L}_X R_z \to \bar{L}_{X\bar{z}}\,,
\end{aligned}
$$

where

$$
\begin{aligned}
\bar{R}_{Xz} &= R_z\bar{R}_X + R_X\bar{L}_z - L_z\bar{R}_X\,,\\
L_{Zx} &= L_Z L_x + [L_Z, R_x]\,,\\
L_{x\bar{Z}} &= L_x L_{\bar{Z}} + [L_x, R_{\bar{Z}}\,,\\
\bar{L}_{X\bar{z}} &= L_X\bar{L}_{\bar{z}} + L_X\bar{X}\bar{R}_{\bar{z}} - R_{\bar{z}}\bar{L}_X\,.
\end{aligned}
$$

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Chapter 4

Dirac equation and self-duality problem in hypercomplex formalism of octonions and sedenions

In this chapter some particular but fundamental problems are discussed applying the nonassociative algebra formalism of octonions and sedenions developed in the preceding chapter.

In Sec. 4.1 we give some introductory remarks about the problems to be treated. In Sec. 4.2 Dirac equation and its generalizations are formulated as Cauchy-Riemann equations (monogenity conditions) of the hypercomplex analysis. Then the Dirac equation is formulated in terms of regular birepresentation of octonions with the subsequent introduction of color-type degrees of freedom and construction of the spectrum of fundamental fermions (Sec. 4.3). In Sec.4.4 self-duality conditions and hypercomplex analyticity in four and eight dimensions are studied. The 4-dimensional case is treated classically in terms of quaternions. Then an attempt is made to parallel this treatment of the two-index field tensor by 8-dimensional octonion formalism. This leads to the broken $SO(8)$-structure of the self-duality condition discussed also by several other authors (by different methods).

In Sec 4.5 self-duality conditions for the Yang-Mills field strength (curvature) tensor in dimensions d=4,8 are investigated in the octonion and sedenion formalisms, respectively. In both cases the hypercomplex formalism leads to the unification of the space-time and internal space through the (generalized) t'Hooft coefficients. In case $d=8$, hypercomplex formalism of ternary sedenions ensures the treatment of self-duality fully analogous to the case $d=4$. The situation is also described from the standpoint of topology and hypercomplex analysis.

4.1 Introductory remarks

In the theoretical elementary particle physics, usually the algebras representable by matrices (i.e. the associative and Lie algebras) are exploited. For physicists the calculable final results of their theories are important and therefore, if some sophisticated mathematical apparatus is used, there must be a possibility to perform calculations with real numbers or with some sets of them (vectors, matrices, etc.) at the final stage.

However, attempts have been made to use nonassociative (non-Lie) algebras, especially the *octonion algebra* in the theory of relativistic-invariant equations and some other topics (see Secs. 1.9.1 and 1.10). The preference of octonions is understandable from both standpoints, mathematics and physics. In mathematics, a well known fact is the uniqueness of octonions among the division algebras over real numbers. The uniqueness and exceptional properties of octonions also seem to have very important consequences for the physical world. We think that this highly exceptional mathematical system where many different structures of mathematics are overlapping, serves as the foundation of highly-determined, perhaps the only possible physical world. The appearance of the group E_8, the last of the five exceptional groups closely related to octonions, in superstring theory also underlines the importance of octonions. In the preceding chapter we constructed in addition to the octonions a ternary system, *ternary sedenions* which is a very natural generalization of Cayley-Dickson algebras into the realm of ternary algebras. If the nonassociativity plays some fundamental role in the structure of matter, it implies some kind of ternary interrelations between some basic "constituents" of matter and a very natural extension of unique mathematical system of binary division algebras terminating with octonions into the region of ternary systems seems very interesting.

In particle physics most of the well-known relativistic-invariant particle equations may be formulated in terms of quaternions. The use of quaternions is principally equivalent with the use of certain systems of matrices, therefore, they do not lead to any essentially new result except for elegant formulation of equations. As octonions are the direct nonassociative generalization of quaternions (for example, through the Cayley-Dickson process), then there is a hope to get some new knowledge from the octonionic formulation of relativistic-invariant equations.

If in the nonassociative (hypercomplex) formulations the hypercomplex units enter directly the equation, there may arise difficulties with their operational properties and interpretation – we simply do not know how to manage them, there is no matrix theory so common for us. It is conceivable that the hypercomplex entities and the operations between them may have some physical relevance, but in order to have a connection with our physical world through the calculability requirement we must have some kind of *associative projection* of the theory with nonassociative hypercomplex entities. In the orthodox theory, this connection with reality is provided through the apparatus of matrix representations.

The octonions have no matrix representation in the usual sense, but they have *bimodule representations* defined for every linear nonassociative algebra. We have already some acquaintance with this theory from the previous chapter. This theory, well known among specialists-mathematicians, is not very familiar to physicists, and it is not elaborated to such degree as the theory of representations of groups and Lie algebras. However, it provides the apparatus for the associative projection of nonassociative hypercomplex entities. The usual matrix representation theory is a particular (associative) case of this general theory.

In the present chapter we shall apply the mathematical apparatus of bimodule representations of nonassociative algebras, especially of octonions and sedenions to some particular but quite important and fundamental problems – to the *Dirac equation* and the *problem of self-duality* in dimensions 4 and 8. It appears that we can formulate the 8-component real Dirac equation in terms of R-matrices of the regbirep of octonions, the other half of regbirep, L-matrices, acting upon intrinsic degrees of freedom (*color*) (Sec. 4.3). Before Dirac equation and its generalizations had been formulated rather from some general principle – as the *Cauchy-Riemann monogenity conditions* of hypercomplex (quaternion, octonion or sedenion) analysis (Sec. 4.2).

There exists a well-known classical masterpiece in modern theoretical physics – the *self-dual static soliton (instanton) solutions* of the Yang–Mills equations with finite action in the Euclidean 4-space [2, 53]. The problems of self-dual Yang-Mills equations have fascinated mathematicians as well as physicists. Mathematicians have found there profound relations between the topology of the 4-space and the self-dual Yang-Mills theory (see, e.g. [17]).

It has also been the reason for a recent interest in duality relations in 8-space, particularly in the octonions formalism. We shall treat the cases dim=4 and 8 in octonion and sedenion formalism, respectively, in Sec. 4.4, [35]. Here we give a brief overview of the problem.

Gürsey and Tze [25] have a detailed treatment of two series of σ-models with projective spaces $CP(n)$ and $H(n)$, with the lowest Yang-Mills groups $U(1)$ and $Sp(1)(\equiv SU(2))$. So we have here two series of models with the gauge groups $SU(n+1)$ and $Sp(n+1)$ respectively. As to these models, it must be said that the lowest groups $U(1)$ and $SU(2)$ determine the general character of the whole series both physically and mathematically (see, e.g. [55, 44]).

Yoneya [57] considers the *instanton* and *monopole* solutions also in finite series of models, where in the case of even dimensions there are instantons and and in the case of odd dimensions there are monopoles. This treatment is topological, in terms of Chern classes. The lowest models (and the respective Yang-Mills groups) coincide with those of the lowest models of Gürsey and Tze. In Yoneya's treatment the number of indices of the field tensor grows with the dimension of the space, so it is undertandable that the proof of possibility of the Yang-Mills instanton only in four dimensions does not apply here. Dual pairs of tensors are here related with general Levy-Civita ϵ-symbols.

In our discussion we want to preserve the two-index field tensor $F_{\mu\nu}$ also for higher dimensions, because it describes vector boson and because it corresponds to the case dim=4, which apparently is our real space in which we live.

It appears that only in the 8-space it is possible to consider a two-index field tensor due to "octonionic structure" of this space. Levy-Civita ϵ-symbols are here replaced with the octonion commutator and associator strucure constants.

Situation with duality relations in dim=8 was clarified in papers by Corrigan et. al. [10, 15] and Fubini, Nicolai [19]. The *self-duality* and *anti-self-duality* occur here *nonsymmetrically*. The self-dual and anti-self-dual cases are realized in 7-dimensional subspaces of the 28-dimensional Lie algebra D_4

of the $SO(8)$-group. These subspaces do not generate any group, but form the Mal'tsev algebra M_7 corresponding to the octonionic Moufang loop. The cases of eigenvalues ± 3 are realized by 21 generators of the $SO(8)$-group, which generate the subgroup $SO(7)$ (in two distinct manners).

Such a branching or *asymmetry* here means that the self-duality (anti-self-duality) in the dim=8 *is broken*, but nevertheless it deserves a thorough and careful investigation.

In the present paper, we propose a different method of constucting the broken duality relations in dim=8.

Following Sommerfeld [49], we expose the proper field tensor $F_{\mu\nu}$ of the 4-dimensional (Euclidean) electrodynamics by quaternion 4×4-matrices, using the regular representation and the corresponding "inverse" representation (Sec. 4.4.1).

This treatment may be generalized to the case of dim=8, where the the tensor $F_{\mu\nu}$ may be constructed in terms of matrices of the regular bimodule representations of the octonion algebra (Sec. 4.4.2).

In the case of dim $= 4$ the construction of the field tensor resembles the construction of vector generators by spinor generators (see e.g. [39] for $SO(4)$ and [18] for $SO(8)$).

Finally, in Sec. 4.5 a general treatment of the Yang-Mills field strength (curvature) tensor in dimensions 4 and 8 is given in octonion and sedenion formalisms, respectively. In both cases extended hypercomplex formalism (from quaternions to octonions, from octonions to sedenions) leads to the natural unification of the space-time and internal space through the (generalized) t'Hooft coefficients. In the case dim=8 the hypercomplex formalism of ternary sedenions ensures the treatment of self-duality fully analogous to the case of dim=4. The situation is also described from the standpoint of topology and hypercomplex analysis.

BIBLIOGRAPHY (resumed). • Hypercomplex monogenity conditions: [30, 20, 21, 33, 38, 24, 25, 27, 26] • Dirac equation: [30, 21, 7, 41, 32, 4, 5, 6, 56, 34, 51, 56] (about equations see also bibliography at the end of Sec. 1.10.2) • Self-duality: [57, 10, 15, 13, 14, 11, 19, 34, 51, 48, 45, 46] [22, 23, 12, 29, 28].

4.2 Dirac equation as the monogenity condition of hypercomplex analysis

Here we demonstrate how the well known *Cauchy-Riemann* (*d'Alembert-Euler*) equations of the complex analysis may be generalized for the Cayley-Dickson algebras. With certain identifications we may consider these equations as a result of nullifying the product components in (3.16) or (3.18); the motivation here is the *holomorphism condition* $\frac{\partial z}{\partial \bar{x}} = 0$, see e.g. [9].

For $p, P, q, Q \in \mathbf{R}$ we have

$$qp - Qp = 0\,, \quad Qp + qP = 0\,,$$

and the identification $p = u$, $P = v$, $q = \frac{\partial}{\partial \xi}$, $Q = \frac{\partial}{\partial \eta}$ gives

$$\frac{\partial u}{\partial \xi} - \frac{\partial v}{\partial \eta} = 0\,, \quad \frac{\partial u}{\partial \eta} + \frac{\partial v}{\partial \xi} = 0\,,$$

the usual form of the Cauchy-Riemann (CR-) equations for a complex function $z = u + iv$ of a complex argument $x = \xi + i\eta$, $u = u(\xi, \eta)$, $v = v(\xi, \eta)$.

In a general case we have from (3.18)

$$pq - \mu\bar{Q}P = 0\,, \quad Qp + P\bar{q} = 0\,, \tag{4.1}$$

or in common notations

$$u\frac{\partial}{\partial \xi} - \mu\frac{\partial}{\partial \bar{\eta}}v = 0\,, \quad \frac{\partial}{\partial \eta} + v\frac{\partial}{\partial \bar{\xi}} = 0\,. \tag{4.2}$$

Equations (4.1) (or (4.2)) are the *generalized CR-equations* of the *hypercomplex analysis in CD-algebras*, [50].

Proceeding from complex numbers (and taking $\mu_2 = 1$), we get a variant of *quaternion analysis*, at a time thoroughly investigated by R.Fueter [20, 21] (a more complete list of references may be found in [50]). The CR-equations of this analysis constitute the *Weyl equation* (the massless Dirac equation).

Now let us start with quaternions and write down the CR-equations for the *octonion analysis*. Identifying $u = u_\mu e_\mu$, $v = v_\mu e_\mu$ (where e_μ, $\mu = 0, 1, 2, 3$ are quaternion units), we get

$$\frac{\partial}{\partial \xi} = \frac{\partial}{e_\mu \partial \xi_\mu} = \bar{e}_\mu \frac{\partial}{\partial \xi_\mu}\,, \quad \frac{\partial}{\partial \bar{\eta}} = \frac{\partial}{\bar{e}_\nu \partial \eta_\nu} = e_\nu \frac{\partial}{\partial \eta_\nu}\,. \tag{4.3}$$

Substituting these quantities into (4.1), we get after replacing the quaternion units by their regular representation matrices $r_\mu, l_\mu, \bar{r}_\mu, \bar{l}_\mu$ the following equations

$$\bar{r}_\mu \frac{\partial}{\partial \xi_\mu}[u] = l_\mu \frac{\partial}{\partial \eta_\mu}[v], \tag{4.4}$$

$$r_\mu \frac{\partial}{\partial \xi_\mu}[v] = -\bar{l}_\mu \frac{\partial}{\partial \eta_\mu}[u], \tag{4.5}$$

where $[u]$ and $[v]$ are 4-columns representing the quaternions u and v, respectively. Introducing a 8-column wave function composed from $[u]$ and $[v]$, we may write

$$\begin{pmatrix} \bar{r}_\mu & 0 \\ 0 & r_\mu \end{pmatrix} \frac{\partial}{\partial \xi_\mu} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{pmatrix} 0 & l_\mu \\ -\bar{l}_\mu & 0 \end{pmatrix} \frac{\partial}{\partial \eta_\mu} \begin{bmatrix} u \\ v \end{bmatrix}. \tag{4.6}$$

This is an equation in the form of free Dirac equation, if the right-hand side is interpreted as some kind of mass operator, and it consists only of R_i-matrices of octonion regbirep, see Appendix 3.

We may arrange a correspondence with the equation (4.12) (without the interaction term) if we choose $\eta_\mu(\mu = 0,1,2,3)$ in (4.6) for space-time variables.

To get equations in which octonion regbirep R-,L-matrices are simultaneously taking part, we may formulate CR-equations of the *(binary) sedenion analysis*. Identifying $u = u_\alpha e_\alpha$, $v = v_\alpha e_\alpha$ $(\alpha = 0,1,...,7)$ and interpreting the derivatives as in the octonion analysis (4.4),(4.5), we get an equation analogous to (4.6):

$$\begin{pmatrix} \bar{R}_\alpha & 0 \\ 0 & R_\alpha \end{pmatrix} \frac{\partial}{\partial \xi_\mu} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{pmatrix} 0 & L_\alpha \\ \bar{L}_\alpha & 0 \end{pmatrix} \frac{\partial}{\partial \eta_\mu} \begin{bmatrix} u \\ v \end{bmatrix}. \tag{4.7}$$

It is also a Dirac-type equation in 8-space, as we may write

$$\left[\begin{pmatrix} R_0 & 0 \\ 0 & R_0 \end{pmatrix} \frac{\partial}{\partial \xi_0} - \begin{pmatrix} R_i & 0 \\ 0 & -R_i \end{pmatrix} \frac{\partial}{\partial \xi_i} \right] \begin{bmatrix} u \\ v \end{bmatrix} = \hat{M} \begin{bmatrix} u \\ v \end{bmatrix}, \tag{4.8}$$

where $\hat{M}$ is a mass operator, consisting of L-matrices. Eqs. (4.6) and (4.8) are free equations, the external field may be introduced by replacing the derivatives with the *covariant* ones, $\frac{\partial}{\partial \xi_\alpha} - ieA_\alpha$:

$$\begin{pmatrix} \bar{R}_\alpha & 0 \\ 0 & R_\alpha \end{pmatrix} \left(\frac{\partial}{\partial \xi_\mu} - ieA_\alpha \right) \begin{bmatrix} u \\ v \end{bmatrix} = \hat{M}' \begin{bmatrix} u \\ v \end{bmatrix}. \tag{4.9}$$

Now we must emphasize that the binary sedenion algebra, the result of the fourth step in the nonmodified CD-process, appears to be quite formal and nonsuitable for physical applications. It must be underlined that the CD-procedure on each of the first steps introduces some modification irrelevant in the previous steps (see, e.g. Sec. 3.4). For real numbers the involution is trivial and the order of factors is not important, on constructing quaternions from complex numbers the involution is already nontrivial, in the third step, on constructing octonions from quaternions both the nontrivial involution and noncommutativity of quaternions must be taken into account. Now proceeding to the sedenions we must take into account the nonassociativity of octonions, but this cannot be done as the operation is binary (some good properties of alternative algebras are, however, preserved in the *ternary sedenion algebra* (Sec. 3.4, [50]). Therefore, the usual CD-process develops quite formally and the further algebras lose all their previous good properties.

Because of this formality the CR-equations of the (binary) sedenion analysis, interpreted as generalized Dirac equations (4.8), do not provide the Laplace equation in the common form corresponding to the Klein-Gordon equation but contain an additional superfluous associator, [50]. In the case of ternary sedenions, no such associator appears. On the other and, some kind of associator appears in the Laplace (Klein-Gordon) equation after introducing covariant derivatives.

4.3 Dirac equation in the regular birepresentation of octonions

4.3.1 Dirac equation written by R-matrices

We proceed from the Dirac equation with the electromagnetic term and γ-matrices in the representation where the "helicity operator" γ^5 is diagonal (the *Pauli representation*, see e.g. [40]):

$$-i\hbar\gamma^{\mu}\frac{\partial\psi}{\partial x_{\mu}}+\frac{e}{c}\gamma^{\mu}A_{\mu}\psi-imc\psi=0\,, \tag{4.10}$$

where $x_4 = ix_0 = ict$, $iA_0 = A_4$; the connection with α, β-matrices of [40] is $\gamma^4 = \beta$, $\gamma^k = -i\beta\alpha^k$, $\alpha^k = i\gamma^4\gamma^k$, $k = 1, 2, 3$.

The *Dirac matrices* γ^μ satisfy the relations

$$\{\gamma^\mu,\gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\delta_{\mu\nu}\mathrm{I}\,; \quad \mu,\nu = 1,2,3,4. \tag{4.11}$$

A suitable choice of triple products of R_i-matrices gives us 8x8 real γ-matrices which reproduce exactly the 8-dimensional real equation obtainable from the Pauli's one by replacements $1 \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $i \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The results for all fundamental modifications $\tilde{0}$,$\tilde{1}$,...,$\tilde{7}$ are summarized in the Table 9.

Table 9. R-**matrix structure of Dirac equation**

FM\\γ	$i\gamma^1$	$i\gamma^2$	$i\gamma^3$	$i\gamma^4$	γ^1	γ^2	γ^3	γ^4	i
$\tilde{0}$	35	52	51	50	412	413	423	765	67
$\tilde{1}$	24	34	40	41	513	521	476	523	67
$\tilde{2}$	17	70	73	27	623	547	621	316	45
$\tilde{3}$	60	61	62	36	546	237	317	127	45
$\tilde{4}$	17	16	15	14	654	745	647	756	32
$\tilde{5}$	60	70	40	50	263	273	243	325	23
$\tilde{6}$	35	34	73	63	315	314	371	361	1
$\tilde{7}$	24	25	62	27	214	215	126	217	1

Perhaps the simplest form (in R-matrices) of the Dirac equation occurs in the table modification $\tilde{5}$:

$$\begin{aligned} R_6\tfrac{\partial\psi}{\partial x_1} + R_7\tfrac{\partial\psi}{\partial x_2} + R_4\tfrac{\partial\psi}{\partial x_3} + R_5\tfrac{\partial\psi}{\partial x_4} \\ + eR_2R_3(R_6A_1 + R_7A_2 + R_4A_3 + R_5A_4)\psi + mR_2R_3\psi = 0 \end{aligned} \tag{4.12}$$

here $\hbar = c = 1$; $\gamma^5 = R_1R_2R_3$ in all modifications. The other half of fundamental modifications (0,1,...,7) gives us Dirac equations in the representations not used in physics.

Quite interesting form of the Dirac equation occurs also in the modification $\tilde{4}$ where all γ-matrices γ^μ $(\mu = 1,2,3,4)$ are obtainable from the matrices R_i $(i = 4,5,6,7)$, respectively, by the multiplication with the matrix $R_1R_2R_3 = \gamma^5$.

The reflection operators P, C, T in the present context have been investigated in [34].

4.3.2 Construction of color states

If we proceed from some fixed table modification (e.g. 0) for all rows in the Table 9, then we get eight equations differing in explicit form but being nevertheless unitary equivalent. Transitions between them may be performed with the aid of L_i-matrices according to formula (3.56) (here i represents the number of the row in the Table 9). We may write

$$D^{(i,0)}\psi_i = 0 \quad \Longrightarrow \quad -L_i^{(0)} D^{(i,0)} L_i^{(0)} L_i^{(0)} \psi_i \,, \tag{4.13}$$

where $D^{(i,0)}$ denotes the Dirac operator corresponding to the R-matrix structure of the i-th row, but written in the 0-modification; $D^{(i,i)} = -L_i^{(0)} D^{(i,0)} L_i^{(0)}$ denotes the Dirac operator with the same R-structure, but written also in the same i-modification, here $i = 0, 1, ..., 7$ without summation.

As it is sufficient for the generation of the Clifford algebra C_6 of L-matrices to take only 6 generative matrices, for example $L_1, L_2, L_3, L_4, L_5, L_6$, let us consider only 6 equations with six wave functions $L_1\psi_1, L_2\psi_2, ..., L_6\psi_6$. They correspond to the identical equations and, therefore, we may take them equal: $L_1\psi_1 = L_2\psi_2 = ... = L_6\psi_6 = \psi_0$, but then we have $\psi_1 = -L_1\psi_0$, $\psi_2 = -L_2\psi_0, ..., \psi_6 = L_6\psi_0$. From these wave functions we can combine the *color states* q_1, q_2, q_3 for the "quark" q:

$$q_1 = \begin{pmatrix} L_1\psi_0 \\ L_2\psi_0 \end{pmatrix}, \quad q_2 = \begin{pmatrix} L_3\psi_0 \\ L_4\psi_0 \end{pmatrix}, \quad q_3 = \begin{pmatrix} L_5\psi_0 \\ L_6\psi_0 \end{pmatrix} \tag{4.14}$$

(here the two components appear because of real representation).

And now we can elementarily introduce quark-gluon interactions which do not change the color state or transfer from other color states into the given one:

$$\begin{aligned} \{D_{\text{free+el.m.}} + \frac{g}{2}\gamma^\mu[B_\mu^{11}\hat{L}_{11} + B_\mu^{12}\hat{L}_{12} + B_\mu^{13}\hat{L}_{13}]\}q_1 = 0\,, \\ \{D_{\text{free+el.m.}} + \frac{g}{2}\gamma^\mu[B_\mu^{21}\hat{L}_{21} + B_\mu^{22}\hat{L}_{22} + B_\mu^{23}\hat{L}_{23}]\}q_2 = 0\,, \\ \{D_{\text{free+el.m.}} + \frac{g}{2}\gamma^\mu[B_\mu^{31}\hat{L}_{31} + B_\mu^{32}\hat{L}_{32} + B_\mu^{33}\hat{L}_{33}]\}q_3 = 0\,, \end{aligned} \tag{4.15}$$

where

$$\hat{L}_{11} = \hat{L}_{22} = \hat{L}_{33} = \mathrm{I}^{(2)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\hat{L}_{12} = \begin{pmatrix} L_3L_1 & 0 \\ 0 & L_4L_2 \end{pmatrix}, \quad \hat{L}_{21} = \begin{pmatrix} L_1L_3 & 0 \\ 0 & L_2L_4 \end{pmatrix},$$
$$\hat{L}_{13} = \begin{pmatrix} L_5L_1 & 0 \\ 0 & L_6L_2 \end{pmatrix}, \quad \hat{L}_{31} = \begin{pmatrix} L_1L_5 & 0 \\ 0 & L_2L_6 \end{pmatrix},$$
$$\hat{L}_{23} = \begin{pmatrix} L_5L_3 & 0 \\ 0 & L_6L_4 \end{pmatrix}, \quad \hat{L}_{32} = \begin{pmatrix} L_3L_5 & 0 \\ 0 & L_4L_6 \end{pmatrix}.$$

To get exact correspondence with the Dirac equation in quantum chromodynamics

$$\{D_{\text{free+el.m.}} + \frac{g}{2}\gamma^\mu B^a_\mu \lambda^a_{ij}\}q = 0\,, \tag{4.16}$$

$$\lambda^a, \ a = 1,2,...,8 - \text{Gell-Mann matrices; } i,j,k = 1,2,3$$

we must choose the potentials B^{ij}_μ in the form

$$B^{11}_\mu = B^3_\mu + \frac{1}{\sqrt{3}}B^8_\mu, \quad B^{22}_\mu = -B^3_\mu + \frac{1}{\sqrt{3}}B^8_\mu, \quad B^{33}_\mu = -\frac{2}{\sqrt{3}}B^8_\mu,$$

$$B^{12}_\mu = B^1_\mu - iB^2_\mu\,, \quad B^{21}_\mu = B^1_\mu + iB^2_\mu\,,$$

$$B^{13}_\mu = B^4_\mu - iB^5_\mu\,, \quad B^{31}_\mu = B^4_\mu + iB^5_\mu\,,$$

$$B^{23}_\mu = B^6_\mu - iB^7_\mu\,, \quad B^{32}_\mu = B^6_\mu + iB^7_\mu\,.$$

4.3.3 Spectrum of fundamental fermions

In this subsection the spectrum of fundamental fermions is constructed by using the Clifford algebra of the regular bimodule representation of the nonassociative algebra of octonions. Fundamental fermions appear here as an associative projection of some hypothetical nonassociative substructure lying in the very basis of matter.

The mathematical prerequisite for the following presentation is the material about the octonion regbirep and related Clifford algebra (Sec. 3.6.2).

Let us start with the formula (3.54), where the regbirep $*$-operation (3.50) for R_i-matrices is represented in a "projective" form:

$$R_i * R_j = \Gamma_{(ijk)} R_j R_i = R_{ij}\,, \tag{4.17}$$

where (ijk) denotes the positive cycle of the octonion units containing the indices i, j. The products $\Gamma_{(ijk)} = R_i R_j R_k$ with ijk positive cycles (3.3) are diagonal matrices of a *reflection type* (i.e. with ± 1 on the main diagonal):

$$\begin{aligned}
\Gamma_{(123)} &= \operatorname{diag}(1,1,1,1,-1,-1,-1,-1)\,, \\
\Gamma_{(145)} &= \operatorname{diag}(1,1,-1,-1,1,1,-1,-1)\,, \\
\Gamma_{(167)} &= \operatorname{diag}(1,1,-1,-1,-1,-1,1,1)\,, \\
\Gamma_{(246)} &= \operatorname{diag}(1,-1,1,-1,1,-1,1,-1)\,, \\
\Gamma_{(257)} &= \operatorname{diag}(1,-1,1,-1,-1,1,-1,1)\,, \\
\Gamma_{(347)} &= \operatorname{diag}(1,-1,-1,1,1,-1,-1,1)\,, \\
\Gamma_{(365)} &= \operatorname{diag}(1,-1,-1,1,-1,1,1,-1)\,.
\end{aligned} \tag{4.18}$$

Any three of them may be regarded as generic ones, e.g. $\Gamma_{(123)}$ ($\equiv \Gamma_2$), $\Gamma_{(145)}$ ($\equiv \Gamma_3$), $\Gamma_{(246)}$ ($\equiv \Gamma_4$), the other ones being expressed as their products (see Table 10).

*** Clifford algebra C_6 from R-matrices**

The Clifford algebra of R-matrices can be constructed straightforwardly from the first 6 R-matrices as generic elements (any six anticommuting matrices are suitable for it). From physical considerations, however, some rearrangement of generic elements is desirable as in Table 10 (where R-matrices and their products are represented by indices only, e.g. 674 meaning (denoting) $R_6 R_7 R_4$, etc.).

Table 10. **A "physical" basis for Clifford algebra C_6**

			Γ_2	Γ_3	$\Gamma_2\Gamma_3$	Γ_4	$\Gamma_2\Gamma_4$	$\Gamma_3\Gamma_4$	$\Gamma_2\Gamma_3\Gamma_4$
		R_0	$\Gamma_{(123)}$	$\Gamma_{(145)}$	$\Gamma_{(167)}$	$\Gamma_{(246)}$	$\Gamma_{(572)}$	$\Gamma_{(347)}$	$\Gamma_{(653)}$
		0	123	145	167	246	572	347	653
		1	32	54	67	753	643	652	472
21	$1 \cdot 21 =$	2	13	763	345	64	75	561	741
31	$1 \cdot 31 =$	3	21	672	542	571	461	74	56
432	$1 \cdot 432 =$	$765=\gamma^1$	4	352	15	631	26	271	37
532	$1 \cdot 532 =$	$674=\gamma^2$	5	432	41	27	317	63	621
632	$1 \cdot 632 =$	$754=\gamma^3$	6	71	732	341	42	35	251
732	$1 \cdot 732 =$	$645=\gamma^4$	7	16	362	52	351	421	43

In the first column between horizontal lines there are 6 generic elements (expressed as products of R-matrices) for the Lie algebra $SO(4,2)$ isomorphic to the Lie algebra of the space-time *conformal group*, the latter representing the maximum symmetry of massless particles (our fundamental fermions are also massless).

The matrix R_1 anticommuting with all 6 conformal elements, gives us by multiplication a set of anticommuting matrices (the 2nd column) generating (through multiplication) a 64-basis for the Clifford algebra C_6, represented by right eight 8-columns. This basis may also be obtained multiplying the third column by the reflection-type matrices $\Gamma_{(123)}, \Gamma_{(145)}, \ldots$, (4.18).

Further we shall see that R_1 will play the role of *charge operator*, so it means that in the generic row there are represented space-time and charge operators. The entire Clifford algebra is the result of multiplication of this column by 7 reflection-type Γ-matrices. It occurs that this rearrangement (or representation) of the Clifford algebra C_6 is especially fitting for the classification of fundamental fermions.

Dirac γ-matrices in the Dirac equation (in the 8-dim. real version, [51]) can be represented as the quadruples of the triple products $R_4R_3R_2$, $R_5R_3R_1$, $R_6R_3R_2$, $R_7R_3R_2$ (in the first two columns) or $R_7R_6R_5$, $R_6R_7R_4$, $R_7R_5R_4$, $R_6R_4R_5$ (in the third column). The first reflection-type matrix $\Gamma_{(123)} = R_1R_2R_3$ is the Dirac's $\gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4$, it gives us the *helicity* projection operators $1 \pm \gamma^5$. The other 6 reflection-type matrices can be interpreted as giving *color helicity* projection operators $1 + \Gamma_{(ijk\neq 123)}$. But after the choosing of definite helicity states there remain only *three* essentially different helicity operators.

The definite choice of helicity states also guarantees that all internal quantum numbers will be equal for all space-time components, otherwise we should not attach definite quantum numbers to particles. It is the reason why in unified models particles occur in definite helicity states (as Weyl spinors).

* Enlarging the Clifford algebra

Up to now we have been dealing with the Clifford algebra formed from R-matrices only. To grasp all the essence of regbirep of the octonion regbirep (as of nonassociative algebra) we must pass over to the full Clifford algebra C_7 formed from both R- and L-matrices. For this it is expedient to pass over to 16x16-matrices of the following structure:

$$\mathcal{R}_0 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \quad \mathcal{R}_i = \begin{pmatrix} R_i & 0 \\ 0 & L_i \end{pmatrix}, \tag{4.19}$$

where $i = 1, 2, \ldots, 7$.

Now we have a new reflection-type matrix as seven-product

$$\hat{\Gamma}_1 = \mathcal{R}_1 \mathcal{R}_2 \ldots \mathcal{R}_7 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{4.20}$$

also the previous 8x8 Γ-matrices may be doubled (with equal signs for the diagonal 8x8-blocks), so we have

$$\hat{\Gamma}_2 = \begin{pmatrix} \Gamma_2 & 0 \\ 0 & \Gamma_2 \end{pmatrix}, \quad \hat{\Gamma}_3 = \begin{pmatrix} \Gamma_3 & 0 \\ 0 & \Gamma_3 \end{pmatrix}, \quad \hat{\Gamma}_4 = \begin{pmatrix} \Gamma_4 & 0 \\ 0 & \Gamma_4 \end{pmatrix}. \tag{4.21}$$

Matrices $\hat{\Gamma}_1, \hat{\Gamma}_2, \hat{\Gamma}_3, \hat{\Gamma}_4$ generate by multiplication 15 reflection-type 16x16-matrices.

Now as a decisive step towards the final classification of *fundamental fermion states* we shall form 4 *projection-reflection operators* as diagonal 16x16-matrices with 1,-1,0 on the main diagonal:

$$\hat{\Gamma}_1 \mp \hat{\Gamma}_{(123)}, \ \hat{\Gamma}_1 \mp \hat{\Gamma}_{(145)}, \ \hat{\Gamma}_1 \mp \hat{\Gamma}_{(246)}, \ \hat{\Gamma}_1 \mp \hat{\Gamma}_{(653)}, \tag{4.22}$$

where

$$\hat{\Gamma}_{(ijk)} = \begin{pmatrix} \Gamma^{(R)}_{(ijk)} & 0 \\ 0 & \Gamma^{(L)}_{(ijk)} \end{pmatrix},$$

$\Gamma^{(L)}_{(ijk)}$ being reflection-type matrices formed from L_i-matrices of (3.47).

In terms of $\hat{\Gamma}$-matrices of (4.21)

$$\hat{\Gamma}_{(123)} = \hat{\Gamma}_2, \quad \hat{\Gamma}_{(145)} = \hat{\Gamma}_3, \quad \hat{\Gamma}_{(246)} = \hat{\Gamma}_4, \quad \hat{\Gamma}_{(653)} = \hat{\Gamma}_2 \hat{\Gamma}_3 \hat{\Gamma}_4. \tag{4.23}$$

We call the first operator in (4.22) *helicity operator*, the other three *color-type helicity operators.*

∗ Fundamental fermions

From diagonal operators (4.22) we can now read out quantum numbers of fundamental fermions (Table 11).

Table 11. **Spectrum of fundamental fermions**

	Particles $\hat{\Gamma}_1 - \hat{\Gamma}_{(ijk)}$				Antiparticles $\hat{\Gamma}_1 + \hat{\Gamma}_{(ijk)}$				
$ijk \triangleright$	145	246	653	123	145	246	653	123	$\triangleleft ijk$
	k_1	k_2	k_3	k_4	k_1	k_2	k_3	k_4	
(ν_R)	0	0	0	0	1	1	1	1	e^+_R
	0	1	1	0	1	0	0	1	
u_R	1	0	1	0	0	1	0	1	$\bar{d}_R$
	1	1	0	0	0	0	1	1	
ν_L	0	0	0	1	1	1	1	0	e^+_L
	0	1	1	1	1	0	0	0	
u_L	1	0	1	1	0	1	0	0	$\bar{d}_L$
	1	1	0	1	0	0	1	0	
e^-_R	−1	−1	−1	0	0	0	0	−1	$\bar{\nu}_R$
	−1	0	0	0	0	−1	−1	−1	
d_R	0	−1	0	0	−1	0	−1	−1	$\bar{u}_R$
	0	0	−1	0	−1	−1	0	−1	
e^-_L	−1	−1	−1	−1	0	0	0	0	$(\bar{\nu}_L)$
	−1	0	0	−1	0	−1	−1	0	
d_L	0	−1	0	−1	−1	0	−1	0	$\bar{u}_L$
	0	0	−1	−1	−1	−1	0	0	

We have got here "infovectors" $(k_1k_2k_3k_4)$ or *particle codons* for 32 states of fundamental fermions forming separate **irreducible** 4-blocks incorporating a *lepton* and a *quark* in 3 color sates. These blocks are irreducible (remaining together) when different signs are combined in

$$(\pm)\hat{\Gamma}_1(\pm)\hat{\Gamma}_{(ijk)}\,.$$

Until the helicity states are not specified (projected out) all codon numbers (all helicities) are structurally equivalent. However, if the definite helicity states are projected out, then *color together with electric charge*, i.e. $(k_1k_2k_3)$, separate from the *weak charge* (k_4).

Electric charge

$$Q = \frac{1}{3}(k_1 + k_2 + k_3)\,, \tag{4.24}$$

the 3rd component of the *weak isospin*

$$T_3^w = \frac{1}{2} k_4 \,. \tag{4.25}$$

The *Colors R (red), G (green)*, and *B (blue)* are determined by the first three components k_1, k_2, k_3 of the codon vector, respectively. The color of a quark is specified by the quantum number k_i differing from the other two (e.g. the quark is *green* if $k_1, k_2, k_3 = 1, 0, 1$ (or 0,−1,0), and *anti-green* if $k_1, k_2, k_3 = 0, 1, 0$ (or −1,0,−1), the two diffrent possibilities related to the fact that the quantum numbers k_1, k_2, k_3 also determine the electric charge). For the ***observable (nonconfined)*** fundamental fermions (leptons)

$$k_1 = k_2 = k_3 \,. \tag{4.26}$$

Our model somewhat differs from the common Standard Model, because we have unified in the first stage the electromagnetic and strong interactions (related to the codons k_1, k_2, k_3), the weak interactions represented by k_4. So our model represents the stage where the electroweak model is already violated and gluons and the photon are treated on the same footing.

∗ The family problem

According to CERN-LEP 1989-90 experiments, fundamental fermions appear in nature in the form of 3-fold repeated patterns of *families*. The 2nd (ν_μ, μ, c, s) and the 3rd (ν_τ, τ, t, b) families reproduce exactly the pattern of the 1st one (ν_e, e, u, d), except the growing masses (as established for $m_e < m_\mu < m_\tau$). Nevertheless the 1st family forms the ordinary matter.

From our point of view the families are manifestations of some *external color* quantities, which in fact are flavor-type, i.e. *nonconfined*, as they are formed in the same region where the measuring apparatus is placed. We use here "color" because of the probable technical similarity.

To introduce the characteristics (quantum numbers) for families we must extend our spinor space, introducing, e.g. two new operators $\hat{\Gamma}_1', \hat{\Gamma}_1''$ analogous to the $\hat{\Gamma}_1$ introduced in (4.20), thus enlarging the dimension of the spinor space to 32 and 64. The spectrum of fundamental fermions will be enlarged by introducing these new operators into (4.22).

This enlarging (doubling) of the spinor space can be described mathematically consistently through the introduction of a new octonion algebra **O'**, which means the describing of all the fundamental particles by the last

Rosenfeld algebra [47] $\mathbf{O} \times \mathbf{O}'$ closely related to the group E_8, the automorphism group of the Jordan algebra $M_3(\mathbf{O} \times \mathbf{O}')$.

4.3.4 Additional remarks

Our approach to the Dirac equation in Sec. 4.3 has been quite cautious and nonpretentious. Proceeding from the well-known Pauli representation we have demonstrated that the Dirac equation may be formulated in terms of R-matrices of the octonion regbirep. It is nothing in particular because the only algebraic condition for the Dirac algebra is the *anticommutativity.* The usual Dirac equation use up 4x4 γ-matrices generating the 16-dimensional Clifford algebra C_4 (usually called *Dirac algebra*). The transition to the R-matrices of the octonion regbirep extends the full algebra to the 64-dimensional Clifford algebra C_6. In this connection there appear some interesting new possibilities, as the possibility of introduction of complex conjugation as an operator etc. If we grasp also L-matrices of the octonion regbirep, we enlarge the algebra into the 128-dimensional Clifford algebra C_7 [1, 3].

Mathematically R- and L-matrices participate as equal partners in the regbirep of octonions. But physically, choosing the R-matrices for the formulation of the Dirac equation we have to interpret L-matrices as acting on some *internal* degrees of freedom (in some internal space), in full agreement with the actual action of L-matrices on R-matrices (according to (3.56)).

The choice is of course free, but it seems that the R-matrices are more suitable for the formulation of the space-time part of the problem. In Sec. 4.4.2 we have succeeded to demonstrate that L-matrices may be used for the construction of *color states*, for the introduction of *color potentials* into the Dirac equation. Therefore, the *noncommutativity* of L- and R-matrices, reflecting the *nonassociativity* of octonions, give rise to some new kind of *uncertainty relations* between the notion of *free particle* (with definite kinematics and spin), and the *colored particle*, the last representing the pure color state with exact *color symmetry*, leading to the physical *confinement* of colored particles.

In Sec. 4.3.3 some new aspect of the *associative projection* of the nonassociativity has been explored for obtaining the *spectrum of fundamental fermions.* Here the interplay of L- and R-matrices of the regbirep has been entirely translated into the Clifford algebra constructed from L- and R-matrices.

A special interest deserves the passage to *sedenions*. We are repeatedly emphasized the unnaturalness of binary sedenions as the extension of the Cayley-Dickson procedure. In Sec. 4.2 we were witnessing the appearance of additional *associators* in the corresponding Laplace equations. In what follows (in Sec. 4.5.2) we shall encounter the *antiassociators* in the derivation of the harmonicity condition. In both cases the passage to the *ternary sedenions* leads to the disappearance of additional ("superfluous") terms, and this circumstance serves as the evidence of naturalness of ternary sedenions.

And finally, let have some remarks about the Dirac equation as the monogenity condition of hypercomplex analysis. Already in 1930 D. Iwanenko and K. Nikolsky [30] have treated the Dirac equation as the Cauchy-Riemann system of a hypercomplex (biquaternion) analysis. Fueter's holomorphism condition for quaternion functions is in fact the Weyl equation for massless leptons. Also the Maxwell equations may be represented in quaternion form (in fact in this form they were derived by Maxwell himself). The problem of a hypercomplex formulation of fundamental physical equations (including Einstein's equations of gravity, see e.g. [31] has been treated by many authors [24] (see references therein), but this hypercomplexity has been mostly of an associative nature (with some exceptions [7, 8, 32, 16], it is also understandable because the dimension of the physical space-time suits well the dimension of quaternion or biquaternion algebra.

4.4 Hypercomplex formulation of self-duality in $d = 4$ and $d = 8$

In this section self-duality conditions and hypercomplex analyticity in four and eight dimensions are investigated. The four-dimensional case is treated classically in terms of quaternions. Following Sommerfeld [49] we expose the proper field tensor $F_{\mu\nu}$ of the 4-dimensional (euclidean) electrodynamics by 4x4-matrices of quaternion regbirep and the corresponding "inverse" representation. The construction of field tensor here resembles the construction of vector generators from spinor ones (see e.g. [39] for $SO(4)$ and [18] for $SO(8)$).

An analogous treatment is given for the case dim=8 where the tensor $F_{\mu\nu}$ may be constructed in terms of 8x8-matrices of octonion regbirep (Sec.

4.4.2). In the eight-dimensional case we start with the construction of seven-dimensional subspaces (self-dual and anti-self-dual "sectors") from the complete set of seven Cartan subalgebras of the D_4-algebra by Hadamard-type matrices. This resembles the construction of vector generators by spinor ones for $SO(8)$ by Freudenthal [18]. However, our construction is somewhat different, because we do not deal with pure generators, but with those multiplied with "field coefficients". The construction of additional sectors corresponding to the eigenvalues ± 3 may be then performed by suitable combinations of operators.

4.4.1 Self-duality and hypercomplex (quaternion) analyticity in four dimensions

Let us introduce necessary quantities and notions by the well-known example of 4-dimensional electromagnetism (see, e.g. [49]).

The *field tensor* is

$$[F_{\mu\nu}] = \begin{pmatrix} 0 & F_{01} & F_{02} & F_{03} \\ F_{10} & 0 & F_{12} & F_{13} \\ F_{20} & F_{21} & 0 & F_{23} \\ F_{30} & F_{31} & F_{32} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & H_3 & -H_2 \\ E_2 & -H_3 & 0 & H_1 \\ E_3 & H_2 & -H_1 & 0 \end{pmatrix}, \tag{4.27}$$

defined by electromagnetic *field potentials* A_μ,

$$F_{\mu\nu} = \frac{\partial A_\mu}{\partial x_\nu} - \frac{\partial A_\nu}{\partial x_\mu}. \tag{4.28}$$

The *dual field tensor* F^* is defined as

$$F^*_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F_{\alpha\beta}, \tag{4.29}$$

where $\varepsilon_{\mu\nu\alpha\beta}$ is antisymmetric and $\varepsilon_{0123} = +1$, then in components

$$[F^*_{\mu\nu}] = \begin{pmatrix} 0 & F^*_{01} & F^*_{02} & F^*_{03} \\ F^*_{10} & 0 & F^*_{12} & F^*_{13} \\ F^*_{20} & F^*_{21} & 0 & F^*_{23} \\ F^*_{30} & F^*_{31} & F^*_{32} & 0 \end{pmatrix} = \begin{pmatrix} 0 & H_1 & H_2 & H_3 \\ -H_1 & 0 & -E_3 & E_2 \\ -H_1 & 0 & -E_3 & E_2 \\ -H_2 & E_3 & 0 & -E_1 \\ -H_3 & -E_2 & E_1 & 0 \end{pmatrix}. \tag{4.30}$$

The transition $F \to F^*$ then leads to the transitions for physical quantities

$$\vec{E} \leftrightarrow -\vec{H} \,. \tag{4.31}$$

The tensor F is called *self-dual* if

$$F = F^* \,, \tag{4.32}$$

and *anti-self-dual* if

$$F = -F^* \,. \tag{4.33}$$

We can construct self-dual and anti-self-dual tensors for every field tensor F:

$$\tilde{F} \;=\; F + F^* \quad \text{(self-dual)} \tag{4.34}$$

$$\bar{F} \;=\; F - F^* \quad \text{(anti-self-dual)} \tag{4.35}$$

and *vice versa*

$$F = \frac{1}{2}(\tilde{F} + \bar{F}) \,, \quad F^* = \frac{1}{2}(\tilde{F} - \bar{F}) \,. \tag{4.36}$$

Let us write the self-dual tensor $\tilde{F}$ componentwise,

$$[\tilde{F}_{\mu\nu}] = \begin{pmatrix} 0 & -E_1 + H_1 & -E_2 + H_2 & -E_3 + H_3 \\ E_1 - H_1 & 0 & -E_3 + H_3 & E_2 - H_2 \\ E_2 - H_2 & E_3 - H_3 & 0 & -E_1 + H_1 \\ E_3 - H_3 & -E_2 + H_2 & E_1 - H_1 & 0 \end{pmatrix} \tag{4.37}$$

and introducing short notations

$$\vec{a}(a_1, a_2, a_3) = \vec{E} - \vec{H} \,,$$

$$[\tilde{F}_{\mu\nu}] = \begin{pmatrix} 0 & -a_1 & -a_2 & -a_3 \\ a_1 & 0 & -a_3 & a_2 \\ a_2 & a_3 & 0 & -a_1 \\ a_3 & -a_2 & a_1 & 0 \end{pmatrix} = a_1 l_1 + a_2 l_2 + a_3 l_3 \,, \tag{4.38}$$

where l_1, l_2, l_3 are 4x4-matrices of (inverse) *regular representation* of the *vector-quaternion* units i, j, k.

Analogously we get for anti-self-dual field tensor

$$[\bar{F}_{\mu\nu}] = \begin{pmatrix} 0 & -b_1 & -b_2 & -b_3 \\ b_1 & 0 & b_3 & -b_2 \\ b_2 & -b_3 & 0 & b_1 \\ b_3 & b_2 & -b_1 & 0 \end{pmatrix} \tag{4.39}$$

where r_1, r_2, r_3 are 4x4-matrices of the *regular representation* of vector-quaternion units, and

$$\vec{b}(b_1, b_2, b_3) = \vec{E} + \vec{H} \, . \tag{4.40}$$

Substituting $F_{\mu\nu}$ (4.28) into the Eq. (4.32), we get the *Fueter's monogenity conditions* [20] for quaternion-valued functions of a quaternion variable:

$$\begin{aligned}
\frac{\partial A_0}{\partial x_1} - \frac{\partial A_1}{\partial x_0} - \frac{\partial A_2}{\partial x_3} + \frac{\partial A_3}{\partial x_2} &= 0 \, , \\
\frac{\partial A_0}{\partial x_2} - \frac{\partial A_2}{\partial x_0} - \frac{\partial A_3}{\partial x_1} + \frac{\partial A_1}{\partial x_3} &= 0 \, , \\
\frac{\partial A_0}{\partial x_3} - \frac{\partial A_3}{\partial x_0} - \frac{\partial A_2}{\partial x_1} + \frac{\partial A_1}{\partial x_2} &= 0 \, , \\
\frac{\partial A_0}{\partial x_0} - \frac{\partial A_1}{\partial x_1} - \frac{\partial A_2}{\partial x_2} - \frac{\partial A_3}{\partial x_3} &= 0 \, ,
\end{aligned} \tag{4.41}$$

the last equation is the *Lorentz condition.*

4.4.2 Self-duality and hypercomplex (octonion) analyticity in eight dimensions

Now let us proceed analogously to the case of $d = 8$. The mathematics used here is mainly that of Freudenthal's [18].

In the 4-dimensional case, the construction of the field tensor (in self-dual or anti-self-dual form) was performed as the construction of vector rotations from spinor-rotations (represented by l- and r-matrices). In the 8-dimensional case, there also exists a construction (in [18]) where vector-rotations are expressed by spinor-rotations (represented by L- and R-matrices of octonion regular bimodule representation) through *Hadamard-type matrices* representing corresponding outer automorphisms of the D_4 Lie algebra of the $SO(8)$-group. These matrices act on *Cartan subalgebras*, and we may take the complete system of seven different Cartan subalgebras to get the full 28-component two-index field tensor $F_{\alpha\beta}$ $(\alpha, \beta = 0, 1, ..., 7)$. Seven of these 28 components will be identified as electric and 21, as magnetic components. For every electric component there are 3 magnetic components labelled by additional ("color") upper index (taking values 1,2,3). Then for the field tensor the following structure may be naturally manifested:

$$[F_{\alpha\beta}] = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 & -E_4 & -E_5 & -E_6 & -E_7 \\ E_1 & 0 & H_3^1 & -H_2^1 & H_5^1 & -H_4^1 & -H_7^3 & H_6^1 \\ E_2 & -H_3^1 & 0 & H_1^1 & H_6^2 & H_7^1 & -H_4^2 & -H_5^3 \\ E_3 & H_2^1 & -H_1^1 & 0 & H_7^2 & -H_6^3 & H_5^2 & -H_4^3 \\ E_4 & -H_5^1 & H_6^2 & -H_7^2 & 0 & H_1^2 & H_2^2 & H_3^3 \\ E_5 & H_4^1 & -H_7^1 & H_6^3 & -H_1^2 & 0 & -H_3^2 & H_2^3 \\ E_6 & H_7^3 & H_4^2 & -H_5^2 & -H_2^2 & H_3^2 & 0 & -H_1^3 \\ E_7 & -H_6^1 & H_5^3 & H_4^3 & -H_3^3 & -H_2^3 & H_1^3 & 0 \end{pmatrix} . \quad (4.42)$$

Now we are writing the correspondence between field tensor elements and physical fields in what follows, the index combinations at F-quartets are important

$$\begin{aligned} &F_{10},\ F_{23},\ F_{45},\ F_{76} &\leftrightarrow\quad &E_1,\ H_1^1,\ H_1^2,\ H_1^3\ ; \\ &F_{20},\ F_{31},\ F_{46},\ F_{57} &\leftrightarrow\quad &E_2,\ H_2^1,\ H_2^2,\ H_2^3\ ; \\ &F_{30},\ F_{12},\ F_{65},\ F_{47} &\leftrightarrow\quad &E_3,\ H_3^1,\ H_3^2,\ H_3^3\ ; \\ &F_{40},\ F_{51},\ F_{62},\ F_{73} &\leftrightarrow\quad &E_4,\ H_4^1,\ H_4^2,\ H_4^3\ ; \\ &F_{50},\ F_{14},\ F_{36},\ F_{72} &\leftrightarrow\quad &E_5,\ H_5^1,\ H_5^2,\ H_5^3\ ; \\ &F_{60},\ F_{17},\ F_{24},\ F_{53} &\leftrightarrow\quad &E_6,\ H_6^1,\ H_6^2,\ H_6^3\ ; \\ &F_{70}\ ,F_{25},\ F_{34},\ F_{61} &\leftrightarrow\quad &E_7,\ H_7^1,\ H_7^2,\ H_7^3\ . \end{aligned} \qquad (4.43)$$

There are seven corresponding *Cartan subalgebras*:

$$\begin{aligned} &G_{10},\ G_{23},\ G_{45},\ G_{76}\ ; \\ &G_{20},\ G_{31},\ G_{46},\ G_{57}\ ; \\ &G_{30},\ G_{12},\ G_{65},\ G_{47}\ ; \\ &G_{40},\ G_{51},\ G_{62},\ G_{73}\ ; \\ &G_{50},\ G_{14},\ G_{36},\ G_{72}\ ; \\ &G_{60},\ G_{17},\ G_{24},\ G_{53}\ ; \\ &G_{70},\ G_{25},\ G_{34},\ G_{61}\ . \end{aligned} \qquad (4.44)$$

According to Freudenthal [18] the vector-generators G_{ij} (here $G_{ij}e_j = -e_i$, $G_{ij}e_i = e_j$) of these subalgebras may be expressed through the *spinor* *generators* R_{ij} by the Hadamard-type matrix

$$\pi = \frac{1}{4}\begin{pmatrix} -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \tag{4.45}$$

$$\begin{aligned} 4G_{10} &= -R_{10} - R_{23} - R_{45} - R_{76}\,, \\ 4G_{23} &= R_{10} + R_{23} - R_{45} - R_{76}\,, \\ 4G_{45} &= R_{10} - R_{23} + R_{45} - R_{76}\,, \\ 4G_{76} &= R_{10} - R_{23} - R_{45} + R_{76}\,. \end{aligned} \tag{4.46}$$

Instead of R_{ij} we may use also L_{ij}-matrices, but then the index pairs must be reversed (see definitions in the preceding chapter, Sec. 3.6.3).

Let us now associate a *field* with each (of four) spinor generators $R_{01}, R_{23}, R_{45}, R_{76}$, denote them a_1, b_1, c_1, d_1 (in general a_i, b_i, c_i, d_i; $i = 1, 2, ..., 7$ for every Cartan subalgebra in (4.44). These fields (or "field coefficients") must be adjusted to get *physical fields* $E_i; H_i^1, H_i^2, H_i^3$ through the process resembling the formation of vector-generators from spinor ones:

$$\begin{aligned} &\frac{1}{4}(a_i R_{i0} + b_i R_{kl} + c_i R_{mn} + d_i R_{pq}) = {}^iF^{(0)}\,, \\ &\frac{1}{4}(a_i R_{i0} + b_i R_{kl} - c_i R_{mn} - d_i R_{pq}) = {}^iF^{(1)}\,, \\ &\frac{1}{4}(a_i R_{i0} - b_i R_{kl} + c_i R_{mn} - d_i R_{pq}) = {}^iF^{(2)}\,, \\ &\frac{1}{4}(a_i R_{i0} - b_i R_{kl} - c_i R_{mn} + d_i R_{pq}) = {}^iF^{(3)}\,, \end{aligned} \tag{4.47}$$

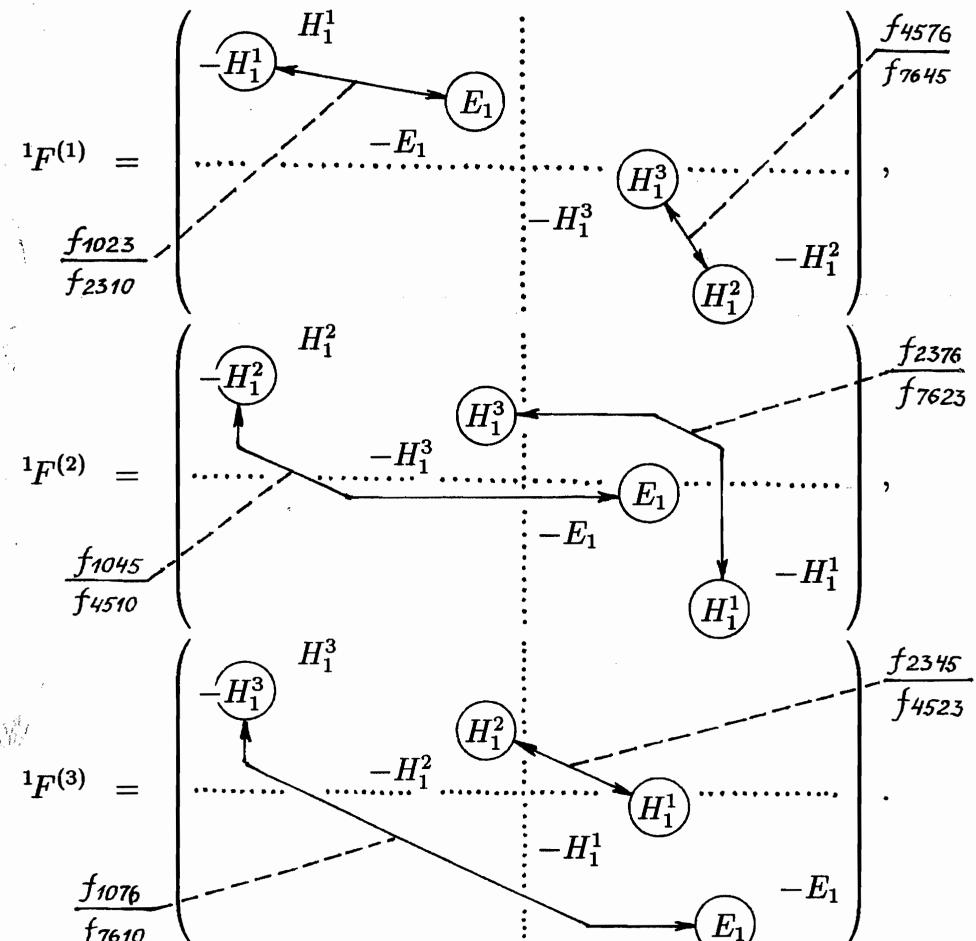
where the index pairs $i0, kl, mn, pq$ are taken from the i-th row of (4.44).

Particularly, demanding that the first equation of (4.47) should give the structure of the initial field tensor (${}^iF^{(0)}$ being the tensor (4.42) with elements E_i, H_i^1, H_i^2, H_i^3 only), we get the relations

$$\begin{aligned} E_i &= \frac{1}{4}(a_i + b_i + c_i + d_i)\,, \\ H_i^1 &= \frac{1}{4}(a_i + b_i - c_i - d_i)\,, \end{aligned}$$

$$
\begin{aligned}
H_i^2 &= \frac{1}{4}(a_i - b_i + c_i - d_i)\,, \\
H_i^3 &= \frac{1}{4}(a_i - b_i - c_i + d_i)\,.
\end{aligned} \tag{4.48}
$$

Now also the tensors ${}^iF^{(1)}$, ${}^iF^{(2)}$, ${}^iF^{(3)}$ $(i = 12, ..., 7)$ may be found in explicit form if necessary. They consist of the fields E, H permuted in certain ways. For an explicit example we give ${}^1F^{(1)}$, ${}^1F^{(2)}$, ${}^1F^{(3)}$ (coefficients $f_{\alpha\beta\gamma\delta}$ defined by (4.52),(4.53)):



Applying the inverse matrix to the (4.47) we get

$$
{}^iF^{(0)} +{}^i F^{(1)} +{}^i F^{(2)} +{}^i F^{(3)} \;=\; a_i R_{i0} \equiv {}^i\bar{F}^{(0)}\,,
$$

$$
\begin{aligned}
{}^iF^{(0)} +{}^i F^{(1)} -{}^i F^{(2)} -{}^i F^{(3)} &= b_i R_{kl} \equiv {}^i\bar{F}^{(1)}\,, \\
{}^iF^{(0)} -{}^i F^{(1)} +{}^i F^{(2)} -{}^i F^{(3)} &= c_i R_{mn} \equiv {}^i\bar{F}^{(2)}\,, \\
{}^iF^{(0)} -{}^i F^{(1)} -{}^i F^{(2)} +{}^i F^{(3)} &= d_i R_{pq} \equiv {}^i \bar{F}^{(3)}\,,
\end{aligned}
\tag{4.49}
$$

these formulae are analogs of (4.34), (4.35) of the case d=4. The first equation from (4.49) gives us the analog of the *self-duality condition* for the tensor

$$
\bar{F}^{(0)} = \sum_{i=1}^{7} {}^i\bar{F}^{(0)}\,, \tag{4.50}
$$

if the *duality* itself is defined as

$$
(+3)\bar{F}^{(0)*}_{\alpha\beta} = \frac{1}{2}\sum_{\gamma\delta} f_{\alpha\beta\gamma\delta}\bar{F}^{(0)}_{\gamma\delta}\,, \tag{4.51}
$$

where the coefficients

$$
f_{\alpha 0\beta\gamma} = -c_{\alpha\beta\gamma}\,, \quad \frac{1}{2}[e_\alpha, e_\beta] = c_{\alpha\beta\gamma}e_\gamma\,; \tag{4.52}
$$

$$
f_{\alpha\beta\gamma\delta} = c_{\alpha\beta\gamma\delta}\,, \quad \frac{1}{2}(e_\alpha, e_\beta, e_\gamma) = c_{\alpha\beta\gamma\delta}e_\delta \tag{4.53}
$$

are the structure constants for the commutator $[e_\alpha, e_\beta]$ and the associator $(e_\alpha, e_\beta, e_\gamma)$ respectively, see also [14] in this connection.

The following equations are satisfied for elements:

$$
\begin{aligned}
\bar{F}_{10} &= \bar{F}_{23} = \bar{F}_{45} = \bar{F}_{76}\,, \\
\bar{F}_{20} &= \bar{F}_{31} = \bar{F}_{46} = \bar{F}_{57}\,, \\
\bar{F}_{30} &= \bar{F}_{12} = \bar{F}_{65} = \bar{F}_{47}\,, \\
\bar{F}_{40} &= \bar{F}_{51} = \bar{F}_{62} = \bar{F}_{73}\,, \\
\bar{F}_{50} &= \bar{F}_{14} = \bar{F}_{36} = \bar{F}_{72}\,, \\
\bar{F}_{60} &= \bar{F}_{17} = \bar{F}_{24} = \bar{F}_{53}\,, \\
\bar{F}_{70} &= \bar{F}_{25} = \bar{F}_{34} = \bar{F}_{61}\,.
\end{aligned}
\tag{4.54}
$$

When the elements of the field strength tensor F are expressed through the potentials analogously to the (4.28), the system (4.54) represents the *octonionic monogenity conditions of Krylov-Meilichson type* [33, 38]. In the quaternion level still the Fueter-type conditions hold.

The seven quantities of (4.54) form 7-dimensional subspace in the 28-dimensional Lie algebra D_4 of the Lie group $SO(8)$.

Let us define now the *duality property* for the other three tensors $\bar{F}^{(k)}$ in (4.49):

$$(-1)\bar{F}^{(k)*}_{\alpha\beta} = \frac{1}{2}\sum_{\gamma\delta} f_{\alpha\beta\gamma\delta}\bar{F}^{(k)}_{\gamma\delta}\,. \tag{4.55}$$

Here we have a 21-dimensional subalgebra formed by the generators $b_i R_{kl}$, $c_i R_{mn}$, $d_i R_{pq}$ where the index pairs kl, mn, pq are taken from the i-th row of the list of Cartan subalgebras (4.44).

Inversing the formulas (4.48) we get

$$\begin{aligned} a_i &= E_i + H_i^1 + H_i^2 + H_i^3\,, \\ b_i &= E_i + H_i^1 - H_i^2 - H_i^3\,, \\ c_i &= E_i - H_i^1 + H_i^2 - H_i^3\,, \\ d_i &= E_i - H_i^1 - H_i^2 + H_i^3\,. \end{aligned} \tag{4.56}$$

Considering the elements of the tensor ${}^i\bar{F}^{(k)}$ with the overmentioned index pairs from (4.44) it is easy to verify that their sum equals zero, so we have the following relations

$$\begin{aligned} {}^1\bar{F}^{(k)}_{10} + {}^1\bar{F}^{(k)}_{23} + {}^1\bar{F}^{(k)}_{45} + {}^1\bar{F}^{(k)}_{76} &= 0\,, \\ {}^2\bar{F}^{(k)}_{20} + {}^2\bar{F}^{(k)}_{31} + {}^2\bar{F}^{(k)}_{46} + {}^2\bar{F}^{(k)}_{57} &= 0\,, \\ {}^3\bar{F}^{(k)}_{30} + {}^3\bar{F}^{(k)}_{12} + {}^3\bar{F}^{(k)}_{65} + {}^3\bar{F}^{(k)}_{47} &= 0\,, \\ {}^4\bar{F}^{(k)}_{40} + {}^4\bar{F}^{(k)}_{51} + {}^4\bar{F}^{(k)}_{62} + {}^4\bar{F}^{(k)}_{73} &= 0\,, \\ {}^5\bar{F}^{(k)}_{50} + {}^5\bar{F}^{(k)}_{14} + {}^5\bar{F}^{(k)}_{36} + {}^5\bar{F}^{(k)}_{72} &= 0\,, \\ {}^6\bar{F}^{(k)}_{60} + {}^6\bar{F}^{(k)}_{17} + {}^6\bar{F}^{(k)}_{24} + {}^6\bar{F}^{(k)}_{53} &= 0\,, \\ {}^7\bar{F}^{(k)}_{70} + {}^7\bar{F}^{(k)}_{25} + {}^7\bar{F}^{(k)}_{34} + {}^7\bar{F}^{(k)}_{61} &= 0\,. \end{aligned} \tag{4.57}$$

These relations (if expressed through derivatives of potentials as in (4.28)) represent the *Fueter-type monogenity conditions* for an octonion-valued function of an octonion variable.

An analogous construction may be performed with the L_{ij}-matrices of the regular birepresentation of octonions.

Let us write an analog of (4.47) using a Hadamard-type matrix

$$\lambda = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix}, \tag{4.58}$$

then we have

$$\begin{aligned}
\frac{1}{4}(a_i L_{i0} + b_i L_{kl} + c_i L_{mn} + d_i L_{pq}) &= {}^{i}F^{(0)}, \\
\frac{1}{4}(-a_i L_{i0} - b_i L_{kl} + c_i L_{mn} + d_i L_{pq}) &= {}^{i}F^{(1)\prime}, \\
\frac{1}{4}(-a_i L_{i0} + b_i L_{kl} - c_i L_{mn} + d_i L_{pq}) &= {}^{i}F^{(2)\prime}, \\
\frac{1}{4}(-a_i L_{i0} + b_i L_{kl} + c_i L_{mn} - d_i L_{pq}) &= {}^{i}F^{(3)\prime}.
\end{aligned} \tag{4.59}$$

From the first equation we get relations between the field coefficients and physical fields:

$$\begin{aligned}
E_i &= \frac{1}{4}(a_i + b_i + c_i + d_i), \\
H_i^1 &= \frac{1}{4}(-a_i - b_i + c_i + d_i), \\
H_i^2 &= \frac{1}{4}(-a_i + b_i - c_i + d_i), \\
H_i^3 &= \frac{1}{4}(-a_i + b_i + c_i - d_i).
\end{aligned} \tag{4.60}$$

Inversing the Eqs. (4.60) we have

$$\begin{aligned}
a_i &= E_i - H_i^1 - H_i^2 - H_i^3, \\
b_i &= E_i - H_i^1 + H_i^2 + H_i^3, \\
c_i &= E_i + H_i^1 - H_i^2 + H_i^3, \\
d_i &= E_i + H_i^1 + H_i^2 - H_i^3.
\end{aligned} \tag{4.61}$$

Applying the inverse matrix to (4.59) we get the formulae analogous to (4.49)

$$ {}^{i}F^{(0)} + {}^{i}F^{(1)\prime} + {}^{i}F^{(2)\prime} + {}^{i}F^{(3)\prime} = a_i L_{i0} \equiv {}^{i}\tilde{F}^{(0)}, $$

$$\begin{aligned} {}^iF^{(0)} + {}^iF^{(1)\prime} - {}^iF^{(2)\prime} - {}^iF^{(3)\prime} &= b_i L_{kl} \equiv {}^i\tilde{F}^{(1)}\,, \\ {}^iF^{(0)} - {}^iF^{(1)\prime} + {}^iF^{(2)\prime} - {}^iF^{(3)\prime} &= c_i L_{mn} \equiv {}^i\tilde{F}^{(2)}\,, \\ {}^iF^{(0)} - {}^iF^{(1)\prime} - {}^iF^{(2)\prime} + {}^iF^{(3)\prime} &= d_i L_{pq} \equiv {}^i\tilde{F}^{(3)}\,, \end{aligned} \tag{4.62}$$

about indices see (4.49).

Now it is quite obvious that we have *self-duality property* (the first equation in (4.62)) for the tensor

$$\tilde{F}^{(0)} = \sum_{i=1}^{7} {}^i\tilde{F}^{(0)}\,, \tag{4.63}$$

if the dual tensor is defined analogously to (4.51) with the coefficient (-3) on the left-hand side:

$$(-3)\tilde{F}^{(0)*}_{\alpha\beta} = \frac{1}{2}\sum_{\gamma\delta} f_{\alpha\beta\gamma\delta}\tilde{F}^{(0)}_{\gamma\delta}\,. \tag{4.64}$$

For the other three tensors $\tilde{F}^{(k)}$, $k = 1,2,3$ we have *self-duality relations* if the *duality* is defined as

$$(+1)\tilde{F}^{(k)*}_{\alpha\beta} = \frac{1}{2}\sum_{\gamma\delta} f_{\alpha\beta\gamma\delta}\tilde{F}^{(k)}_{\gamma\delta}\,. \tag{4.65}$$

There is full agreement between our results and those of papers [10, 15, 19]. Corrigan et. al. [10, 15] have solved the *secular equation*

$$\lambda F_{\alpha\beta} = \frac{1}{2} f_{\alpha\beta\gamma\delta} F_{\gamma\delta} \tag{4.66}$$

and found the solutions $\lambda = 1, -3$ and $\lambda = -1, 3$. Fubini and Nicolai [19] have got the same result by the method of *projection operators*.

As a final remark we want to emphasize the *character of time reversal* in the eight-dimensional octonion formalism. In the four-dimensional Maxwell theory the time reversal (inversion) leads to the inversion of magnetic field. If this is the case also in the eight-dimensional theory then we can conclude (according to the construction of the field tensor from L- and R-matrices of octonion regbirep) that the time reversal here leads to the transition to the inverse representation ($L \leftrightarrow R$).

4.5 The Ansatz for potentials and the condition of harmonicity in dimensions 4 and 8

Here we investigate self-duality conditions for the Yang-Mills *field strength (curvature) tensor* in dimensions 4 and 8, [35, 36]. In both cases the hypercomplex formalism leads to the natural unification of the space-time and internal space through the (generalized) t'Hooft coefficients. In the case $d = 8$, the hypercomplex formalism of *ternary sedenions* ensures the treatment of the anti-self-dual sector fully analogous to the case $d = 4$.

4.5.1 The case of $d = 4$, octonion formalism

A general element X of the octonion algebra $\mathbf{O}$ may be represented as

$$X = x_0 e_0 + x_1 e_1 + \cdots + x_7 e_7 \,, \qquad x_0, x_1, \ldots, x_7 \in \mathbf{R} \,, \tag{4.67}$$

in what follows we use the terminology introduced in Sec. 3.2.

We choose now for Euclidean 4-space the proper octonionic part $x_4 e_4 + x_5 e_5 + x_6 e_6 + x_7 e_7$, where e_4, e_5, e_6, e_7 may be regarded as independent basis vectors, their nonassociativity being dismissed for a moment. Then x_4, x_5, x_6, x_7 are the coordinates of a vector in Euclidean 4-space. It appears that the pure quaternion part $x_1 e_1 + x_2 e_2 + x_3 e_3$ may be regarded as an internal space. Now the space indices of all Yang-Mills expressions have the values 4,5,6,7, the internal space indices, the values 1,2,3.

The Yang-Mills *field (curvature) tensor* is

$$F_{ab} = \partial_a A_b - \partial_b A_a + [A_a, A_b] \,, \tag{4.68}$$

the corresponding dual tensor F^* and self-duality are defined by Eqs. (4.29) and (4.32).

In our formalism, we take the *t'Hooft Ansatz* (see e.g. [55, 44]) in the form

$$A_a = \frac{1}{2}[e_a, e_r]\partial_r \ln \varphi \,, \tag{4.69}$$

where $a, r = 4, 5, 6, 7$ and r is the summation index.

We can introduce the *internal* degrees of freedom, stating that

$$A_a = \frac{1}{2} e_m A_a^m, \quad A_a^m = \eta_{ar}^m a_r , \tag{4.70}$$

where $a, r = 4,5,6,7$; $m = 1,2,3$; $a_r = \partial_r \ln\varphi$; and η_{ar}^m are the *t'Hooft coefficients* relating relating the space-time and internal degrees of freedom. In our formalism,

$$[e_a, e_r] = \eta_{ar}^m e_m , \tag{4.71}$$

i.e. the t'Hooft coefficients are the *structure constants* of the commutator algebra of octonions, the 7-dimensional simple *Mal'tsev algebra* M_7.

Substituting A_a (4.70) into the self-duality Eq. (4.32) we come after some algebra to the *harmonicity condition* or the *instanton equation*

$$\partial_c \partial_c (\ln\varphi) + (\partial_c \ln\phi)^2 = 0 . \tag{4.72}$$

4.5.2 The case $d = 8$, sedenion formalism

A general element X of the (binary) *sedenion algebra* has the form

$$X = x_0 e_0 + x_1 e_1 + \cdots + x_7 e_7 + x_8 e_8 + \cdots + x_{15} e_{15} , \tag{4.73}$$

where $x_0, x_1, ..., x_{15} \in \mathbf{R}$ and $e_0, e_1, ..., e_{15}$ are *sedenion units.* About sedenions see Secs. 3.3 and 3.4 (and [50, 37]).

Now we take for the Euclidean 8-space the *proper sedenion* part $x_8 e_8 + x_9 e_9 + ... + x_{15} e_{15}$ of the general sedenion (4.73), the pure octonionic part $x_1 e_1 + x_2 e_2 + \cdots + x_7 e_7$ remaining for the internal space. So, further the space-time indices take values 8,9,...,15, and the internal space indices, 1,2,...,7.

The field tensor F_{ab} has the form (4.68) (where $a, b = 8, 9, \ldots, 15$), the dual field tensor is now defined as

$$\tilde{F}_{ab} = \frac{1}{2} c_{abcd} F_{cd} , \tag{4.74}$$

where c_{abcd} are *associator structure constants* for binary sedenions (for proper sedenion units), their octonionic form is given by (4.53),

$$(e_a, e_b, e_c) = (e_a e_b) e_c - e_a (e_b e_c) = c_{abcd} e_d . \tag{4.75}$$

We now choose the *Ansatz* in the form

$$A_a = \frac{1}{12}[e_a, e_r]\partial_r \ln\varphi\,. \tag{4.76}$$

Substituting A_a (4.76) into (4.68), we get

$$\begin{aligned} F_{ab} = {} & \frac{1}{12}[e_a, e_b]\partial_a\partial_c \ln\varphi - \frac{1}{12}[e_a, e_c]\partial_c \ln\varphi + \\ & + \frac{1}{2\cdot 2\cdot 36}[[e_a, e_c], [e_b, e_d]](\partial_c \ln\varphi)(\partial_d \ln\varphi)\,. \end{aligned} \tag{4.77}$$

Now we must carry through some algebraic computation, [35]. In these calculations, we use mainly anticommutativity and *modified Moufang identity* in the form

$$(e_c, e_a)(e_b e_c) = \underbrace{e_c(e_a e_b)e_c}_{=e_c((e_a e_b)e_c)} + \{e_c, e_a, e_b e_c\}\,. \tag{4.78}$$

In the case of antiassociative units as for octonions and ternary sedenions the antiassociator term in (4.78) disappears and the usual *Moufang identity* is satisfied. In final results we pass over to the ternary sedenions, and therefore we omit in what follows all antiassociator terms (a.a.t.) originated by (4.78). Substituting the results of computations into (4.77), after some recombination we get the final result

$$\begin{aligned} F_{ab} + \tilde{F}_{ab} = {} & -\frac{1}{36}[e_a e_b]9\partial_c \ln\varphi)^2 \\ & -\frac{1}{12}[e_a, e_b](\partial_c\partial_c \ln\varphi - \frac{1}{3}(\partial_c \ln\varphi)^2) - \frac{1}{12}[e_a, e_b]\partial_c \ln\varphi)^2 = \\ = {} & -\frac{1}{12}[e_a, e_b](\frac{1}{3}(\partial_c \ln\varphi)^2 + \partial_c\partial_c \ln\varphi - \frac{1}{3}(\partial_c \ln\varphi)^2 + (\partial_c \ln\varphi)^2) = \\ = {} & -\frac{1}{12}[e_a, e_b](\partial_c\partial_c \ln\varphi + (\partial_c \ln\varphi)^2)\,. \end{aligned} \tag{4.79}$$

Then the anti-self-duality leads to the *harmonicity condition*

$$\partial_c\partial_c \ln\varphi + (\partial_c \ln\varphi)^2 = 0 \tag{4.80}$$

in analogy of the case d=4, (4.72). We must emphasize that to get rid of an enormous bulk of a.a.t. we must use ternary sedenions instead of binary ones. In any case, binary product xy may always be interpreted as a ternary one if we write $xy = xye_0$.

The exact correspondence between the sedenion duality and the broken duality relations in 8-space deserves a special discussion.

4.5.3 The topological aspect of the problem. Hopf maps

We have been dealing here with self-duality problem in octonion and sedenion formalisms. This context may be extended into a full complex of exceptional objects, consisting of all *division algebras* [43], *Hopf fibrations* [52, 42], and *monogenity equations* of the corresponding hypercomplex analyses (see refs. in the Table 12; the latter problems were discussed briefly in Sec. 4.4, 4.5). All this is expressed in a brief and compact form in the Table 12, [35], where Hopf fibrations play a central role. A very interesting material about Hopf maps and their applications with numerous references may be found in [42, 54]. A more complete list of references on quaternion analysis is in [50].

We already have been acquainted with exceptional algebraic systems as *division algebras* over $\mathbf{R}$ and some accompanying exceptional objects as exceptional groups $(G_2, F_4, E_6, E_7, E_8)$ and corresponding geometries, and the monogenity conditions in corresponding hypercomplex analyses. In topology, in the theory of *fibre bundles* we also have an imprint of these exceptional systems in the form of exceptional *Hopf fibrations* or *Hopf maps* of spheres $S^1 \mapsto S^1, S^3 \mapsto S^2, S^7 \mapsto S^4, S^{15} \mapsto S^8$.

Hopf fibrations (maps) are naturally connected with pairs of elements from the division algebras **R**, **C**, **H**, **O**. On the left hand side of the map there are spheres S^{2n-1} $(n = 1, 2, 4, 8)$ which may be represented by a common equation $z_1\bar{z}_1 + z_2\bar{z}_2 = 1$ in the planes over division algebras **R**, **C**, **H**, **O** respectively. The spheres S^n $(n = 1, 2, 4, 8)$ on the right hand side may be represented as *projective lines* $\mathbf{R}P^1, \mathbf{C}P^1, \mathbf{H}P^1, \mathbf{O}P^1$ over the same division algebras, respectively, with the *harmonic coordinates* $[z_1 : z_2] \equiv [\lambda z_1, \lambda z_2]$.

The Hopf map is given by the correspondence $p((z_1, z_2)) = [z_1 : z_2]$. In the case $n = 2$ it is the common *stereographical projection* relating spinorial and vectorial coordinates.

The essential feature of this construction is the fact that the spheres and projective lines in the Hopf fibrations are given by pairs of corresponding hypercomplex numbers, one member of the pair representing the *basis*, the other a *fibre*. The CD-process (Sec. 3.2) combines the members of the pair into the new unified object – to the higher hypercomplex number of the doubled dimensionality. We have also seen, that this process describes the unification of space-time (the basis) and internal space (a fibre).

Table 12. **Summary about Hopf maps**

Monogenity equations	Hopf fibration ▼ Projective space ▼	Physical system
	$S^1 \xrightarrow{Z_2} S^1 \quad \sim RP^1$	Kink of Sine-Gordon [54]
Cauchy-Riemann	$S^3 \xrightarrow{S^1} S^2 \quad \sim \mathbb{C}P^1$	Dirac monopole [54]
Fueter [20,21]	$S^7 \xrightarrow{S^3} S^4 \quad \sim \mathbb{H}P^1$	d=4 instanton [54]
Gürsey, Tze [25] Dündarer et al. [14] Sorgsepp, Lõhmus (Ref. [331] of Ch. 1)	$S^{15} \xrightarrow{S^7} S^8 \quad \sim \mathbb{O}P^1$	d=8 instanton. (Ref. [117] of Ch. 1 and [35])
Fibration structure ▶	fibre bundle fibre basis	

basis: Compactified physical (euclidean) space

fibre: Compactified internal space

fibre bundle: The unifying fibre bundle

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Chapter 5

Moufang-Mal'tsev symmetry

Moufang-Mal'tsev symmetry is an extension and generalization of the classical Lie group symmetry concept (Sec. 5.1), it rests on the notions of Moufang loop and Mal'tsev algebra which are straightforward and natural generalizations of Lie groups and Lie algebras (Sec. 5.2). For physical applications these algebraic systems are realized in terms of birepresentations (Sec. 5.3). The generators of continuous Moufang transformations form open Moufang-Mal'tsev algebra, closing it appears to be an interesting and nontrivial procedure (Sec. 5.4.2). In the field theoretical context a complete set of conservation laws may be found, part of them having their origin in the nonassociativity (Secs. 5.4.1, 5.4.3). Various symmetry types of the field Lagrangian may be considered (Secs. 5.4.4, 5.4.5).

5.1 General introduction – the philosophy of the subject

Moufang loops and Mal'tsev algebras are straightforward and very natural (minimal) generalizations of Lie groups and Lie algebras, respectively. Because of the uniqueness of octonions also octonionic Moufang loop and the corresponding simple (non-Lie) Mal'tsev algebra are of exceptional importance for mathematics and physics (unfortunately this very interesting aspect remains not elaborated in the present monograph).

Recently Mal'tsev algebras have found their way into physics through the investigation of *anomalies* in quantum field theories. Calculations [12] based on the Bjorken-Johnson-Low limit theorem [4, 13] demonstrate that some canonical *equal time commutators* (ETC) of nonabelian chiral gauge theory appeared to be anomalous and the modified ETC had to be satisfied the *Mal'tsev identity* instead of Jacobi's one. The same result was achieved in [29, 44] by different methods.

In the last centennial the theory of Moufang loops and Mal'tsev algebras has been quite well-developed [2, 9, 18, 27]. Finite simple nonassociative Moufang loops and finite-dimensional non-Lie Mal'tsev algebras have been classified. Existence theorems have been proved relating Mal'tsev algebras with analytic Moufang loops. The structure theory of Mal'tsev algebras paralleling the structure theory of Lie algebras also has been quite well-developed. The geometric meaning of analytic Moufang loops and Mal'tsev algebras is clarified. Also a representation theory, Eilenberg bimodule representation theory for Mal'tsev algebras has been considerably advanced. Some perspectives of applications make the theory of Moufang loops and Mal'tsev algebras attractive also for physicists. For physical applications nevertheless some reformulation of the present theory is needed with elaboration of some specific additional aspects.

Groups are used in physics as tools for symmetry considerations, they serve as a synonym of symmetry, describing and **measuring** it. Groups form a well-established and elaborated in details apparatus, beginning with Erlangen Program in geometry, continuing with Noether theorem and lasting (but not terminating) with recent new topics of infinite-dimensional groups and noninvariance groups.

On the other hand, representation theory of Moufang loops was a rare discussed subject and even now after some research it is yet far from such a level of elaboration as it is common for Lie groups. Unfortunately the existing Eilenberg type representation theory for Mal'tsev algebras is not oriented for physical applications but only for the study of structure of Mal'tsev algebras themselves. For physics these bimodule representations are quite awkward and nonattractive because one of the criteria of adequacy of the physical theories is their beauty and simplicity. Just the elaboration of physically applicable representation theory is the very aim of Moufang-Mal'tsev symmetry. Here in this chapter we only give a preliminary analysis of one of the possibilities of advancing the theory for applications, which has been studied in [20, 21, 22, 30, 31, 32, 33, 34, 35, 38].

In brief the concept of the *Moufang-Mal'tsev (MM-) symmetry* lies in the following. If some system or object (quantity, variable) is invariant with respect of Moufang transformations realizing some birepresentation of a Moufang loop, then it is natural to regard this system (quantity, variable) *Moufang invariant* and the transformation itself as Moufang symmetry. We call this type of symmetry extension *Moufang-Mal'tsev symmetry* if it is realized through birepresentations of an analytic Moufang loop.

The concept of MM-symmetry represents some kind of compromise between the traditional (classical) group theoretical symmetry and the attempt to extend it beyond the region (realm) of Lie groups. As we shall see in this chapter the MM-symmetry explores Lie group formalism but has features imprinted in it having their origin in non-Lie nonassociativity (as additional conservation laws, etc.). We have already noted some justification of extension of symmetry treatment arising from anomalies in QFT but there are also some more general suggestions (e.g. [15]) that *strict identification of symmetries with groups perhaps is not absolutely incontestable (or adequate?) and needs some revision*, and there is serious reasons for the extensions of the group theoretical symmetry method.

In the general mathematical framework the concept of MM-symmetry has been introduced in [34]. In this paper *complete* system of differential equations (DE) has been found for the determination of invariants of continuous Moufang transformations. The completeness conditions of the system of DE are tightly related to the structure of birepresentations of Mal'tsev algebras and with the Sagle-Yamaguti identity in particular. Also the concepts of *weak* and *hidden* MM-symmetries are introduced with the installment of *complete* systems of DE for the corresponding specific (weak, hidden) invariants. The completeness of the system is quaranteed by the fact that generators of weak and hidden MM-symmetries realize *weak* and *generalized* representations of Mal'tsev algebras in sense of Yamaguti [49].

In [20] MM-symmetry was formulated in the physical field theoretical context. Introducing Noether currents and charges *complete* system of conservation laws corresponding to continuous MM-symmetry has been derived. For the time components of Noether currents *equal time commutators* (ETC) have been calculated generalizing the corresponding entities [6] based on group formalism in QFT. Results of calculations can be formulated in well-determined terms of birepresentations. Noether charges generated by continuous MM-symmetries realize birepresentations of the corresponding Mal'tsev algebra. In the case of *weak* or *hidden* MM-symmetries the corresponding

Noether charges realize *weak* or *generalized* representations of the Mal'tsev algebra (in sense of Yamaguti [49]), respectively. *Complete* sets of conservation laws have been found for both these particular cases.

It should be mentioned that unlike of papers [12, 29, 44] where Mal'tsev algebras appeared as the manifestation of *anomaly* (i.e. as a result of *quantum-mechanical* symmetry breaking), in our approach Mal'tsev algebras appear already on the *classical* level (before quantization). It seems natural that in quantization of a system with MM-symmetry the problem of anomalies also will be present. We have no experience to decide which is less complicated to deal with – systems with group symmetries and anomalies, or systems with MM-symmetries but without anomalies. In this context it seems very interesting to investigate birepresentations of Mal'tsev algebras encountered in [12, 29, 44].

5.2 Moufang loops and Mal'tsev algebras

5.2.1 Moufang loops

A *Moufang loop* [5, 28] is a set G with a binary operation (*multiplication*) denoted by juxtaposition so that the following axioms are satisfied:

1) in the equation $gh = k$, the knowledge of any two of $g, h, k \in G$ specifies the third one uniquely;

2) there is a distinguished *unit* or *identity element* e of G with the property $eg = ge = g$, $\forall g \in G$;

3) the *Moufang identity* holds:

$$(gh)(kg) = g((hk)g)\,, \qquad \forall g, h, k \in G\,. \tag{5.1}$$

A set G with such a binary operation that only axioms 1) and 2) are satisfied is called a *loop*. An element e is then the *identity* element of the loop G. The most familiar kind of loops are those with the *associative law* $(gh)k = g(hk)$ and they are called *groups*. Thus, roughly speaking, loops are the "nonassociative groups". A (Moufang) loop is said to be *commutative* if $gh = hg$ $\forall g, h \in G$, and only commutative associative (Moufang) loops are said to be *abelian*.

The most remarkable property of Moufang loops is their *diassociativity*: the subloop generated by any two elements in a Moufang loop is associative (group). Hence, for any g, h in a Moufang loop G one has the following mild associative laws:

$$(hg)g = hg^2 \,, \quad g(gh) = g^2 h \,, \tag{5.2}$$

$$(gh)g = g(hg) \,. \tag{5.3}$$

Here we note that thanks to (5.3), the Moufang identity (5.1) can be written in the symmetric form

$$(gh)(kg) = g(hk)g \,. \tag{5.4}$$

As in the case of groups, one can define the notion of the *inverse element* of $g \in G$. The unique solution of the equation $gx = e$ ($xg = e$) is called the *right (left) inverse element* of $g \in G$ and denoted as g_R^{-1} (g_L^{-1}). From the diassociativity of a Moufang loop G one can easily infer the following elementary properties:

$$\begin{aligned} g_R^{-1} &= g_L^{-1} = g^{-1} \,, \\ g^{-1}(gh) &= (hg)g^{-1} = h \,, \\ (g^{-1})^{-1} &= g \,, \\ (gh)^{-1} &= h^{-1}g^{-1} \,; \quad \forall g, h \in G \,. \end{aligned}$$

5.2.2 Analytic Moufang loops and Mal'tsev algebras

The Moufang loop G is said to be *analytic* [26] if G is a real analytic manifold so that both the Moufang *loop operation* $G \times G \to G : (g, h) \mapsto gh$ and the *inversion map* $G \to G : g \mapsto g^{-1}$ are analytic ones. We shall denote the dimension of G by r. The *local coordinates* of $g \in G$ are denoted (in a fixed chart of e) by $g^1, g^2, \dots, g^r$, and the local coordinates of the identity element e of G are supposed to be zero: $e^i = 0$ ($i = 1, 2, \dots, r$). As in the case of Lie groups, we can consider the Taylor expansions

$$(gh)^i = g^i + h^i + a^i_{jk} g^j h^k + \dots \,; \quad i, j, k = 1, 2, \dots, r \tag{5.5}$$

and introduce the antisymmetric quantities

$$c^i_{jk} := a^i_{jk} - a^i_{kj} = -c^i_{jk} \,, \quad i, j, k = 1, \dots r \,, \tag{5.6}$$

called the *structure constants* of G.

The *tangent algebra* of G can be defined [26, 40] similarly to the tangent (Lie) algebra of the Lie [39] group and we denote it by Γ. Geometrically, this algebra is the tangent space of G at e. The product of X, Y in Γ will be denoted by $[X, Y]$:

$$[X,Y]^i := c^i_{jk} X^j Y^k = -[Y,X]^i\,; \quad i,j,k = 1,2,\ldots,r\,. \tag{5.7}$$

The tangent algebra of G need not be a Lie algebra. In other words, there may be triples $X, Y, Z \in \Gamma$, such that the Jacobi identity fails in Γ:

$$J(X,Y,Z) := [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] \neq 0\,.$$

Instead we have for all $X, Y, Z \in \Gamma$ [26] a more general identity

$$[[X,Y],[Z,X]] + [[X,Y],Z],X] + [[[Y,Z],X],X] + [[[Z,X],X],Y] = 0\,, \tag{5.8}$$

called *Mal'tsev identity.* The tangent algebra Γ of G is hence said to be the *Mal'tsev algebra.* This identity concisely reads [41]

$$J(X,Y,[X,Z]) = [J(X,Y,Z),X]\,, \tag{5.9}$$

from which it can be easily seen that every Lie algebra is a Mal'tsev algebra as well. Every finite-dimensional real Mal'tsev algebra is known to be [1, 14, 17] the tangent algebra of some analytic Moufang loop.

5.2.3 Mal'tsev algebras as Lie triple systems

In a Mal'tsev algebra Γ the *Yamaguti triple product (triple brackets)* [48, 49] may be defined as follows

$$[x,y,z] := [x,[y,z]] - [y,[x,z]] + [[x,y],z]\,. \tag{5.10}$$

In components

$$[x,y,z]^i = 6d^i_{jkl} x^j y^k z^l\,; \quad i,j,k,l = 1,2,\ldots,r\,, \tag{5.11}$$

where $6d^i_{jkl} = c^i_{js} c^s_{kl} - c^i_{ks} c^s_{jl} + c^i_{sl} c^s_{jk}$.

In every Mal'tsev algebra also the following identities are satisfied [48, 49]:

$$[x, x] = 0\,, \tag{5.12}$$

$$[x, x, y] = 0\,, \tag{5.13}$$

$$\sum_{cycl}[x, y, z] + \sum_{cycl}[[x, y], z] = 0\,, \tag{5.14}$$

$$[[x, y], z, w] + [[y, z], x, w] + [[z, x], y, w] = 0\,, \tag{5.15}$$

$$[x, y, [z, w]] = [[x, y, z], w] + [z, [x, y, w]]\,, \tag{5.16}$$

$$[x, y, [z, u, v]] = [[x, y, z], u, v] + [z, [x, y, u], v] + [z, u, [x, y, v]], \tag{5.17}$$

$$\forall x, y, z, u, v, w \in \Gamma\,.$$

A linear space with bilinear (commutator) operation [x,y] and trilinear operation (5.10), satisfying identities (5.12- 5.17) is called *generalized Lie triple system* (GLTS) [47]. Every Mal'tsev algebra is a GLTS. If (5.10) is considered as an relation between the operations, it is a necessary and sufficient condition for the GLTS to be a Mal'tsev algebra, [49]. The equation (5.16) is called *Sagle-Yamaguti identity.* Yamaguti [48] proved that in anticommutative algebras (over fields with char $\neq 2, 3$) the Mal'tsev identity (5.8) and the identity (5.16) are equivalent, so the Sagle-Yamaguti identity can be taken as for the definitory identity instead of the Mal'tsev identity.

K.Yamaguti [48, 49] proved also the possibility of embedding a Mal'tsev algebra into a Lie algebra. It occurs that every Mal'tsev algebra can be realized as a subspace of some Lie algebra so that the Mal'tsev algebra operation is a projection of the Lie algebra operation to this subspace. An analogous realization (embedding) for GLTS was proved in [47]. In terms of such an embedding the Sagle-Yamaguti identity may be interpreted as a Jacobi identity for the overwhelming Lie algebra.

In every Mal'tsev algebra also *Loos triple product (Loos brackets)* [23] may be defined by

$$3\{x, y, z\} = [x, [y, z]] - [y, [x, z]] + 2[[x, y], z]\,. \tag{5.18}$$

In components,

$$\{x, y, z\}^i = a^i_{jkl} x^j y^k z^l\,; \quad i, j, k, l = 1, 2, \ldots, r\,, \tag{5.19}$$

where

$$3a^i_{jkl} = c^i_{js}c^s_{kl} - c^i_{ks}c^s_{jl} + 2c^i_{sl}c^s_{jk}\,, \tag{5.20}$$

and c^i_{jk} are structure constants of the Mal'tsev algebra under consideration.

The Loos and Yamaguti brackets are related by the equation

$$3\{x,y,z\} = [x,y,z] + [[x,y],z]\,, \tag{5.21}$$

or in components

$$3a^i_{jkl} = 6d^i_{jkl} + c^i_{sl}c^s_{jk}\,. \tag{5.22}$$

The following identities are satisfied [23] in every Mal'tsev algebra Γ

$$\{x,x,y\} = 0\,, \tag{5.23}$$

$$\{x,y,z\} + \{y,z,x\} + \{z,x,y\} = 0\,, \tag{5.24}$$

$$\{x,y,\{z,u,v\}\} = \{\{x,y,z\}u,v\} + \{z,\{x,y,u\},v\} + \{z,u,\{x,y,v\}\}\,, \tag{5.25}$$

$$\forall x,y,z,u,v \in \Gamma\,.$$

A linear space with the trilinear (Loos) product {x,y,z} satisfying identities (5.23), (5.24), (5.25) is called *Lie triple system* (LTS), [8, 10, 11, 19, 24, 46]. Every Mal'tsev algebra is also a LTS. Lie triple systems have quite a definite geometric nature, they serve as tangent algebras for *symmetric spaces*, [8, 24, 46].

5.3 Birepresentations of Moufang loops and Mal'tsev algebras

5.3.1 Birepresentations of Moufang loops

Let $\mathfrak{X}$ be a set and let $\mathfrak{T}(\mathfrak{X})$ be *transformation group* of $\mathfrak{X}$, i.e. the group of bijective maps of $\mathfrak{X}$ onto $\mathfrak{X}$. Elements of $\mathfrak{T}(\mathfrak{X})$ are called the *transformations* of $\mathfrak{X}$. Multiplication in $\mathfrak{T}(\mathfrak{X})$ is defined as the *composition* of transformations, and *identity element* of $\mathfrak{T}(\mathfrak{X})$ coincides with identity transformation I of $\mathfrak{X}$.

A pair (S,T) of the maps $g \to S_g$, $g \to T_g$ of a Moufang loop G into the group $\mathfrak{T}(\mathfrak{X})$ is said [31] to be an *action* of G on $\mathfrak{X}$ if

$$S_e = T_e = I\,, \tag{5.26}$$

and

$$S_gT_gS_h \overset{(a)}{=} S_{gh}T_g\,, \quad S_gT_gS_h \overset{(b)}{=} T_{hg}S_g \tag{5.27}$$

hold for all $g, h \in G$. We call the pair (S, T) a *birepresentation* of G, and the transformations $S_g, T_g \in \mathfrak{T}(\mathfrak{X})$ $(g \in G)$ *G-transfromations (Moufang-transformations)* of $\mathfrak{X}$.

The properties of such transformations were recently discussed in [20, 30, 31, 32, 33, 34, 35, 36]. Let us raise here only the most elementary ones:

$$\begin{aligned} &S_g T_g = T_g S_g\,, \quad S_g^{-1} = S_{g^{-1}}\,, \quad T^{-1} = T_{g^{-1}}\,, \\ &S_g S_h T_h T_g = T_h T_g S_g S_h\,, \quad \forall g, h \in G \end{aligned} \tag{5.28}$$

A birepresentation (S, T) of G is said to be *linear* if $\mathfrak{X}$ is a linear space and S_g, T_g $(g \in G)$ are linear transformations of $\mathfrak{X}$. It is an easy excercise to show that if (S, T) is an action of G on a set $\mathfrak{X}$, and $\mathfrak{J}(\mathfrak{X})$ denotes the linear space of functions on $\mathfrak{X}$, then the pair (S', T') of the maps

$$g \mapsto S'_g : \ (S'_g f)(x) = f(S_g^{-1} x)\,, \qquad g \mapsto T'_g : \ (T'_g f)(x) = f(T_g^{-1} x)\,,$$

$$g \in G\,, \quad x \in X\,, \quad f \in J(X)$$

defines a *linear* action, i.e. a birepresentation of G on $\mathfrak{J}(\mathfrak{X})$.

5.3.2 Generalized Lie equations

Let G be an analytic Moufang loop and let $\mathfrak{X}$ be a differentiable manifold. The dimensions of G and $\mathfrak{X}$ will be denoted by r and n, respectively.

An action (S, T) of G on $\mathfrak{X}$ is said to be *differentiable (smooth, analytic)* if the local coordinates of the points $S_g A$ and $T_g A$ are differentiable (smooth, analytic) functions of the local coordinates of the points $g \in G$ and $A \in \mathfrak{X}$. In this case, (S, T) is called *differentiable (smooth, analytic) birepresentation* as well.

Now, let GL_n denote the *general linear group* of n-dimensional (real or complex) vector space V_n. Let G be an analytic Moufang loop with the identity element e, and let (S, T) be a differentiable linear birepresentation of G in GL_n. By fixing a base in V_n, one can represent an element g of G by two nonsingular matrices $S_g, T_g \in GL_n$ which will further be assumed to be differentiable with respect to g as many times as needed. It is obvious that we should define the *generators* of (S, T) as follows:

$$S_i = \partial_i S_g|_{g=e}\,, \quad T_i = \partial_i T_g|_{g=e}\,; \quad i = 1, 2, \ldots, r\,. \tag{5.29}$$

We have here denoted $\partial_i := \partial/\partial g^i$.

With the following theorem, the *generalized Lie equations* for differentiable linear Moufang transformations are stated.

Theorem 5.1., [37]). *Let (S,T) be a differentiable linear birepresentation of an analytic Moufang loop G. Then the birepresentation matrices S_g, T_g ($g \in G$) satisfy the system of simultaneous differential equations (generalized Lie equations)*

$$v_i^n(g)\partial_n S_g = S_g T_g S_i T_g^{-1}\,, \quad v_i^n(g)\partial_n T_g = S_g^{-1} T_i S_g T_g\,, \tag{5.30}$$

$$j = 1, 2, \ldots, r\,.$$

Proof. We obtain the first equation at once by differentiating the first equation in (5.27) with respect to the local coordinates h^i, $(i = 1, 2, \ldots, r)$ of $h \in G$ at $h = e$. To show the 2nd equation in (5.30), we must first note that $S_h T_h T_g = T_{gh} S_h$. Differentiating the latter with respect to h^i at $h = e$, we get

$$S_i T_g + T_i T_g = v_i^n(g)\partial_n T_g + T_g S_i\,,$$

from which follows that

$$v_i^n(g)\partial_n T_g = S_i T_g - T_g S_i + T_i T_g\,. \tag{5.31}$$

So we must show that

$$S_i T_g - T_g S_i + T_i T_g = S_g^{-1} T_i S_g T_g\,.$$

This identity reads

$$S_g^{-1} T_i S_g + T_g S_i T_g^{-1} = S_i T_i\,, \quad i = 1, 2, \ldots, r\,, \tag{5.32}$$

and can easily be obtained by differentiating (5.28) with respect to h^i at $h = e$. □

Corollary. By means of (5.32), we can rewrite (5.30) as follows

$$v_i^n(g)\partial_n S_g = S_g S_i + [S_g, S_i]\,, \tag{5.33}$$

$$v_i^n(g)\partial_n T_g = T_i T_g + [S_i, T_g]\,, \tag{5.34}$$

$$i = 1, 2, \ldots, r\,.$$

When $S_g T_h = T_h S_g$ for all $g, h \in G$, then (5.30) and (5.34) return the familiar *Lie equations.*

Let us now rewrite the generalized Lie equations (5.30) as

$$v_i^n(g)\partial_n S_g = S_g S_i'(g)\,, \quad v_i^n(g)\partial_n T_g = T_i'(g) T_g\,, \tag{5.35}$$

with

$$S_i' :\overset{(a)}{=} T_g S_i T_g^{-1}\,, \quad T_i' :\overset{(b)}{=} S_g^{-1} T_i S_g\,. \tag{5.36}$$

Keeping close to the terminology of [35, 36], the matrices $S_i'(g), T_i'(g)$ are called the *derivative generators* of (S, T). We have the obvious initial conditions

$$S_i'(e) = S_i\,, \quad T_i'(e) = T_i\,, \qquad j = 1, 2, \ldots, r\,. \tag{5.37}$$

The identities (5.32) read

$$S_i'(g) + T_i'(g) = S_i + T_i\,, \qquad i = 1, 2, \ldots, r\,. \tag{5.38}$$

Theorem 5.2. *The derivative generators of (S, T) satisfy the system of simultaneous Heisenberg-like equations*

$$v_k^n(g)\partial_n S_i'(g) = [T_k'(g), S_i'(g)]\,, \quad v_k^n(g)\partial_n T_i'(g) = [T_i'(g), S_k'(g)]\,, \tag{5.39}$$
$$i, k = 1, 2, \ldots, r\,.$$

For the proof see [37].

Corollary. The derivative generators of (S, T) satisfy the commutation relations (CR)

$$[S_i'(g), T_k'(g)] = [T_i'(g), S_k'(g)]\,, \quad i, k = 1, 2, \ldots, r\,. \tag{5.40}$$

For the proof we should add equations (5.39) and then use (5.38).

Now let us define the *structure functions* $c_{ik}^n(g)$ of G by

$$v_i^n(g)\partial_n v_k^j(g) - v_k^n(g)\partial_n v_i^j(g) = c_{ik}^n(g) v_n^j(g)\,. \tag{5.41}$$

The direct computation show that $c_{ik}^j(e) = c_{ik}^j$.

Theorem 5.3. *The integrability conditions of the generalized Lie equations* (5.30) *read as commutation relations*

$$[S_i'(g), S_k'(g)] = c_{ik}^n(g) S_n'(g) - 2[S_i'(g), T_k'(g)]\,, \tag{5.42}$$
$$[T_i'(g), T_k'(g)] = -c_{ik}^n(g) T_n'(g) - 2[T_i'(g), S_k'(g)]\,, \tag{5.43}$$

$$i, k = 1, 2 \ldots, r\,.$$

Proof. As an example we prove (5.42). Differentiating the first equation of (5.35) with respect to g^m, we obtain

$$\begin{aligned}
v_k^m(g)\partial_m v_i^n(g)\partial_n S_g \;&+\; v_k^m(g)v_i^n(g)\partial_m\partial_n S_g \\
&= v_k^m(g)\partial_m S_g S'(g) + v_k^m(g) S_g \partial_m S_i'(g) \\
&= S_g S'(g) S_k'(g) S_i'(g) + S_g[T_k'(g), S_i'(g)]\,.
\end{aligned}$$

Exchange now the indices i and k, and subtract the resulting equality from the original one. Reducing then the *second-order partial derivatives*, we get

$$\begin{aligned}
c_{ki}^m(g)v_m^n(g)\partial_n S_g &= S_g[S_k'(g), S_i'(g)] + S_g[T_k'(g), S_i'(g)] - \\
&- S_g[T_i'(g), S_k'(g)] = \\
&= S_g[S_k'(g), S_i'(g)] + 2S_g[T_k'(g), S_i'(g)]\,.
\end{aligned}$$

We must use GLE once more on the left-hand side of this equality. Then, after the left-division by S_g and rearranging the terms, we obtain the required CR (5.42). CR (5.43) can be proved similarly starting from the second eq. of (5.35). □

Corollary. The generators of (S,T) satisfy the CR

$$[S_i, S_k] = c_{ik}^n S_n - 2[S_i, T_k]\,, \tag{5.44}$$

$$[T_i, T_k] = -c_{ik}^n T_n - 2[T_i, S_k]\,, \tag{5.45}$$

$$i, k = 1, 2, \ldots, r\,.$$

Remark. A more general (nonlinear) version of CRs (5.44),(5.45) can be established in another way [32, 34] as well. It is also easy to see that CR (5.40) can now be re-obtained from the identity $[S_i'(g), S_k'(g)] = -[S_k'(g), S_i'(g)]$.

5.3.3 Birepresentations of Mal'tsev algebras

The representation theory of *Mal'tsev algebras* which follows the concept of birepresentation introduced by S.Eilenberg [7] has been well elaborated [16, 49] (see also reviews [2, 18] where the main results of this theory are outlined. In this section, the following two *non-Eilenbergian* but equivalent definitions of a *birepresentation* of the Mal'tsev algebra are formulated. These definitions seem also to be supremely natural from the point of view of the theory of *alternative algebras* [42, 43]. In the next subsection (Theorem 5.5) it will be shown that the integrability conditions of GLE are related to the non-Eilenbergian representations of Mal'tsev algebras.

Let M be a *Mal'tsev algebra* and let L be a *Lie algebra*.

Definition I. A pair (S,T) of linear maps $M \to L: \ x \mapsto Sx, x \mapsto Tx$ of M into L is said to be a *birepresentation* of the Mal'tsev algebra M if for all $x,y,z \in M$ the following identities hold (in L):

$$[Sx,Sy] = S[x,y] - 2[Sx,Ty]\,, \tag{5.46}$$
$$[Tx,Ty] = -T[x,y] - 2[Tx,Sy]\,, \tag{5.47}$$
$$6[F(x;y),Sz] = S[x,y,z]\,, \tag{5.48}$$
$$6[F(x;y),Tz] = T[x,y,z]\,. \tag{5.49}$$

where $F(x;y)$ is defined by

$$3F(x;y) := S[x,y] - T[x,y] - 3[Sx,Ty]\,, \tag{5.50}$$

and $[x,y,z]$ is the *Yamaguti triple product* [48, 49] in M defined by (5.10).

The identities (5.48), (5.49) are called *reductivity conditions* for the birepresentation (S,T).

Definition II. A pair (S,T) of linear maps $x \mapsto Sx, x \mapsto Tx$ of M into L is said to be a *birepresentation* of the Mal'tsev algebra M if the identities (5.46), (5.47) and

$$[[Sx,Sy],Sz] = S\{x,y,z\}\,, \tag{5.51}$$
$$[[Tx,Ty],Tz] = T\{x,y,z\} \tag{5.52}$$

hold for all $x,y,z \in M$, where the *Loos triple product* $\{x,y,z\}$ [23] is defined in M by (5.18).

These definitions can be motivated by the fact that the birepresentations in the above sense appear as *differentials of continuous birepresentations* of the analytic Moufang loops, [30, 32, 34]. Also, it must be noted that the Jacobi identities in the Lie algebra L are quaranteed [30] by the identities of the *Lie* and *general Lie triple systems* of the Mal'tsev algebra M, [23, 48, 49].

The connection of these birepresentations with the Eilenbergian representations is not clear.

5.3.4 Derivative Mal'tsev algebras

For $x, y \in T_e(G)$, define their new product $[x, y]_a$ in $T_e(G)$ by

$$[x, y]_a^i := c_{jk}^i(a) x^j y^k = -[y, x]_a^i \,, \tag{5.53}$$

$$i, j, k = 1, 2, \ldots, r \,.$$

The tangent space $T_e(G)$ with such a multiplication is said to be the *derivative* of the tangent algebra Γ of G and is denoted by Γ'_a. For $g, h \in G$, define [5, 35, 36] their *derivative product*

$$(gh)'_a = (ga^{-1})(ah) \,, \qquad g, h, a \in G \,. \tag{5.54}$$

satisfying also [3] all the axioms of a Moufang loop. This loop with the new multiplication (6.1) is called *derivative loop* of G and is denoted as G'_a. The identity element of G'_a is also e, and the inverse element of $g \in G'_a$ is g^{-1}.

There are some straightforward results connecting derivative loops and algebras and their birepresentations, [37]. The situation may be described by the following theorems.

Theorem 5.4. *Let G'_a be a derivative loop of an analytic Moufang loop G, and let Γ be the tangent algebra of G. Then the derivative algebra Γ'_a of Γ is the tangent algebra of the derivative Moufang loop G'_a of G.*

It would be natural to call Γ'_a the *derivative Mal'tsev algebra* of Γ.

Theorem 5.5. *Let (S, T) be a differentiable linear birepresentation of an analytic Moufang loop G, and let Γ be the tangent Mal'tsev algebra of G. Then, for each $a \in G$, the pair $*(S, T)'_a$ of the maps*

$$x \mapsto (Sx)'_a := x^i S'_i(a) \,, \quad x \mapsto (Tx)'_a := x^i T'_i(a) \qquad (x \in \Gamma'_a) \tag{5.55}$$

is a birepresentation of the derivative Mal'tsev algebra Γ'_a of Γ.

5.4 Moufang-Mal'tsev symmetry

5.4.1 Moufang symmetries and currents

Let us now introduce some necessary field-theoretic notations. The *coordinates* of a *space-time* point x are labelled as x^μ ($\mu = 0, 1, \ldots, D-1$) with a *time* coordinate $x^0 = t$. The *Lagrangian* (density) $L(\psi)$ is supposed to depend on a set of independent *fields* $\psi^A(x)$ $(A = 1, 2, \ldots, N)$ and their derivatives $\partial_\mu \psi^A(x) = \psi^A_\mu(x)$. The canonical *D-momenta* are denoted as $\pi^\mu_A := \partial L / \partial \psi^A_\mu$. The *field equations* read

$$L_A := \partial_\mu \frac{\partial L}{\partial \psi^A_\mu} - \frac{\partial L}{\partial \psi^A} = 0\,, \qquad A = 1, 2, \ldots, N\,. \tag{5.56}$$

In the following we shall denote the *row*-vector of canonical D-momenta as π^μ, and ψ will label the *column vector* of fields.

Let us consider a pair (S, T) of the linear transformations

$$\psi(x) \mapsto S_g \psi(x) \;=\; \psi(x) + g^i S_i \psi(x) + O(g^2)\,, \tag{5.57}$$
$$\psi(x) \mapsto T_g \psi(x) \;=\; \psi(x) + g^i T_i \psi(x) + O(g^2)\,, \tag{5.58}$$

with S_i and T_i as the *infinitesimal operators* of (S, T). Here g is not assumed to be an element of a Lie group, instead, it is supposed to be an element of some analytic Moufang loop G. Once more discarding the group-centered treatments, the following identities are assumed to hold for all $g, h \in G$:

$$S_g T_g S_h = S_{gh} T_g\,, \quad S_g T_g T_h = T_{hg} S_g\,. \tag{5.59}$$

Nevertheless, $S_e = T_e = I$ (*identity transformation*) are assumed to survive. The pair (S, T) with such properties was called (Sec.5.2.1) the *birepresentation* of G, and the transformations (5.57, 5.58) were called *G-transformations* or Moufang-transformations. Some very elementary algebraic properties have been given by formulas (5.28).

The action of $g \in G$ on the Lagrangian $L(\psi)$ is defined by

$$L(\psi) \mapsto L(S_g^{-1}\psi) \;=\; L(\psi) + g^i S'_i L(\psi) + O(g^2)\,, \tag{5.60}$$
$$L(\psi) \mapsto L(T_g^{-1}\psi) \;=\; L(\psi) + g^i T'_i L(\psi) + O(g^2)\,, \tag{5.61}$$

where S'_i and T'_i are corresponding infinitesimal operators. Such pair of transformations is said to be *induced* by (S,T) and denoted as (S',T'). It is quite easy to check [20] that the latter turns out to be a birepresentation of G as well.

In general, the Moufang-transformations need not be the symmetries of $L(\psi)$. The Lagrangian $L(\psi)$ is said to be *G-invariant (Moufang invariant)* if

$$L(S_g\psi) = L(T_g\psi) = L(\psi)\,, \quad \forall g \in G\,, \tag{5.62}$$

which infinitesimally read

$$S'_i L(\psi) = T'_i L(\psi) = 0\,, \quad i = 1,2,\ldots,r\,. \tag{5.63}$$

By rearranging the terms according to the canonical prescription, the latter can be in turn rewritten as *Noether identities*,

$$-S'_i L(\psi) = S^A_{iB}\psi^B L_4 + \partial_\mu s^\mu_i = 0\,, \tag{5.64}$$

$$-T'_i L(\psi) = T^A_{iB}\psi^B L_4 + \partial_\mu t^\mu_i = 0\,, \tag{5.65}$$

with s^μ_i and t^μ_i as the *Noether currents* generated by the Moufang transformations:

$$s^\mu_i(x) := \pi^\mu(x) S_i \psi(x)\,, \quad t^\mu_i(x) := \pi^\mu(x) T_i \psi(x)\,. \tag{5.66}$$

At this point, it is useful to remind that every infinitesimal transformation $\delta\psi$ of ψ generates its Noether current $\pi^\mu \delta\psi$. Thus, continuous Moufang transformations produce their own currents without the slightest hurt to the field theoretic formalism. In our case, the *Noether charges* read

$$\sigma_i(t) : \;=\; -i \int s^0_i(x) dx^1 \ldots dx^{D-1}\,, \tag{5.67}$$

$$\tau_i(t) : \;=\; -i \int t^0_i(x) dx^1 \ldots dx^{D-1}\,, \tag{5.68}$$

$$i = 1,2,\ldots,r$$

and the corresponding *conservation laws*

$$\frac{d}{dt}\sigma_i(t) = \frac{d}{dt}\tau_i(t) = 0\,, \quad i = 1,2,\ldots,r\,. \tag{5.69}$$

can be obtained from (5.64),(5.65) under the ordinary assumptions that field equations (5.56) hold and G-currents s^μ_i, t^μ_i vanish on the spatial integration boundary.

5.4.2 Moufang-Mal'tsev algebra and its closing

Suppose for a moment that the birepresentation (S,T) of G obeys the common homomorphism properties of group representations:

$$S_g S_h = S_{gh}\,, \quad T_g T_h = T_{hg}\,, \quad S_g T_h = T_h S_g\,, \qquad \forall g,h \in G$$

Such birepresentation of G is called *associative.* It can be easily shown that these identities are *equivalent.* The Moufang loop G may nevertheless remain non-associative, but one must remember [31] that *nonassociative* Moufang loops do not have *faithful* associative birepresentations.

It is well known, [39] that infinitesimal associative transformations obey the *Lie algebra* CR

$$[S_i,S_j] = c^k_{ij}S_k\,, \quad [T_i,T_j] = -c^k_{ij}T_k\,, \quad [S_i,T_j] = 0\,, \tag{5.70}$$

$$i,j,k = 1,2,\ldots,r\,,$$

with the structure constants c^k_{ij} of G. Likewise, the induced infinitesimal symmetry transformations of the Lagrangian $L(\psi)$ follow this algebra. Now, if the Moufang-transformations are not assumed to be associative, a question about the corresponding modification of CR (5.70) arises. Answer is given by the theorem (cf. (5.44), (5.45)).

Theorem 5.6., [32]. *Let (S,T) be a differentiable representation of an analytic Moufang loop G and c^k_{ij} be the structure constants of G. Then, the infinitesimal operators of (S,T) obey the CR*

$$[S_i,S_j] = c^p_{ij}S_p - 2[S_i,T_j]\,, \tag{5.71}$$

$$[T_i,T_j] = -c^p_{ij}T_p - 2[T_i,S_j]\,, \tag{5.72}$$

$$[S_i,T_j] = [T_i,S_j]\,, \quad i,j,p = 1,2,...,r\,, \tag{5.73}$$

We call the algebra satisfying CR (5.71-5.73) *Moufang-Mal'tsev (MM-) algebra.* We can see that the Moufang-Mal'tsev algebra is in some sense a *minimal* though an *open* extension (generalization) of the Lie algebra (5.70). The induced infinitesimal Moufang transformations of the Lagrangian $L(\psi)$ obey this algebra as well. Since c^k_{ij} are structure constants of the tangent Mal'tsev algebra of the analytic Moufang loop G, *continuous* Moufang symmetries are natural to call the *Moufang-Mal'tsev (MM-) symmetries.*

To close the Moufang-Mal'tsev algebra it must be embedded into a larger-dimensional Lie algebra. This is a nontrivial procedure (cf. [30, 32] and we summarize here only some principal points.

Infinitesimal G-invariance conditions (5.63) can be considered as a constraining set of partial differential equations for the Lagrangian $L(\psi)$ to have truly the G-symmetry. CR (5.70) are the *closure conditions* for the system (5.63) when G-transformations are *associative.* To close the system in *general (non-associative)* case, we must extend it with equations

$$[S'_i, T'_j]L(\psi) = 0\,, \quad i, j = 1, 2, \ldots, r\,. \tag{5.74}$$

Now a way of closing [30, 32] the Moufang-Mal'tsev algebra (which in fact means its *embedding* into a *Lie algebra*) is outlined.

Let start by rewriting the MM-algebra as follows:

$$[S_i, S_j] = 2F_{ij} + \frac{1}{3}c^k_{ij}S_k + \frac{2}{3}c^k_{ij}T_k\,, \tag{5.75}$$

$$[S_i, T_j] = -F_{ij} + \frac{1}{3}c^k_{ij}S_k - \frac{1}{3}c^k_{ij}T_k\,, \tag{5.76}$$

$$[T_i, T_j] = 2F_{ij} - \frac{2}{3}c^k_{ij}S_k - \frac{1}{3}c^k_{ij}T_k\,, \tag{5.77}$$

Here every equation of the three can be assumed to be the defining identity for the *Yamaguti operator* F_{ik}. It turns out that these operators are not linearly independent, since

$$c^n_{ij}F_{nk} + c^n_{jk}F_{ni} + c^n_{ki}F_{nj} = 0\,, \tag{5.78}$$

$$F_{ij} + F_{ji} = 0\,, \tag{5.79}$$

$$i, j, k, n = 1, 2, \ldots, r\,.$$

Constraints (5.79) trivially descend from the *anti-symmetry* of the commutator bracketing, but the proof of (5.78) is not so trivial [30]. The following two theorems are of crucial significance.

Theorem 5.8. *Let (S, T) be a differentiable birepresentation of an analytic Moufang loop G and c^k_{ij} be the structure constants of G. Then, the infinitesimal operators of (S, T) obey the CR*

$$[F_{ij}, S_k] = d^n_{ijk}S_n\,, \quad [F_{ij}, T_k] = d^n_{ijk}T_n\,, \tag{5.80}$$

where d^l_{ijk} are *Yamaguti constants* given by (5.11). Equations (5.80) are called *reductivity conditions* of the birepresentation (S, T). Operators F_{ij} are defined by (5.75) (or by (5.76), or by (5.77)).

Theorem 5.9. *Let (S, T) be a differentiable birepresentation of an analytic Moufang loop G and d^n_{ijk} be the Yamaguti constants of G. Then, the Yamaguti operators of (S, T) obey the Lie algebra CR*

$$[F_{ij}, F_{kl}] = d^s_{ijk} F_{sl} + d^s_{ijl} F_{ks} \,, \tag{5.81}$$

$$i, j, k, l, s = 1, 2, \ldots, r \,(= dimG) \,.$$

Computations which in fact prove these theorems have been carried out in [30, 32]. Dimension of the MM-algebra formed by operators $\{S_i, T_i, F_{jk}\}$ obeying CR (5.75–5.81) does not exceed $2r + r(r-1)/2$, while the dimension of the F-subalgebra does not exceed $r(r-1)/2$. *Jacobi identities* of this Lie algebra are quaranteed [30] by the identities of the *Lie triple* [23] and *generalized Lie triple systems* [48, 49] associated with the tangent Mal'tsev algebra of G.

5.4.3 Moufang-Mal'tsev symmetries and currents (continued)

We now turn again to our discussion of G-symmetry (started in Sec. 5.3.1) by noting that equations (5.63) can be in fact closed by

$$F'_{ij} L(\psi) = 0 \,, \qquad i, j = 1, 2, \ldots r \,, \tag{5.82}$$

where the Yamaguti operators F'_{ij} are defined as

$$F'_{ij} := -[S'_i, T'_j] + \frac{1}{3} c^n_{ij} S'_n - \frac{1}{3} c^n_{ij} T'_n = -F'_{ji} \,. \tag{5.83}$$

Also, to find the closing set of *conservation laws*, (5.82) must be rewritten as the Noether identities as well:

$$F'_{ij} L(\psi) = F^A_{ijB} \psi^B L_A + \partial_\mu f^\mu_{ji} = 0 \,, \tag{5.84}$$

where the additional currents

$$f^\mu_{ij}(x) := \pi^\mu(x) F_{ij} \psi(x) = -f^\mu_{ji}(x) \tag{5.85}$$

obey the linear constraints

$$c_{ij}^{n} f_{nk}^{\mu}(x) + c_{jk}^{n} f_{ni}^{\mu}(x) + c_{ki}^{n} f_{nj}^{\mu}(x) = 0 \,,$$

following from (5.78). By introducing now the charges

$$\Phi_{ij} := -i \int f_{ij}^{0}(x) dx^{1} \ldots dx^{D-1} = -\Phi_{ji}(t) \,, \tag{5.86}$$

with the obvious constraints

$$c_{ij}^{n} \Phi_{nk}(t) + c_{jk}^{n} \Phi_{ni} + c_{ki}^{n} \Phi_{nj}(t) = 0 \,,$$

we can obtain from (5.84) the desired *closing* set of conservation laws:

$$\frac{d}{dt} \Phi_{ij}(t) = 0 \,, \quad i, j = 1, 2, \ldots, r \,. \tag{5.87}$$

Note that the *charges* Φ_{ij} *and conservation laws* (5.87) *have their origin in* **nonassociativity**. Our method used up the fact that infinitesimal Moufang transformations generate the Lie algebra with CR (5.75)-(5.81). In this sense, we can state that the collection of conservation laws (5.69) and (5.87) is closed (complete) as well.

5.4.4 Weak Moufang symmetries

The structure of the algebra of infinitesimal Moufang transformations enables to introduce some natural generalizations [20, 34] of the Moufang symmetry.

The Lagrangian $L(\psi)$ is said to be *weakly G-invariant* if

$$L(S_g \psi) = L(T_g) \,, \quad \forall g \in G \,, \tag{5.88}$$

which infinitesimally reads

$$S_i' L(\psi) = T_i'(\psi) \,, \quad i = 1, 2, \ldots, r \,. \tag{5.89}$$

Rearranging the terms according to the canonical prescription, we can get from (5.89) the *weak* Noether identities

$$S_{iB}^{A} \psi^{B} L_{A} + \partial_{\mu} s_{i}^{\mu} = T_{iB}^{A} \psi^{B} L_{A} + \partial_{\mu} t_{i}^{\mu} = 0 \,, \tag{5.90}$$

from which the *weak* conservation laws

$$\frac{d}{dt}\sigma_i(t) = \frac{d}{dt}\tau_i(t)\,, \quad i = 1,2,\ldots,r\,. \tag{5.91}$$

can in turn be obtained. Denoting

$$Q'_i := S'_i - T'_i\,, \qquad Q_i := S_i - T_i\,, \tag{5.92}$$

$$g_i^\mu := s_i^\mu - t_i^\mu\,, \qquad \theta_i := \sigma_i - \tau_i\,, \tag{5.93}$$

it follows from (5.89) and (5.90) that

$$-\,Q'_i L(\psi) = Q^A_{iB}\psi^B L_A + \partial_\mu q_i^\mu = 0\,, \tag{5.94}$$

and the weak conservation laws (5.91) read

$$\frac{d}{dt}\theta_i(t) = 0\,, \quad i = 1,2,\ldots,r\,. \tag{5.95}$$

To close the system of the weak G-invariance conditions (5.89), we must extend it with equations

$$[Q'_i, Q'_j]L(\psi) = 0\,, \quad i,j = 1,2,\ldots r\,. \tag{5.96}$$

Now we can proceed by computing (using (5.92) and (5.75)-(5.77)) CR for the algebra formed by $\{Q_i, F_{ij}\}$:

$$[Q_i, Q_j] = 6F_{ij} - c^n_{ij}Q_n\,, \tag{5.97}$$

$$[F_{ij}, Q_k] = d^n_{ijk}Q_n\,. \tag{5.98}$$

which in terms of Yamaguti [49] means that the infinitesimal *weak* Moufang symmetry operators realize a *weak representation* of the tangent Mal'tsev algebra of G. Commutation relations (5.98) easily follow from (5.80), while (5.97) arises as a new re-defining identity for F_{ij}. Note that the algebra with CR (5.97),(5.98) can be also closed by (5.81), and the dimension of the resulting Lie algebra does not exceed $r + r(r-1)/2$. The closed collection of infinitesimal *weak* G-invariance conditions (weak Noether identities) consists of Eqs. (5.94) and (5.84).

5.4.5 Hidden Moufang-Mal'tsev symmetries

The Lagrangian $L(\psi)$ is said to be *hidden* G-invariant if $F'_{ij}L(\psi) = 0$ $(i, j = 1, 2, \ldots, r)$. The operators F'_{ij} are defined by (5.83) and obey the Lie algebra

$$[F'_{ij}, F'_{kl}] = d^{s}_{ijk}F'_{sl} + d^{s}_{ijl}F'_{ks}\,, \tag{5.99}$$

$$i, j, k, l, s = 1, 2, \ldots, r\,.$$

The corresponding Noether identities (5.84) give rise to *hidden* conservation laws (5.87). Following K.Yamaguti [49], it can be said that the operators F'_{ij} realize a *generalized representation* of the tangent Mal'tsev algebra of G.

Let us finally remark that all these Moufang symmetry considerations are well acceptable from the point of view of alternative algebras and octonions [25, 42, 43, 45]. Also, it is quite trivial to foresee the Noether *charge density algebras* generated by continuous Moufang transformations.

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Chapter 6

Nonassociativity as a fundamental principle in physics. Towards the associator quantization

A general concept of the third infobarrier connected with the color-type interactions (confinement) is formulated in terms of associators as a fundamental principle in physics (Sec. 6.1). The other two infobarriers are those of special relativity and quantum mechanics. These infobarriers are associated with three unique number algebras: complex numbers, quaternions, and octonions. To the third infobarrier also corresponds the third fundamental constant, the *fundamental length* characterizing the size of the confinement region.

It appears that there are interesting possibilities to introduce the nonassociative treatment in the most fundamental region of quantum gravity (Sec. 6.2). The spontaneous symmetry breaking due to nonassociativity enables gravity to be dealt from a new viewpoint where the difficulties on the Planck scale disappear. Various possibilities for the deep level associator quantization are proposed.

6.1 Nonassociativity and the third infobarrier

The *nonassociativity* here means that in some algebraic system the associativity law fails, i.e. $(ab)c \neq a(bc)$ in general. The precedent of quantum mechanics (QM) where the *noncommutativity* plays fundamental role demonstrates that such a violation of a general law of usual numbers may have a good sense.

Among the huge variety of general (nonassociative) algebras there are very few with such good properties as our usual and traditional number systems have. From the theory of algebras we know that there is a unique complex of four *division algebras* – real numbers $\mathbf{R}$, complex numbers $\mathbf{C}$, quaternions $\mathbf{H}$, and octonions $\mathbf{O}$, which may be obtained from the real numbers by the *Cayley-Dickson procedure* (Sec. 3.2).

The role of real numbers perhaps needs no comment here, as their application is so wide that we are not able to specify any especially fundamental context. Complex numbers and quaternions in some contexts have more principal roles. Complex numbers seem to have a fundamental connection with the special theory of relativity (STR) allowing to introduce the light cone and a nondetermined interval for events. The noncommutativity of QM is related to the uncertainty relations and the complementarity principle, and is characteristically represented by the quaternion units satisfying the same commutation relations as do the angular momentum operators.

In these two examples there exist certain limits in obtaining information. We call them *infobarriers* and state that they are intimatelly connected with the (hyper-) complex nature of the number algebras $\mathbf{C}$ and $\mathbf{H}$, and also with the fundamental constants c and h respectively. In STR we cannot transfer information over the spacelike intervals, in QM we cannot perform precise measurements of the complementary pairs of observables.

Now we may ask *what fundamental role octonions may have and whether there is also some new infobarrier and some new fundamental constant connected with nonassociativity?* [4]

As a heuristic principle we have kept in mind the *quark confinement* which nowadays is formulated in quantum chromodynamics (QCD) through the *exact color symmetry.* The color properties formed in strong interactions lead to a new kind of forces, which do not allow quarks to be hit out of hadrons. *The nonobservability (confinement) of colored particles is a new infobarrier related to some nonassociativity properties.*

If the information is obtained (by some measurement) in the same region where the measurable properties are formed, we can measure these properties taking into account only the limitations of STR and QM. The Universe as the whole forms all measurable quantities (inertial properties (Mach principle), flavors etc.). If the observer and his apparatus is placed outside the system or the region where the properties are formed, they are not directly measurable by this observer.

Applying some quite natural analogies and additional hypotheses, we can give some model consideration (mathematical formulation) of the situation in terms of expressions where at some associator there stands some fundamental constant related to the new infobarrier and having a physical meaning of the range of that spatial region where the third infobarrier is maintained.

We present here an example where we use the resemblance between the magnetic charge (Dirac monopole) and color charges.

Jacoby identity for the components of the covariant 4-momentum of the Dirac monopole is violated [3]:

$$\sum_{cycl}[[p^i, p^j], p^k] = -\frac{eh^2}{c}\nabla B\,, \tag{6.1}$$

where ∇B is the magnetic flux. Therefore p^i are not the elements of a Lie algebra.

Passing over to the 8-dimensional formalism (where in the 8x8 field tensor "colored" magnetic fields appear naturally, see (4.42)) we conjecture that p^i are the elements of the *Mal'tsev algebra* [5] M_7, the commutator algebra of octonions. Then

$$\sum_{cycl}[[p^i, p^j], p^k] = 6(p^i, p^j, p^k) = \frac{h^2}{\lambda^2}p^l\,. \tag{6.2}$$

The constant λ is a new fundamental constant which characterizes the range of the region where color-type forces act and where the third infobarrier (nonobservability, confinement) affects. If we want λ to be truly fundamental, we must develop our considerations on the *superstring* or *quantum gravity* level. The third infobarrier then results from the reparametrization invariance which does not allow the observation of the internal structure of a small piece $(\lambda \cdot c\tau)$ of the world sheet. We give here a very brief discussion only. At first let us define

$$p \to q = \frac{2\alpha'}{c^2} p \,,$$

where α' is the Regge slope and the quantities q are of the length dimension. Then

$$6(q^i, q^j, q^k) = \lambda^2 q^l \,, \tag{6.3}$$

where $\lambda = \sqrt{2\alpha' h}/c$. In the case $\kappa = 2c\alpha'$ (κ is the gravitational constant) we get

$$\lambda = \lambda_{\text{Planck}} = \sqrt{\kappa h / c^3} \,. \tag{6.4}$$

This indicates an intimate connection with gravitation. From (6.3), (6.4) it follows that the quantities q may be characteristic observables in the "associator-quantized" theory (quantum gravity?).

6.2 Towards the associator quantization

***1** *Associator quantization* here means the introduction of *nonassociativity* for some elements of the deep structure of matter. This property is measured by the *associator*

$$(a, b, c) = (ab)c - a(bc) \,; \quad a, b, c \in \mathcal{A} \,,$$

where $\mathcal{A}$ is some nonassociative algebra with a binary operation $a, b \to ab$, describing the entities in some deep level.

At the end of preceding section we had an example demonstrating that some new nonassociative features may have their appearance in the Planck scale of space-time, the characteristic region of quantum gravity. Here we proceed by some general considerations of the gravitational field showing that for gravity the usual commutator quantization fails and the associator quantization may have some perspectives. Here the fundamental length is the *Planck length* $l_{\text{P}} = \sqrt{h\kappa/c^3} \approx 10^{-33}$ cm the ultimate quantum of space which plays the role of a lattice constant for some space-time "crystal-like" regular discrete substructure. In the Planck region $x < l_{\text{P}}$ this lattice structure decays and the gravity interpreted as the elasticity of this substructure forfeits any meaning

***2** In [2] Freeman Dyson emphasizes the innovation of James Maxwell when introducing his famous equations of the electromagnetic field. It consists of a *two-layer* structure of his theory where in the "upper" layer there are directly measurable classical physical quantities, such as energy, forces, etc. and in the "lower' or "deeper" layer there are *fields*, not directly measurable "square roots" of the upper layer entities. The expedience of these field quantities lies in the simple and unifying form of field equations.

This 2-layerness acquires principal importance in QM, where measurability or nonmeasurability have a principal meaning. Matter (fermion) fields in QM are purely ststistical and may be described by probability amplitudes only.

In what follows our basic physical fields will be those of carrying spins 1/2, 1 and 2 — the matter (fermion) fields, the force agents (intermediate bosons), and the gravitational field (gravitons). Among these matter fields are purely nonclassical. The Maxwell field as a zero mass boson field may be condensated to get a macroscopical classical field. Historically this field was the first one that was rigorously treated both as classical and the quantum field theory, the most modern and far-reaching of the latter is now the *gauge theory of interactions.* The gravitation field differs from fermion fields even more than the electromagntic (Maxwell) field, as it has no quantum theory except the spin 2 quantum field theory (QFT) beyond the Planck region.

For a long time since the birth of QM theoreticians have been racking their brains over the quantization of gravity. All these attempts have been unsuccessful insofar and this leads to a natural question — is gravity quantizable at all?

***3** Let us examine some properties of the gravitational field differing it from other quantum fields.

Firstly, the *dimensionless coupling constant* $\alpha_{\mathrm{gr}} = \kappa M_1 M_2 (c\hbar)^{-1}$ (where the Newton constant has the value $\kappa = 6.70711(86) \times 10^{-39} c\hbar\ (\mathrm{GeV}/\mathrm{c}^2)^{-2}$) is not a true constant, but depends on masses and increases with energy so that the perturbation expansion is not possible in the most interesting Planck region where the energies become comparable or higher than the *Planck mass (energy)* $M_{\mathrm{P}} = \sqrt{c\hbar/\kappa} \approx 10^{-5}\mathrm{g} = 10^{19}\mathrm{Gev}/\mathrm{c}^2$.

In contrast, the dimensionless coupling constant of the electromagnetic interaction does not depend on energy and the perturbation calculation is applicable because of $\alpha_{\mathrm{el.m.}} = e^2/c\hbar \approx 1/137$. For the gravity perturbation calculus fails at the Planck scale.

Secondly, there is a well-known complication of applying the QFT to the gravitation field. According to Bohr and Rosenfeld [1] the mean value of the quantum field strength may be measured with an arbitrary given accuracy if the mass of the probe particle is taken sufficiently large. Then

$$\Delta E (\Delta L)^3 \geq \frac{Q}{M} \cdot \frac{h}{c},$$

where M is the mass of the probe particle, Q is its charge, L the size, and E the field strength.

For the gravity $Q = M_{\mathrm{gr}}$ (gravitational mass), according to the *weak equivalence principle* $M_{\mathrm{gr}} = \kappa M_{\mathrm{in}}$ (M_{in} – inertial mass) and the accuracy of measurement cannot be improved because of $Q/M = M_{\mathrm{gr}}/M_{\mathrm{in}} = \kappa$ (const.). This means that gravity has, in addition to the Heisenberg uncertainty relations, the uncertainties related to the *nonlocality* of the gravitational field. If there is a quantum theory of gravity, it must be principally different from other QFTs.

From the changing coupling constant it is evident that we have two regions for quantum gravity — the *weak coupling* region and the *strong coupling* region.

In the weak coupling region where the coupling constant $\alpha_{\mathrm{gr}} < 1$, the quantum theory of gravity may be developed without any restriction; it is a common spin 2 boson field theory. However, at high energies where the quantum effects become significant (fluctuations of metric, etc.), quantum gravity is lacking. Perhaps now it is logical to try with some kind of strong coupling theory applying it to gravity.

***4** There is one quite interesting version of strong coupling theories, based on the *Bogolyubov method of collective coordinates*, which takes its origin from the solid state theory, but has later been reformulated into the strong coupling theory of particles with a boson field, invariant with respect to some symmetry group, [9]. If the QM ground state wave function is constructed from a classical ground state solution, it appears that it does not realize a representation of the symmetry group — the original symmetry of the problem is *spontaneously broken*. The essence of the Bogolyubov method lies in the restoration of the original symmetry through the canonical Bogolyubov transformation of variables. The method involves L- and R-operators of the left and right shifts (i.e. multiplications from left and right) in the symmetry group. Due to their commutativity the original symmetry is restored for the ground state wave function.

The Bogolyubov method is especially suitable for our purposes because 1) it is a *strong coupling theory* , i.e. the theory without perturbation expansion, 2) here L- and R-operators of the left and right multiplications are used, for groups these are commuting, but there is a very straightforward possibility for passing over to some nonassociative system with noncommuting L- and R-operators as *regular birepresentation* operators (see Sec. 3.6). Birepresentation of the nonassociative algebra $\mathcal{A}$ by L- and R-operators, as we have already seen, in fact means some kind of "associative projection" of this nonassociative system, generalizing the common notion of homomorphic representation of groups, and associative (and Lie) algebras. It is the only way to have a bridge between our world and the deep regions of matter with nonassociative fundamental entities.

***5** Perhaps the most natural way to treat gravity is to regard it as some kind of "metrical elasticity" (Sakharov [7], see also [6] about the history and development of this viewpoint). Wheeler (in [6]) has pointed out that is natural to derive this elasticity from some kind of structure, as it was in the case of crystals — that the properties of crystals such as elasticity, etc. are determined by their atomic structure and not conversely.

Proceeding from these ideas we propose for gravity the following general picture. Soon after the Big Bang in the course of phase transition continuous homogeneity (translational symmetry) was spontaneously broken and some kind of space-time ("preon") *lattice* was formed. We regard the gravity as some kind of *elasticity* of this lattice and gravitons as *quasiparticles.* It means that there is no gravity at all without this lattice. The lattice constant is the Planck length, it is formed during the Planck time when the matter is cooled below the Planck temperature and energy becomes lower than the Planck energy. We can compare the Planck energy with the Debye energy for crystals.

The main point here is that the continuous translational symmetry is spontaneously broken due to nonassociativity, the formation of the preon lattice is essentially the associator quantization. The commutator quantization quantizes the action, the associator quantization quantizes the length.

***6** For both the commutator and the associator quantization there exist several different versions and possibilities. For QM we can bring out at least three main approaches: 1) the *Schrödinger formalism* with differential operators and wave functions; 2) the *Heisenberg formulation* with (matrix) operators, commutators, the time evolution equations in the Hamiltonian form, etc.; 3) the *Feynman path integral method.* We can now say that all these approaches are principally equivalent.

Finally let us try to formulate several different possibilities and perspectives for the associator quantization.

1. We may simply declare that some physical quantities in some particular model are nonassociative. In [4] (see Sec. 6.1) we introduced a version of associator quantization through the 8-dimensional theory of a "colored" magnetic monopole, which in some sense is an analog of the soliton-type solution for the strong-coupling theory. The starting point here was the Jackiw's consideration [3] of monopole momenta not satisfying the Jacobi identity.

We have argued that gravity demands perhaps, the introduction of nonassociative entities, so we may identify the new fundamental constant with the Planck length intimately related to gravity and we have simply postulated that for length-dimensional momenta there holds an associator equation (see the previous section, Sec. 6.1).

2. Nonassociativity may be introduced through the noncommuting L- and R-operators of the regular birepresentation of some nonassociative algebra $\mathcal{A}$, as mentioned already above in *4 (and in Secs. 3.5, 4.3.4). The advantage of this version is the absence of pure nonassociative entities, nonassociativity is transferred to the noncommutativity of L-, R-operators, as mentioned already, it is an *associative projection* of the nonassociative theory.

This possibility resembles closely the second (matrix) approach to QM and is naturally connected with spontaneous symmetry breaking and with the method of collective coordinates. The method is general and applicable for all types of nonassociative algebras, so the previous case may be also be formulated in terms of noncommuting L-,R-matrices.

3. There is also quite straightforward possibility of introducing nonassociative entities into the Hamiltonian time evolution equation of QM through an additional associator term:

$$\frac{dF}{dt_{12}} = \frac{\partial F}{\partial t_{12}} + \frac{1}{i\hbar}[[H_1, H_2], F] - \frac{1}{i\hbar}\frac{3m}{M_P}(H_1, F, H_2) \quad . \tag{6.5}$$

This follows rigorously from the derivation formulas of alternative algebras, [8]. In the associative case this formula reduces to the common Heisenberg formula, then $[H_1, H_2] = H$ and $(H_1, F, H_2) = 0$; where H_1, H_2 are "factor-Hamiltonians" acting in the deepest (third) layer of nonassociative entities. We also have an associative conformal invariant Penrose-type theory if the mass m of a particle vanishes.

It must be remarked that the Hamiltonian formalism is quite appropriate in QM and QFT, allowing the separation of physical and nonphysical quantities and the formulation of the theory in terms of constraints, but the formalism is inappropriate if we want to emphasize the relativistic invariance.

4. Starting from a fundamental Dirac-type equation in the 16-dimensional *binary sedenion* formalism [10, 11] and passing over to the "square" equation (to an analog of the Klein-Gordon equation) there appears an additional associator term which may also be regarded as the appearance of associator quantization and which breaks the relativistic invariance of the initial equation. In the *ternary sedenion* formalism this associator term is absent and the relativistic invariance remains intact.

There are yet some additional aspects of associator quantization.

(5.) A geometric-topological nature of the spontaneous symmetry breaking lies in the *Hopf fibrations* connected with *Hopf maps* $S^3 \to S^2, S^7 \to S^4, S^{15} \to S^8$.

(6.) One of the fundamental concepts of QM is the *spin* angular momentum. The primary quantum transformations are acting in the spinor space, while the transformations of vectors are expressed in the form of two-sided multiplications. Passing over to nonassociative quantities there arises the bracketing problem and the "factorized" form of transformations of vectors is lost. This "nonfactorizability" puts the density matrix formalism and the Jordan formulation of QM into a more favourable light.

The interrelations between the above mentioned possibilities, versions and approaches must be carefully studied.

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Appendices

A1. Fundamental modifications of octonion multiplication table

Here in Appendices we shall present 16 special versions, *fundamental modifications* (FM), of the multiplication table of octonion units (Table 5 in Sec. 3.1.), and the corresponding L- and R-matrices (Sec. 3.6) of the regular (adjoint) bimodule representation. It may be shown that there are at all 480 multiplicative table versions (see Ref. [64] in Sec. 3.1.). From these only 16 can be chosen, which are deductable from the original Cayley table (Table 5) in a quite simple way and which may be effectively used in various constructions (e.g. all they are engaged in the basic multiplication rule of ternary sedenions, Sec. 3.4).

These simple modifications confine themselves to the changes of signs (reflections) of some octonion units. This is connected with the nonuniqueness of the definition of the antisymmetric tensor ϵ_{ijk} in the definition formula (3.2). The signs for ϵ_{ijk} may be arbitrarily chosen only for 4 cycles (ijk), then the other 3 are uniquely determined. So there are 16 such modifications of the multiplication table of octonion units, which we call *fundamental.* Positive cycles for all modifications are exposed in the Table A1-1 (changes compared to 0-modification are underlined).

In Tables A1-2,3 we shall write out in details only the multiplication table for the 0-modification (Cayley's original table). In the rest of the tables we only hatch the "boxes" where signs at product units are reversed compared with 0-modification. Let us remark that the multiplication tables for FM appear as pairs of mutually *inverse* tables with all cycles inversed, i.e., with opposite signs. We denote these pairs as $0, \tilde{0}$; $1, \tilde{1}$; etc.

Table A1-1. **Positive cycles for fundamental modifications**

0	123	145	176	246	257	347	365
1	123	145	176	$\underline{264}$	$\underline{275}$	$\underline{374}$	$\underline{356}$
2	123	$\underline{154}$	$\underline{167}$	246	257	$\underline{374}$	$\underline{356}$
3	123	$\underline{154}$	$\underline{167}$	$\underline{264}$	$\underline{275}$	347	365
4	$\underline{132}$	145	$\underline{167}$	246	$\underline{275}$	347	$\underline{356}$
5	$\underline{132}$	145	$\underline{167}$	$\underline{264}$	257	$\underline{374}$	365
6	$\underline{132}$	$\underline{154}$	176	246	$\underline{275}$	$\underline{347}$	365
7	$\underline{132}$	$\underline{154}$	176	$\underline{264}$	257	374	$\underline{356}$
$\tilde{0}$	$\underline{132}$	$\underline{154}$	$\underline{167}$	$\underline{264}$	$\underline{275}$	$\underline{374}$	$\underline{356}$
$\tilde{1}$	$\underline{132}$	$\underline{154}$	$\underline{167}$	246	257	347	365
$\tilde{2}$	$\underline{132}$	145	176	$\underline{264}$	$\underline{275}$	347	365
$\tilde{3}$	$\underline{132}$	145	176	246	257	$\underline{374}$	$\underline{356}$
$\tilde{4}$	123	$\underline{154}$	176	$\underline{264}$	257	$\underline{374}$	365
$\tilde{5}$	123	$\underline{154}$	176	246	$\underline{275}$	347	$\underline{356}$
$\tilde{6}$	123	145	$\underline{167}$	$\underline{264}$	257	347	$\underline{356}$
$\tilde{7}$	123	145	$\underline{167}$	246	$\underline{275}$	$\underline{374}$	365

Table A1-2. **Fundamental modifications 0-7**

0 $e_1e_2=e_3,\ e_1e_4=e_5,\ e_2e_4=e_6,\ e_3e_4=e_7$

	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

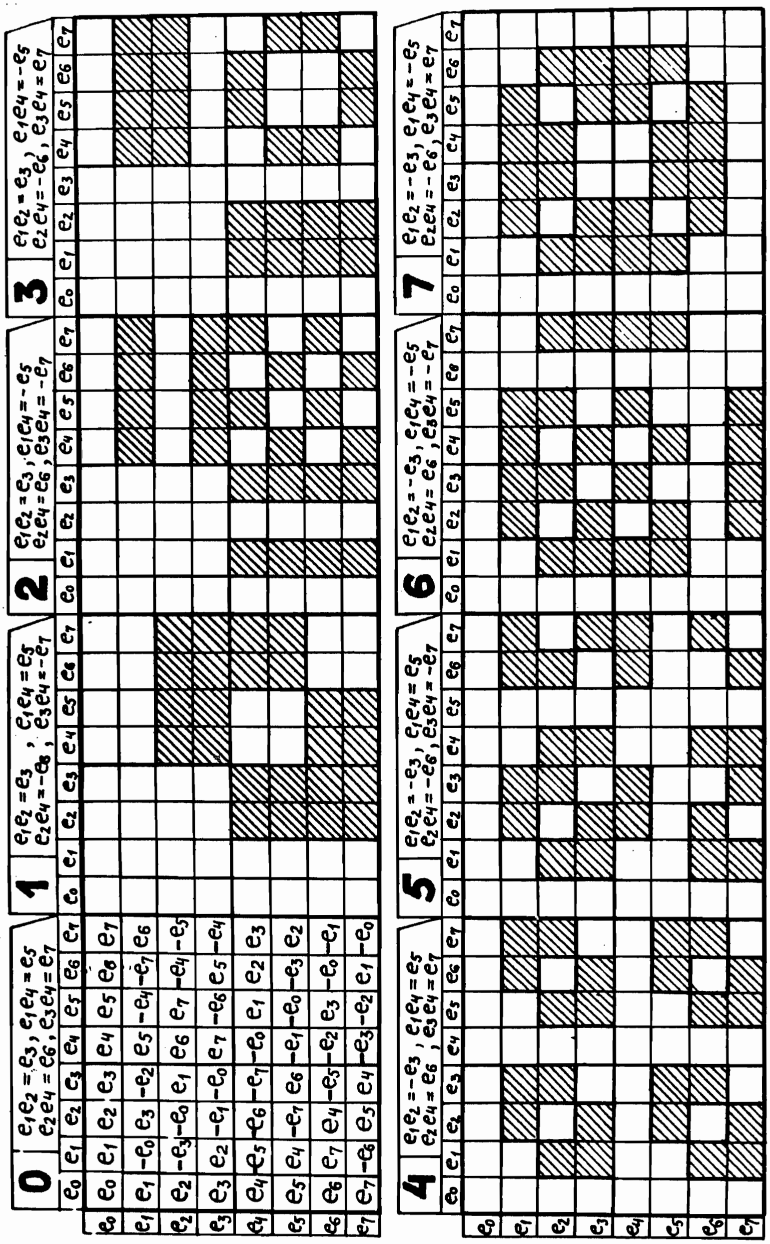
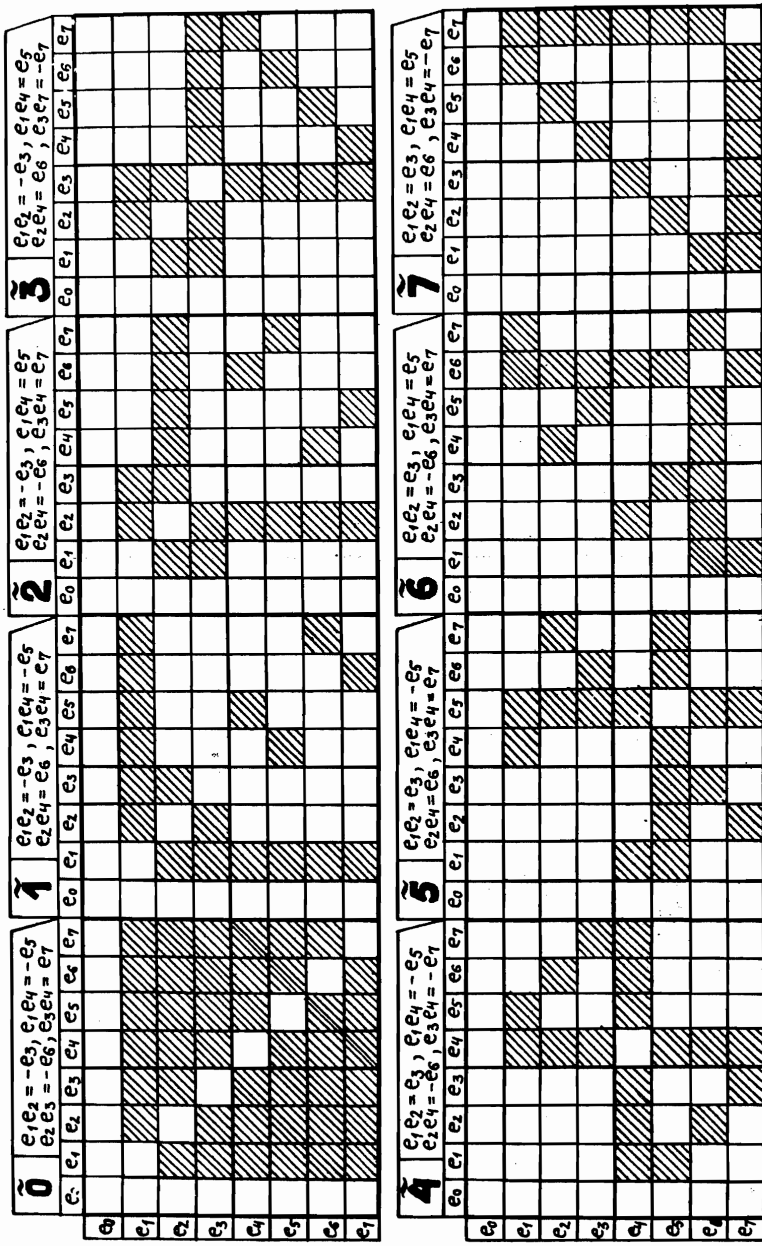


Table A1-3. **Fundamental modifications** $\tilde{0} - \tilde{7}$



A2. Matrices of regular birepresentation of octonions

We present here L_i- and R_i-matrices of regbirep of octonions for all fundamental modifications in Tables A2-1 and A2-2, respectively. For the mutually inverse modifications the L_i- and R_i-matrices are interchangeable. So, for example, L_i-matrices for modifications $\tilde{k}$, $k = 0, 1, \ldots, 7$ are equal to the R_i-matrices of modifications k .

The symbols $L_1, L_2, \ldots, L_7$ and $R_1, R_2, \ldots, R_7$ are written only in left columns of Tables A2-1,2. The numbers ij on the left in other columns indicate that the corresponding matrix of the modification may be obtained as a product L_iL_j (or R_iR_j) of the 0-modification.

Matrices L_i, R_i are defined in Sec. 3.6 (Eqs. (3.47)).

Table A2-1. **Octonion regbirep L-matrices for FM 0-7**

0

L_1

	-1						
1							
			-1				
		1					
					-1		
				1			
							1
						-1	

L_2

		-1					
			1				
1							
	-1						
						-1	
							-1
				1			
					1		

L_3

			-1				
		-1					
	1						
1							
							-1
						1	
					-1		
				1			

L_4

				-1			
					1		
						1	
							1
1							
	-1						
		-1					
			-1				

L_5

					-1		
				-1			
							1
						-1	
	1						
1							
			1				
		-1					

L_6

						-1	
							-1
				1			
					-1		
		1					
			-1				
1							
	1						

L_7

							-1
						1	
					-1		
				-1			
			1				
		1					
	-1						
1							

1

L_1

	-1						
1							
			-1				
		1					
					-1		
				1			
							1
						-1	

31

		-1					
			1				
1							
	-1						
						1	
							1
				-1			
					-1		

12

			-1				
		-1					
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-1							
							1
						-1	
					1		
				-1			

51

				-1			
					1		
						-1	
							-1
1							
	-1						
		1					
			1				

14

					-1		
				-1			
							-1
						1	
	1						
1							
			-1				
		1					

17

						-1	
							-1
				1			
					-1		
		-1					
			1				
1							
	1						

61

							-1
						1	
					1		
				1			
			-1				
		-1					
	-1						
1							

2

23

	-1						
1							
			-1				
		1					
					1		
				-1			
							-1
						1	

L_2

		-1					
			1				
1							
	-1						
						-1	
							-1
				1			
					1		

12

			-1				
		-1					
	1						
1							
							1
						-1	
					1		
				-1			

62

				-1			
					-1		
						1	
							-1
1							
	1						
		-1					
			1				

72

					-1		
				1			
							1
						1	
	-1						
1							
			-1				
		-1					

24

						-1	
							1
				-1			
					-1		
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	-1						

25

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3

23

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							-1
						1	

31

		-1					
			1				
1							
	-1						
						1	
							1
				-1			
					-1		

L_3

			-1				
		-1					
	1						
1							
							-1
						1	
					-1		
				1			

73

				-1			
					-1		
						-1	
							1
1							
	1						
		1					
			-1				

36

					-1		
				1			
							-1
						-1	
	-1						
1							
			1				
		1					

53

						-1	
							1
				1			
					1		
		-1					
			-1				
1							
	-1						

34

							-1
						-1	
					1		
				-1			
			1				
		-1					
	1						
1							

Table A2-1. (continued)

4

$$45\ \begin{bmatrix} & -1 & & & & & & \\ 1 & & & & & & & \\ & & & 1 & & & & \\ & & -1 & & & & & \\ & & & & & -1 & & \\ & & & & 1 & & & \\ & & & & & & & -1 \\ & & & & & & 1 & \end{bmatrix}$$

$$46\ \begin{bmatrix} & & -1 & & & & & \\ & & & -1 & & & & \\ 1 & & & & & & & \\ & 1 & & & & & & \\ & & & & & & -1 & \\ & & & & & & & 1 \\ & & & & 1 & & & \\ & & & & & -1 & & \end{bmatrix}$$

$$47\ \begin{bmatrix} & & & -1 & & & & \\ & & 1 & & & & & \\ & -1 & & & & & & \\ 1 & & & & & & & \\ & & & & & & & -1 \\ & & & & & & -1 & \\ & & & & & 1 & & \\ & & & & 1 & & & \end{bmatrix}$$

$$L_4\ \begin{bmatrix} & & & & -1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \\ 1 & & & & & & & \\ & -1 & & & & & & \\ & & -1 & & & & & \\ & & & -1 & & & & \end{bmatrix}$$

$$14\ \begin{bmatrix} & & & & & -1 & & \\ & & & & -1 & & & \\ & & & & & & & -1 \\ & & & & & & 1 & \\ & 1 & & & & & & \\ 1 & & & & & & & \\ & & & -1 & & & & \\ & & 1 & & & & & \end{bmatrix}$$

$$24\ \begin{bmatrix} & & & & & & -1 & \\ & & & & & & & 1 \\ & & & & -1 & & & \\ & & & & & -1 & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ 1 & & & & & & & \\ & -1 & & & & & & \end{bmatrix}$$

$$34\ \begin{bmatrix} & & & & & & & -1 \\ & & & & & & -1 & \\ & & & & & 1 & & \\ & & & & -1 & & & \\ & & & 1 & & & & \\ & & -1 & & & & & \\ & 1 & & & & & & \\ 1 & & & & & & & \end{bmatrix}$$

5

$$45\ \begin{bmatrix} & -1 & & & & & & \\ 1 & & & & & & & \\ & & & 1 & & & & \\ & & -1 & & & & & \\ & & & & & -1 & & \\ & & & & 1 & & & \\ & & & & & & & -1 \\ & & & & & & 1 & \end{bmatrix}$$

$$57\ \begin{bmatrix} & & -1 & & & & & \\ & & & -1 & & & & \\ 1 & & & & & & & \\ & 1 & & & & & & \\ & & & & & & 1 & \\ & & & & & & & -1 \\ & & & & -1 & & & \\ & & & & & 1 & & \end{bmatrix}$$

$$65\ \begin{bmatrix} & & & -1 & & & & \\ & & 1 & & & & & \\ & -1 & & & & & & \\ 1 & & & & & & & \\ & & & & & & & 1 \\ & & & & & & 1 & \\ & & & & & -1 & & \\ & & & & -1 & & & \end{bmatrix}$$

$$51\ \begin{bmatrix} & & & & -1 & & & \\ & & & & & 1 & & \\ & & & & & & -1 & \\ & & & & & & & -1 \\ 1 & & & & & & & \\ & -1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \end{bmatrix}$$

$$L_5\ \begin{bmatrix} & & & & & -1 & & \\ & & & & -1 & & & \\ & & & & & & & 1 \\ & & & & & & -1 & \\ & 1 & & & & & & \\ 1 & & & & & & & \\ & & & 1 & & & & \\ & & -1 & & & & & \end{bmatrix}$$

$$53\ \begin{bmatrix} & & & & & & -1 & \\ & & & & & & & 1 \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & -1 & & & & & \\ & & & -1 & & & & \\ 1 & & & & & & & \\ & -1 & & & & & & \end{bmatrix}$$

$$25\ \begin{bmatrix} & & & & & & & -1 \\ & & & & & & -1 & \\ & & & & & 1 & & \\ & & & & -1 & & & \\ & & & 1 & & & & \\ & & -1 & & & & & \\ & 1 & & & & & & \\ 1 & & & & & & & \end{bmatrix}$$

6

$$76\ \begin{bmatrix} & -1 & & & & & & \\ 1 & & & & & & & \\ & & & 1 & & & & \\ & & -1 & & & & & \\ & & & & & 1 & & \\ & & & & -1 & & & \\ & & & & & & & 1 \\ & & & & & & -1 & \end{bmatrix}$$

$$46\ \begin{bmatrix} & & -1 & & & & & \\ & & & -1 & & & & \\ 1 & & & & & & & \\ & 1 & & & & & & \\ & & & & & & -1 & \\ & & & & & & & 1 \\ & & & & 1 & & & \\ & & & & & -1 & & \end{bmatrix}$$

$$65\ \begin{bmatrix} & & & -1 & & & & \\ & & 1 & & & & & \\ & -1 & & & & & & \\ 1 & & & & & & & \\ & & & & & & & 1 \\ & & & & & & 1 & \\ & & & & & -1 & & \\ & & & & -1 & & & \end{bmatrix}$$

$$62\ \begin{bmatrix} & & & & -1 & & & \\ & & & & & -1 & & \\ & & & & & & 1 & \\ & & & & & & & -1 \\ 1 & & & & & & & \\ & 1 & & & & & & \\ & & -1 & & & & & \\ & & & 1 & & & & \end{bmatrix}$$

$$36\ \begin{bmatrix} & & & & & -1 & & \\ & & & & 1 & & & \\ & & & & & & & -1 \\ & & & & & & -1 & \\ & -1 & & & & & & \\ 1 & & & & & & & \\ & & & 1 & & & & \\ & & 1 & & & & & \end{bmatrix}$$

$$L_6\ \begin{bmatrix} & & & & & & -1 & \\ & & & & & & & -1 \\ & & & & -1 & & & \\ & & & & & 1 & & \\ & & 1 & & & & & \\ & & & -1 & & & & \\ 1 & & & & & & & \\ & 1 & & & & & & \end{bmatrix}$$

$$61\ \begin{bmatrix} & & & & & & & -1 \\ & & & & & & 1 & \\ & & & & & 1 & & \\ & & & & 1 & & & \\ & & & -1 & & & & \\ & & -1 & & & & & \\ & -1 & & & & & & \\ 1 & & & & & & & \end{bmatrix}$$

7

$$76\ \begin{bmatrix} & -1 & & & & & & \\ 1 & & & & & & & \\ & & & 1 & & & & \\ & & -1 & & & & & \\ & & & & & 1 & & \\ & & & & -1 & & & \\ & & & & & & & 1 \\ & & & & & & -1 & \end{bmatrix}$$

$$57\ \begin{bmatrix} & & -1 & & & & & \\ & & & -1 & & & & \\ 1 & & & & & & & \\ & 1 & & & & & & \\ & & & & & & 1 & \\ & & & & & & & -1 \\ & & & & -1 & & & \\ & & & & & 1 & & \end{bmatrix}$$

$$47\ \begin{bmatrix} & & & -1 & & & & \\ & & 1 & & & & & \\ & -1 & & & & & & \\ 1 & & & & & & & \\ & & & & & & & -1 \\ & & & & & & -1 & \\ & & & & & 1 & & \\ & & & & 1 & & & \end{bmatrix}$$

$$73\ \begin{bmatrix} & & & & -1 & & & \\ & & & & & -1 & & \\ & & & & & & -1 & \\ & & & & & & & 1 \\ 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \end{bmatrix}$$

$$72\ \begin{bmatrix} & & & & & -1 & & \\ & & & & 1 & & & \\ & & & & & & & 1 \\ & & & & & & 1 & \\ & -1 & & & & & & \\ 1 & & & & & & & \\ & & & -1 & & & & \\ & & -1 & & & & & \end{bmatrix}$$

$$17\ \begin{bmatrix} & & & & & & -1 & \\ & & & & & & & -1 \\ & & & & 1 & & & \\ & & & & & -1 & & \\ & & -1 & & & & & \\ & & & 1 & & & & \\ 1 & & & & & & & \\ & 1 & & & & & & \end{bmatrix}$$

$$L_7\ \begin{bmatrix} & & & & & & & -1 \\ & & & & & & 1 & \\ & & & & & -1 & & \\ & & & & -1 & & & \\ & & & 1 & & & & \\ & & 1 & & & & & \\ & -1 & & & & & & \\ 1 & & & & & & & \end{bmatrix}$$

Table A2-2. **Octonion regbirep R-matrices for FM 0-7**

	0	1	2	3
	$R_1 \begin{bmatrix} &-1&&&&&& \\ 1&&&&&&& \\ &&&1&&&& \\ &&-1&&&&& \\ &&&&&1&& \\ &&&&-1&&& \\ &&&&&&&-1 \\ &&&&&&1& \end{bmatrix}$	$R_1 \begin{bmatrix} &-1&&&&&& \\ 1&&&&&&& \\ &&&1&&&& \\ &&-1&&&&& \\ &&&&&1&& \\ &&&&-1&&& \\ &&&&&&&-1 \\ &&&&&&1& \end{bmatrix}$	$32 \begin{bmatrix} &-1&&&&&& \\ 1&&&&&&& \\ &&&1&&&& \\ &&-1&&&&& \\ &&&&&-1&& \\ &&&&1&&& \\ &&&&&&&1 \\ &&&&&&-1& \end{bmatrix}$	$32 \begin{bmatrix} &-1&&&&&& \\ 1&&&&&&& \\ &&&1&&&& \\ &&-1&&&&& \\ &&&&&-1&& \\ &&&&1&&& \\ &&&&&&&1 \\ &&&&&&-1& \end{bmatrix}$
	$R_2 \begin{bmatrix} &&-1&&&&& \\ &&&-1&&&& \\ 1&&&&&&& \\ &1&&&&&& \\ &&&&&&1& \\ &&&&&&&1 \\ &&&&-1&&& \\ &&&&&-1&& \end{bmatrix}$	$13 \begin{bmatrix} &&-1&&&&& \\ &&&-1&&&& \\ 1&&&&&&& \\ &1&&&&&& \\ &&&&&&-1& \\ &&&&&&&-1 \\ &&&&1&&& \\ &&&&&1&& \end{bmatrix}$	$R_2 \begin{bmatrix} &&-1&&&&& \\ &&&-1&&&& \\ 1&&&&&&& \\ &1&&&&&& \\ &&&&&&1& \\ &&&&&&&1 \\ &&&&-1&&& \\ &&&&&-1&& \end{bmatrix}$	$13 \begin{bmatrix} &&-1&&&&& \\ &&&-1&&&& \\ 1&&&&&&& \\ &1&&&&&& \\ &&&&&&-1& \\ &&&&&&&-1 \\ &&&&1&&& \\ &&&&&1&& \end{bmatrix}$
	$R_3 \begin{bmatrix} &&&-1&&&& \\ &&1&&&&& \\ &-1&&&&&& \\ 1&&&&&&& \\ &&&&&&&1 \\ &&&&&&-1& \\ &&&&&1&& \\ &&&&-1&&& \end{bmatrix}$	$21 \begin{bmatrix} &&&-1&&&& \\ &&1&&&&& \\ &-1&&&&&& \\ 1&&&&&&& \\ &&&&&&&-1 \\ &&&&&&1& \\ &&&&&-1&& \\ &&&&1&&& \end{bmatrix}$	$21 \begin{bmatrix} &&&-1&&&& \\ &&1&&&&& \\ &-1&&&&&& \\ 1&&&&&&& \\ &&&&&&&-1 \\ &&&&&&1& \\ &&&&&-1&& \\ &&&&1&&& \end{bmatrix}$	$R_3 \begin{bmatrix} &&&-1&&&& \\ &&1&&&&& \\ &-1&&&&&& \\ 1&&&&&&& \\ &&&&&&&1 \\ &&&&&&-1& \\ &&&&&1&& \\ &&&&-1&&& \end{bmatrix}$
	$R_4 \begin{bmatrix} &&&&-1&&& \\ &&&&&-1&& \\ &&&&&&-1& \\ &&&&&&&-1 \\ 1&&&&&&& \\ &1&&&&&& \\ &&1&&&&& \\ &&&1&&&& \end{bmatrix}$	$15 \begin{bmatrix} &&&&-1&&& \\ &&&&&-1&& \\ &&&&&&1& \\ &&&&&&&1 \\ 1&&&&&&& \\ &1&&&&&& \\ &&-1&&&&& \\ &&&-1&&&& \end{bmatrix}$	$26 \begin{bmatrix} &&&&-1&&& \\ &&&&&1&& \\ &&&&&&-1& \\ &&&&&&&1 \\ 1&&&&&&& \\ &-1&&&&&& \\ &&1&&&&& \\ &&&-1&&&& \end{bmatrix}$	$37 \begin{bmatrix} &&&&-1&&& \\ &&&&&1&& \\ &&&&&&1& \\ &&&&&&&-1 \\ 1&&&&&&& \\ &-1&&&&&& \\ &&-1&&&&& \\ &&&1&&&& \end{bmatrix}$
	$R_5 \begin{bmatrix} &&&&&-1&& \\ &&&&1&&& \\ &&&&&&&-1 \\ &&&&&&1& \\ &-1&&&&&& \\ 1&&&&&&& \\ &&&-1&&&& \\ &&1&&&&& \end{bmatrix}$	$41 \begin{bmatrix} &&&&&-1&& \\ &&&&1&&& \\ &&&&&&&1 \\ &&&&&&-1& \\ &-1&&&&&& \\ 1&&&&&&& \\ &&&1&&&& \\ &&-1&&&&& \end{bmatrix}$	$27 \begin{bmatrix} &&&&&-1&& \\ &&&&-1&&& \\ &&&&&&&-1 \\ &&&&&&-1& \\ &1&&&&&& \\ 1&&&&&&& \\ &&&1&&&& \\ &&1&&&&& \end{bmatrix}$	$63 \begin{bmatrix} &&&&&-1&& \\ &&&&-1&&& \\ &&&&&&&1 \\ &&&&&&1& \\ &1&&&&&& \\ 1&&&&&&& \\ &&&-1&&&& \\ &&-1&&&&& \end{bmatrix}$
	$R_6 \begin{bmatrix} &&&&&&-1& \\ &&&&&&&1 \\ &&&&1&&& \\ &&&&&-1&& \\ &&-1&&&&& \\ &&&1&&&& \\ 1&&&&&&& \\ &-1&&&&&& \end{bmatrix}$	$71 \begin{bmatrix} &&&&&&-1& \\ &&&&&&&1 \\ &&&&-1&&& \\ &&&&&1&& \\ &&1&&&&& \\ &&&-1&&&& \\ 1&&&&&&& \\ &-1&&&&&& \end{bmatrix}$	$42 \begin{bmatrix} &&&&&&-1& \\ &&&&&&&-1 \\ &&&&1&&& \\ &&&&&1&& \\ &&-1&&&&& \\ &&&-1&&&& \\ 1&&&&&&& \\ &1&&&&&& \end{bmatrix}$	$35 \begin{bmatrix} &&&&&&-1& \\ &&&&&&&-1 \\ &&&&-1&&& \\ &&&&&-1&& \\ &&1&&&&& \\ &&&1&&&& \\ 1&&&&&&& \\ &1&&&&&& \end{bmatrix}$
	$R_7 \begin{bmatrix} &&&&&&&-1 \\ &&&&&&-1& \\ &&&&&1&& \\ &&&&1&&& \\ &&&-1&&&& \\ &&-1&&&&& \\ &1&&&&&& \\ 1&&&&&&& \end{bmatrix}$	$16 \begin{bmatrix} &&&&&&&-1 \\ &&&&&&-1& \\ &&&&&-1&& \\ &&&&-1&&& \\ &&&1&&&& \\ &&1&&&&& \\ &1&&&&&& \\ 1&&&&&&& \end{bmatrix}$	$52 \begin{bmatrix} &&&&&&&-1 \\ &&&&&&1& \\ &&&&&1&& \\ &&&&-1&&& \\ &&&1&&&& \\ &&-1&&&&& \\ &-1&&&&&& \\ 1&&&&&&& \end{bmatrix}$	$43 \begin{bmatrix} &&&&&&&-1 \\ &&&&&&1& \\ &&&&&-1&& \\ &&&&1&&& \\ &&&-1&&&& \\ &&1&&&&& \\ &-1&&&&&& \\ 1&&&&&&& \end{bmatrix}$

Table A2-2. (continued)

	4									5									6									7							
		-1									-1									-1									-1						
	1									1									1									1							
				-1									-1									-1									-1				
			1									1									1									1					
54						1			54						1			67						-1			67						-1		
					-1									-1									1									1			
								1									1									-1									-1
							-1									-1									1									1	
			-1									-1									-1									-1					
				1									1									1									1				
	1									1									1									1							
		-1									-1									-1									-1						
64							1		75							-1		64							1		75							-1	
								-1									1									-1									1
					-1									1									-1									1			
						1									-1									1									-1		
				-1									-1									-1									-1				
			-1									-1									-1									-1					
		1									1									1									1						
	1									1									1									1							
74								1	56								-1	56								-1	74								1
							1									-1									-1									1	
						-1									1									1									-1		
					-1									1									1									-1			
					-1									-1									-1									-1			
						-1									-1									1									1		
							-1									1									-1									1	
								-1									1									1									-1
R_4	1								15	1								26	1								37	1							
		1									1									-1									-1						
			1									-1									1									-1					
				1									-1									-1									1				
						-1									-1									-1									-1		
					1									1									-1									-1			
								1									-1									1									-1
							-1									1									1									-1	
41		-1							R_5		-1							63		1							27		1						
	1									1									1									1							
				1									-1									-1									1				
			-1									1									-1									1					
							-1									-1									-1									-1	
								-1									-1									1									1
					1									-1									1									-1			
						1									-1									-1									1		
42			-1						35			1						R_6			-1						71			1					
				-1									1									1									-1				
	1									1									1									1							
		1									1									-1									-1						
								-1									-1									-1									-1
							1									1									-1									-1	
						-1									1									-1									1		
					1									-1									-1									1			
43				-1					52				1					16				1					R_7				-1				
			1									-1									1									-1					
		-1									-1									1									1						
	1									1									1									1							

A3. Regular (bi)representation of Cayley-Dickson algebras – from complex numbers to binary sedenions

According to the formulas in Secs. 3.5, 3.6. we shall write here $n \times n$-matrices of regular (bi)representations of (hyper)complex units of algebras **R**, **C**, **H**, **O**, and **BS** as

$$L_i x = e_i x\,, \quad R_i = x e_i\,, \tag{6.6}$$

where matrices L_i, R_i are acting upon $x \in \mathcal{A}$ as upon vectors of the vector space of the algebra $\mathcal{A}$; here $n = 1, i = 0$ for the real numbers; $n = 2, i = 0, 1$ for complex numbers; $n = 4$, $i = 0, 1, 2, 3$ for quaternions; $n = 8$, $i = 0, 1, \ldots, 7$ for octonions, and $n = 16$, $i = 0, 1, \ldots, 15$ for (binary) sedenions.

For real number algebra **R** we then have quite trivially

$$e_0 = 1\,, \tag{6.7}$$

for complex number algebra **C**

$$e_0(=1) \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1(=i) \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{6.8}$$

In what follows it is useful to have explicit expressions also for "conjugated" matrices $\bar{L}_i, \bar{R}_i$ defined as

$$\bar{L}_i x = L_i \bar{x}\,, \quad \bar{R}_i x = R_i \bar{x}\,. \tag{6.9}$$

For complex numbers we denote

$$\bar{1} \equiv f = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \bar{i} \equiv g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{6.10}$$

As $n \times n$-matrices (n=2,4,8,16) of regbireps of Cayley-Dickson algebras consist of $\frac{n}{2} \times \frac{n}{2}$-matrices of regbireps of the preceding algebra, placed on the main and collateral diagonals, then it is convenient to introduce the following denotations:

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = d(A, B)\,, \quad \begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix} = cod(A, B)\,,$$

where A, B are matrices of equal order.

Then *l-,r-matrices of the regular representation of quaternions* may be expressed in the explicit form

$$
\begin{aligned}
&l_0 = r_0 = I^{(4)}\,; \quad \bar{l}_0 = \bar{r}_0 = d(f,-1)\,;\\
&l_1 = d(i,i)\,, \quad l_2 = cod(f,-f)\,, \quad l_3 = cod(g,-g)\,;\\
&\bar{l}_1 = d(1,-i)\,, \quad \bar{l}_2 = cod(1,f)\,, \quad \bar{l}_3 = cod(i,g)\,;\\
&r_1 = d(i,-i)\,, \quad r_2 = cod(1,-1)\,, \quad r_3 = cod(i,i)\,;\\
&\bar{r}_1 = d(g,i)\,, \quad \bar{r}_2 = cod(f,1)\,, \quad \bar{r}_3 = cod(g,-i)\,.
\end{aligned}
\tag{6.11}
$$

The corresponding *L-,R-matrices of the regular representation of octonions* have the form

$$
\begin{aligned}
&L_0 = R_0 = I^{(8)}\,, \quad L_4 = cod(\bar{l}_0,-l_0)\,, \quad R_4 = cod(r_0,-r_0)\,;\\
&L_k = d(l_k,r_k)\,, \quad L_{4+k} = cod(\bar{l}_k,-\bar{r}_k)\,;\\
&R_k = d(r_k,-r_k)\,, \quad R_{4+k} = cod(l_k,l_k)\,;\\
&\bar{L}_0 = \bar{R}_0 = d(\bar{l}_0,-l_0),\ \bar{L}_4 = cod(l_0,-\bar{l}_0),\ \bar{R}_4 = cod(\bar{r}_0,r_0)\,;\\
&\bar{L}_k = d(\bar{l}_k,-r_k)\,, \quad \bar{L}_{4+k} = cod(l_k,\bar{r}_k)\,;\\
&\bar{R}_k = d(\bar{r}_k,r_k)\,, \quad \bar{R}_{4+k} = cod(\bar{l}_k,-l_k)\,;
\end{aligned}
\tag{6.12}
$$

$$k = 1,2,3\,.$$

In the Appendix 2 explicit forms of octonion regbirep L- and R- matrices are presented for all fundamental modifications. At the end of this Appendix we shall give explicit form of $\bar{L}$- and $\bar{R}$-matrices in 0-modification. These matrices are necessary to write $\mathcal{L}$- and $\mathcal{R}$-matrices of *regular birepresentation of binary sedenions*:

$$
\begin{aligned}
&\mathcal{L}_0 = \mathcal{R}_0 = I^{(16)}\,,\\
&\mathcal{L}_8 = cod(\bar{L}_0,-\bar{L}_0)\,, \quad \mathcal{R}_8 = cod(R_0,-R_0)\,;\\
&\mathcal{L}_m = d(L_m,R_m)\,, \quad \mathcal{L}_{m+8} = cod(\bar{L}_m,-\bar{R}_m)\,;\\
&\mathcal{R}_m = d(R_m,-R_m)\,, \quad \mathcal{R}_{m+8} = cod(L_m,L_m)\,;
\end{aligned}
\tag{6.13}
$$

$$m = 1,2,\ldots,7\,.$$

Formulas above allow to write L-,R-matrices of a regular (bi)representation for all (hyper)complex CD-algebras considered in this monograph – from real numbers to binary sedenions.

Table A3-1. L-,$\bar{L}$-,R- and $\bar{R}$-matrices for octonions

$$L_1\begin{bmatrix}&-1&&&&&&\\1&&&&&&&\\&&&-1&&&&\\&&1&&&&&\\&&&&&-1&&\\&&&&1&&&\\&&&&&&&1\\&&&&&&-1&\end{bmatrix}\quad \bar{L}_1\begin{bmatrix}&1&&&&&&\\1&&&&&&&\\&&&1&&&&\\&&-1&&&&&\\&&&&&1&&\\&&&&-1&&&\\&&&&&&&-1\\&&&&&&1&\end{bmatrix}\quad R_1\begin{bmatrix}&-1&&&&&&\\1&&&&&&&\\&&&1&&&&\\&&-1&&&&&\\&&&&&1&&\\&&&&-1&&&\\&&&&&&&-1\\&&&&&&1&\end{bmatrix}\quad \bar{R}_1\begin{bmatrix}&1&&&&&&\\1&&&&&&&\\&&&-1&&&&\\&&1&&&&&\\&&&&&-1&&\\&&&&1&&&\\&&&&&&&1\\&&&&&&-1&\end{bmatrix}$$

$$L_2\begin{bmatrix}&&-1&&&&&\\&&&1&&&&\\1&&&&&&&\\&-1&&&&&&\\&&&&&&-1&\\&&&&&&&-1\\&&&&1&&&\\&&&&&1&&\end{bmatrix}\quad \bar{L}_2\begin{bmatrix}&&1&&&&&\\&&&-1&&&&\\1&&&&&&&\\&1&&&&&&\\&&&&&&1&\\&&&&&&&1\\&&&&-1&&&\\&&&&&-1&&\end{bmatrix}\quad R_2\begin{bmatrix}&&-1&&&&&\\&&&-1&&&&\\1&&&&&&&\\&1&&&&&&\\&&&&&&1&\\&&&&&&&1\\&&&&-1&&&\\&&&&&-1&&\end{bmatrix}\quad \bar{R}_2\begin{bmatrix}&&1&&&&&\\&&&1&&&&\\1&&&&&&&\\&-1&&&&&&\\&&&&&&-1&\\&&&&&&&-1\\&&&&1&&&\\&&&&&1&&\end{bmatrix}$$

$$L_3\begin{bmatrix}&&&-1&&&&\\&&-1&&&&&\\&1&&&&&&\\1&&&&&&&\\&&&&&&&-1\\&&&&&&1&\\&&&&&-1&&\\&&&&1&&&\end{bmatrix}\quad \bar{L}_3\begin{bmatrix}&&&1&&&&\\&&1&&&&&\\&-1&&&&&&\\1&&&&&&&\\&&&&&&&1\\&&&&&&-1&\\&&&&&1&&\\&&&&-1&&&\end{bmatrix}\quad R_3\begin{bmatrix}&&&-1&&&&\\&&1&&&&&\\&-1&&&&&&\\1&&&&&&&\\&&&&&&&1\\&&&&&&-1&\\&&&&&1&&\\&&&&-1&&&\end{bmatrix}\quad \bar{R}_3\begin{bmatrix}&&&1&&&&\\&&-1&&&&&\\&1&&&&&&\\1&&&&&&&\\&&&&&&&-1\\&&&&&&1&\\&&&&&-1&&\\&&&&1&&&\end{bmatrix}$$

$$L_4\begin{bmatrix}&&&&-1&&&\\&&&&&1&&\\&&&&&&1&\\&&&&&&&1\\1&&&&&&&\\&-1&&&&&&\\&&-1&&&&&\\&&&-1&&&&\end{bmatrix}\quad \bar{L}_4\begin{bmatrix}&&&&1&&&\\&&&&&-1&&\\&&&&&&-1&\\&&&&&&&-1\\1&&&&&&&\\&1&&&&&&\\&&1&&&&&\\&&&1&&&&\end{bmatrix}\quad R_4\begin{bmatrix}&&&&-1&&&\\&&&&&-1&&\\&&&&&&-1&\\&&&&&&&-1\\1&&&&&&&\\&1&&&&&&\\&&1&&&&&\\&&&1&&&&\end{bmatrix}\quad \bar{R}_4\begin{bmatrix}&&&&1&&&\\&&&&&1&&\\&&&&&&1&\\&&&&&&&1\\1&&&&&&&\\&-1&&&&&&\\&&-1&&&&&\\&&&-1&&&&\end{bmatrix}$$

$$L_5\begin{bmatrix}&&&&&-1&&\\&&&&-1&&&\\&&&&&&&1\\&&&&&&-1&\\&1&&&&&&\\1&&&&&&&\\&&&1&&&&\\&&-1&&&&&\end{bmatrix}\quad \bar{L}_5\begin{bmatrix}&&&&&1&&\\&&&&1&&&\\&&&&&&&-1\\&&&&&&1&\\&-1&&&&&&\\1&&&&&&&\\&&&-1&&&&\\&&1&&&&&\end{bmatrix}\quad R_5\begin{bmatrix}&&&&&-1&&\\&&&&1&&&\\&&&&&&&-1\\&&&&&&1&\\&-1&&&&&&\\1&&&&&&&\\&&&-1&&&&\\&&1&&&&&\end{bmatrix}\quad \bar{R}_5\begin{bmatrix}&&&&&1&&\\&&&&-1&&&\\&&&&&&&1\\&&&&&&-1&\\&1&&&&&&\\1&&&&&&&\\&&&1&&&&\\&&-1&&&&&\end{bmatrix}$$

$$L_6\begin{bmatrix}&&&&&&-1&\\&&&&&&&-1\\&&&&-1&&&\\&&&&&1&&\\&&1&&&&&\\&&&-1&&&&\\1&&&&&&&\\&1&&&&&&\end{bmatrix}\quad \bar{L}_6\begin{bmatrix}&&&&&&1&\\&&&&&&&1\\&&&&1&&&\\&&&&&-1&&\\&&-1&&&&&\\&&&1&&&&\\1&&&&&&&\\&-1&&&&&&\end{bmatrix}\quad R_6\begin{bmatrix}&&&&&&-1&\\&&&&&&&1\\&&&&1&&&\\&&&&&-1&&\\&&-1&&&&&\\&&&1&&&&\\1&&&&&&&\\&-1&&&&&&\end{bmatrix}\quad \bar{R}_6\begin{bmatrix}&&&&&&1&\\&&&&&&&-1\\&&&&-1&&&\\&&&&&1&&\\&&1&&&&&\\&&&-1&&&&\\1&&&&&&&\\&1&&&&&&\end{bmatrix}$$

$$L_7\begin{bmatrix}&&&&&&&-1\\&&&&&&1&\\&&&&&-1&&\\&&&&-1&&&\\&&&1&&&&\\&&1&&&&&\\&-1&&&&&&\\1&&&&&&&\end{bmatrix}\quad \bar{L}_7\begin{bmatrix}&&&&&&&1\\&&&&&&-1&\\&&&&&1&&\\&&&&1&&&\\&&&-1&&&&\\&&-1&&&&&\\&1&&&&&&\\1&&&&&&&\end{bmatrix}\quad R_7\begin{bmatrix}&&&&&&&-1\\&&&&&&-1&\\&&&&&1&&\\&&&&1&&&\\&&&-1&&&&\\&&-1&&&&&\\&1&&&&&&\\1&&&&&&&\end{bmatrix}\quad \bar{R}_7\begin{bmatrix}&&&&&&&1\\&&&&&&1&\\&&&&&-1&&\\&&&&-1&&&\\&&&1&&&&\\&&1&&&&&\\&-1&&&&&&\\-1&&&&&&&\end{bmatrix}$$

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ABOUT THE BOOK

This is the only original monograph on nonassociative algebras in physics existing in the recent literature, with an historical perspective and the most advanced recent applications. After due emphasis on the contributions of Sophus Lie, the monograph presents * the work of Albert, Jordan, Santilli and others, * nonassociativity in mathematics and physics, * limit transition between algebras, contractions and deformations, * Cayley-Dikson algebras and their representations, * Dirac equation and the problem of self-duality in the hypercomplex formalism of octonions and sedenions, * Moufang-Mal'tsev symmetries and conservation laws. This is an indispensable text for all scientific libraries.

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