

Lie-Santilli Admissibility using P-hyperoperations on matrices

Thomas Vougiouklis

Democritus University of Thrace, School of Education,

68 100 Alexandroupolis, Greece

tvougiou@eled.duth.gr

Abstract

We present a hyperproduct on non square matrices by using a generalization of the well known P-hopes. This theory is connected with the corresponding classical algebra, mainly with the theory of representations by (hyper) matrices. This can be achieved by using the fundamental relations defined on the hyperstructures.

Key words: hyperstructures, H_v -structures, hopes, P-hopes, e-hyperstructures

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1 Introduction

The largest class of hyperstructures is the one which satisfy the weak properties. These are called H_v -structures introduced in 1990 [13], and they proved to have a lot of applications on several applied sciences such as linguistics, biology, chemistry, physics, and so on. The H_v -structures satisfy the weak axioms where the non-empty intersection replaces the equality. The H_v -structures can be used in models as an organized devise, as well.

Recall some basic definitions:

Definition 1.1. A set H equipped with at least one hyperoperation (we abbreviate hyperoperation by **hope**) $\cdot : H \times H \rightarrow P(H) - \{\emptyset\}$, is called *Hyperstructure*. We abbreviate by WASS the **weak associativity**: $(xy)z \cap x(yz) \neq \emptyset, \forall x, y, z \in H$ and by COW the **weak commutativity**: $xy \cap yx \neq \emptyset, \forall x, y \in H$. The hyperstructure (H, \cdot) is called H_v -semigroup if it is WASS and is called H_v -group if it is reproductive H_v -semigroup, i.e. $xH = Hx = H, \forall x \in H$. The hyperstructure $(R, +, \cdot)$ is called **H_v -ring** if $(+)$ and (\cdot) are WASS, the reproduction axiom is valid for $(+)$ and (\cdot) is weak distributive with respect to $(+)$, i.e. $x(y + z) \cap (xy + xz) \neq \emptyset, (x + y)z \cap (xz + yz) \neq \emptyset, \forall x, y, z \in R$. The **H_v -module** is an H_v -group over an H_v -ring if the weak distributivity and a mixed weak associativity on all hypermultiplications, is valid. In an analogous way the **H_v -vector spaces** and the **H_v -algebra** can be defined [15].

A large class of H_v -structures is the following [21]:

Definition 1.2. Let (G, \cdot) be groupoid (resp. hypergroupoid) and $f : G \rightarrow G$ be a map. We define a hope (∂) , called theta-hope, we write ∂ -hope, on G as follows

$$x\partial y = \{f(x) \cdot y, x \cdot f(y)\}, \forall x, y \in G.$$

$$\text{(resp. } x\partial y = (f(x) \cdot y) \cup (x \cdot f(y)), \forall x, y \in G)$$

If (\cdot) is commutative then ∂ is commutative. If (\cdot) is COW, then ∂ is COW.

For more definitions and results on H_v -structures one can see in books and papers as [1], [3], [15], [16], [19]. An extreme class of the H_v -structures is the following [14]: An H_v -structure is called *very thin* iff all hopes are operations except one, which has all hyperproducts singletons except only one, which has cardinality more than one.

The fundamental relations β^* , γ^* and ϵ^* , are defined, in H_v -groups, H_v -rings and H_v -vector space, respectively, as the smallest equivalences so that the quotient would be group, ring and vector space, respectively [12], [13], [15], [17], [18], [23]. The way to find the fundamental classes is given by analogous theorems to the following one:

Theorem 1.1. Let (H, \cdot) be an H_v -group and denote by \mathbf{U} the set of all finite products of elements of H . We define the relation β in H by setting $x\beta y$ iff $\{x, y\} \subset \mathbf{u}$ where $\mathbf{u} \in \mathbf{U}$. Then β^* is the transitive closure of β .

We present the proof for the analogous to above theorem in the case of an H_v -ring [15], [23]:

Theorem 1.2. *Let $(\mathbf{R}, +, \cdot)$ be an H_v -ring, \mathbf{U} be the set of all finite polynomials of elements of \mathbf{R} . Define the relation γ in \mathbf{R} as follows: $x\gamma y$ iff $\{x, y\} \subset \mathbf{u}$ where $\mathbf{u} \in \mathbf{U}$. Then γ^* is the transitive closure of γ .*

Proof. Let $\underline{\gamma}$ be the transitive closure of γ , and $\underline{\gamma}(a)$ the class of a . First we prove that the quotient set $M/\underline{\gamma}$ is a ring.

In $R/\underline{\gamma}$ the sum (\oplus) and the product (\otimes) are defined in the usual manner:

$$\underline{\gamma}(a) \oplus \underline{\gamma}(b) = \{\underline{\gamma}(c) : c \in \underline{\gamma}(a) + \underline{\gamma}(b)\},$$

$$\underline{\gamma}(a) \otimes \underline{\gamma}(b) = \{\underline{\gamma}(d) : d \in \underline{\gamma}(a) \cdot \underline{\gamma}(b)\}, \quad \forall a, b \in R.$$

Take $a' \in \underline{\gamma}(a)$, $b' \in \underline{\gamma}(b)$. Then we have

$$a' \underline{\gamma} a \text{ iff } \exists x_1, \dots, x_{m+1} \text{ with } x_1 = a', x_{m+1} = a \text{ and } u_1, \dots, u_m \in U$$

such that $\{x_i, x_{i+1}\} \subset u_i$, $i = 1, \dots, m$, and

$$b' \underline{\gamma} b \text{ iff } \exists y_1, \dots, y_{n+1} \text{ with } y_1 = b', y_{n+1} = b \text{ and } v_1, \dots, v_n \in U$$

such that $\{y_j, y_{j+1}\} \subset v_j$, $i = 1, \dots, n$.

From the above we obtain

$$\{x_i, x_{i+1}\} + y_1 \subset u_i + v_1, \quad i = 1, \dots, m - 1,$$

$$x_{m+1} + \{y_j, y_{j+1}\} \subset u_m + v_j, \quad j = 1, \dots, n.$$

$$u_i + v_1 = t_i, \quad i = 1, \dots, m - 1 \text{ and } u_m + v_j = t_{m+j-1}, \quad j = 1, \dots, n$$

are also polynomials, so $t_k \in U, \forall k \in \{1, \dots, m + n - 1\}$.

Now, pick up z_1, \dots, z_{m+n} such that $z_i \in x_i + y_1$, $i = 1, \dots, n$ and $z_{m+j} \in x_{m+1} + y_{j+1}$, $j = 1, \dots, n$, therefore, using the above relations we obtain $\{z_k, z_{k+1}\} \subset t_k$, $k = 1, \dots, m + n - 1$.

Thus, every element $z_1 \in x_1 + y_1 = a' + b'$ is $\underline{\gamma}$ equivalent to every element $z_{m+n} \in x_{m+1} + y_{n+1} = a + b$. Therefore $\underline{\gamma}(a) \oplus \underline{\gamma}(b)$ is a singleton so we can write $\underline{\gamma}(a) \oplus \underline{\gamma}(b) = \underline{\gamma}(c)$ for all $c \in \underline{\gamma}(a) + \underline{\gamma}(b)$. In a similar way we prove that $\underline{\gamma}(a) \otimes \underline{\gamma}(b) = \underline{\gamma}(d)$ for all $d \in \underline{\gamma}(a) \cdot \underline{\gamma}(b)$. The WASS and the weak

distributivity on \mathbf{R} guarantee that the associativity and the distributivity are valid for the quotient R/γ^* . Therefore R/γ^* is a ring.

Now let σ be an equivalence relation in \mathbf{R} such that R/σ is a ring, denote $\sigma(a)$ the class of a . Then $\sigma(a) \oplus \sigma(b)$ and $\sigma(a) \otimes \sigma(b)$ are singletons for all $a, b \in R$, i.e.

$$\sigma(a) \oplus \sigma(b) = \sigma(c), \forall c \in \sigma(a) + \sigma(b), \quad \sigma(a) \otimes \sigma(b) = \sigma(d), \forall d \in \sigma(a) \cdot \sigma(b)$$

Thus we can write, for every $a, b \in R$ and $A \subset \sigma(a)$, $B \subset \sigma(b)$,

$$\sigma(a) \oplus \sigma(b) = \sigma(a + b) = \sigma(A + B), \quad \sigma(a) \otimes \sigma(b) = \sigma(ab) = \sigma(A \cdot B)$$

By induction, we extend these relations on finite sums and products. Thus, for every $u \in U$, we have the relation $\sigma(x) = \sigma(u)$ for all $x \in u$. Consequently $x \in \gamma(a)$ implies $x \in \sigma(a)$.

That means that $\underline{\gamma}$ is the smallest equivalence relation in \mathbf{R} such that $R/\underline{\gamma}$ is a ring, i.e. $\underline{\gamma} = \gamma^*$. \square

An element is called single if its fundamental class is singleton [12], [15].

Definition 1.3. *An H_v -ring $(R, +, \cdot)$ is called H_v -field if R/γ^* is a field.*

From this definition a new class of hyperstructures is defined [18], [19]: The H_v -semigroup (H, \cdot) is called h/v -group if H/β^* is a group. In a similar way the h/v -rings, h/v -fields, h/v -vector spaces etc, are defined.

Let $(H, \cdot), (H, *)$ be H_v -semigroups defined on the same set H . (\cdot) is called *smaller* than $(*)$, and $(*)$ *greater* than (\cdot) , iff there exists an

$$f \in \text{Aut}(H, *) \text{ such that } xy \subset f(x * y), \quad \forall x, y \in H.$$

Then we write $\cdot \leq *$ and we say that $(H, *)$ *contains* (H, \cdot) . If (H, \cdot) is a structure then it is called *basic structure* and $(H, *)$ is called H_b -structure.

Theorem 1.3. *(The Little Theorem). Greater hopes than the ones which are WASS or COW, are also WASS or COW, respectively.*

This obvious theorem leads to a partial order on H_v -structures and mainly to a correspondence between hyperstructures and posets. Using

the partial ordering with the fundamental relations one can give several definitions to obtain constructions used in several applications [17], [18], [19].

The uniting elements method was introduced by Corsini - Vougiouklis [2]. With this method one puts in the same class, two or more elements. This leads to structures satisfying additional properties. Constructions with desired fundamental structures, where the elements of a structure are replaced by sets such that the obtained H_v -structure has the same fundamental structure. The "enlarged" hyperstructures were examined in the sense that an extra element, outside the set, appears in one result. On the other direction one can obtain H_v -vector spaces, by taking out some elements [17], [18].

Let (H, \cdot) be hypergroupoid. We say that *remove* $h \in H$, if consider the restriction of (\cdot) in the set $H - \{h\}$. We say that $\underline{h} \in H$ *absorbs* $h \in H$ if we replace h , whenever it appears, by \underline{h} . We say that $\underline{h} \in H$ *merges* with the $h \in H$, if we take as the product of any $x \in H$ by \underline{h} , the union of the results of x with both h and \underline{h} , and we consider h and \underline{h} as one class with representative \underline{h} .

The representation (abbreviate by rep) problem of H_v -structures by H_v -matrices is defined as follows [15], [18], [19]:

Definition 1.4. H_v -**matrix** is a matrix with entries of an H_v -ring or H_v -field. The hyperproduct of two H_v -matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$, of type $m \times n$ and $n \times r$ respectively, is defined, in the usual manner, but it is a set of $m \times r$ H_v -matrices:

$$A \cdot B = (a_{ij}) \cdot (b_{ij}) = \{C = (c_{ij}) | (c_{ij}) \in \oplus \sum a_{ik} \cdot b_{kj}\},$$

where (\oplus) denotes the n -ary circle hope on the hyperaddition, , i.e. the sum of products of elements of the H_v -ring is the union of the sets obtained with all possible parentheses put on them. The hyperproduct is not WASS. Let (H, \cdot) be an H_v -group, consider an H_v -ring or H_v -field $(R, +, \cdot)$ and a set $M_R = \{(a_{ij}) | a_{ij} \in R\}$ then is called H_v -matrix rep, any map

$$T : H \rightarrow M_R : h \mapsto T(h) \text{ with } T(h_1 h_2) \cap T(h_1)T(h_2) \neq \emptyset, \forall h_1, h_2 \in H.$$

If $T(h_1 h_2) \subset T(h_1)T(h_2)$, then \mathbf{T} is an inclusion rep, if $T(h_1 h_2) = T(h_1)T(h_2)$, then \mathbf{T} is a good rep.

The main theorem of the theory of reps is:

Theorem 1.4. *A necessary condition to have an inclusion rep \mathbf{T} of the H_v -group (H, \cdot) by $n \times n$ H_v -matrices over the H_v -ring $(R, +, \cdot)$ is the following: For all classes $\beta^*(a)$, $a \in H$ there must exist elements $a_{ij} \in R$, $i, j \in \{1, \dots, n\}$ such that*

$$T(\beta^*(a)) \subset \{A = (a'_{ij}) | a'_{ij} \in \gamma^*(a_{ij}), i, j \in \{1, \dots, n\}\}.$$

The rep problem of H_v -groups by H_v -matrices, has more parameters than the classical matrix rep theory. The main difficulty is that there is not only one basic H_v -field as the corresponding field of real numbers. Thus one should have a number of H_v -fields to choose the appropriate one. We can construct, by adding or deleting elements, to take greater or smaller H_v -fields so that a good rep to be reached. The basic Theorem 1.4, leads to the fact that in order that a rep should be obtained, a corresponding rep on the fundamental structures must be valid. Another complicated problem, more than the one in the classical theory, is to find irreducible reps. However, for this problem one has to use some smaller hyperstructures, a non-conventional practice in the classical reps.

2 Hopes on non-square matrices

A general way to define hopes from given operations [11], [15] is the following:

Definition 2.1. *Let (G, \cdot) be a groupoid then for every $P \subset G$, $P \neq \emptyset$, we define the following hopes called **P-hopes**: for all $x, y \in G$*

$$\underline{P} : x\underline{P}y = (xP)y \cup x(Py),$$

$$\underline{P}_r : x\underline{P}_ry = (xy)P \cup x(yP), \quad \underline{P}_l : x\underline{P}_ly = (Px)y \cup P(xy).$$

The (G, \underline{P}) , (G, \underline{P}_r) and (G, \underline{P}_l) are called *P-hyperstructures*. If (G, \cdot) is semigroup, then (G, \underline{P}) is a semihypergroup but we do not know about (G, \underline{P}_r) and (G, \underline{P}_l) . In some cases, depending on the choice of the set P , the (G, \underline{P}_r) and (G, \underline{P}_l) can be associative or WASS. If in G several operations are defined, then for each one operation, P-hopes can be defined. Generalization of P-hopes is the following [4], [5], [9], [22]:

Construction 2.1. Let (G, \cdot) be an abelian group and P any subset of G with more than one elements. We define the hope \times_P as follows:

$$x \times_P y = \begin{cases} x \cdot P \cdot y = \{x \cdot h \cdot y \mid h \in P\} & \text{if } x \neq e \text{ and } c \neq e \\ x \cdot y & \text{if } x = e \text{ or } y = e \end{cases}$$

we call this hope P_e -hope. The hyperstructure (G, \times_P) is an abelian H_v -group.

Definition 2.2. [13], [15], [20] Let $\mathbf{M} = \mathbf{M}_{m \times n}$ be a module of $m \times n$ matrices over an \mathbf{R} and $\mathbf{P} = \{P_i : i \in I\} \subseteq \mathbf{M}$. We define, a kind of, a P -hope \underline{P} on \mathbf{M} as follows:

$$\underline{P} : \mathbf{M} \times \mathbf{M} \rightarrow P(\mathbf{M}) : (A, B) \rightarrow \underline{APB} = \{AP_i^t B : i \in I\} \subseteq \mathbf{M}$$

where P^t denotes the transpose of the matrix P .

The hope \underline{P} , which is a bilinear map, is a generalization of Rees operation where, instead of one sandwich matrix, a set of sandwich matrices is used. The hope \underline{P} is strong associative and the inclusion distributivity with respect to addition of matrices is valid:

$$\underline{AP}(B + C) \subseteq \underline{APB} + \underline{APC} \text{ for all } A, B, C \text{ in } \mathbf{M}$$

Therefore, $(\mathbf{M}, +, \underline{P})$ defines a multiplicative hyperring on non-square matrices. Multiplicative hyperring means that only the multiplication is a hope.

Definition 2.3. Let $\mathbf{M} = \mathbf{M}_{m \times n}$ be a module of $m \times n$ matrices over R and let us take sets

$$\mathbf{S} = \{s_k : k \in K\} \subseteq R, \mathbf{Q} = \{Q_j : j \in J\} \subseteq \mathbf{M}, \mathbf{P} = \{P_i : i \in I\} \subseteq \mathbf{M}.$$

We define three hopes as follows

$$\underline{S} : R \times \mathbf{M} \rightarrow \mathbf{P}(\mathbf{M}) : (r, A) \rightarrow r\underline{SA} = \{(rs_k)A : k \in K\} \subseteq \mathbf{M}$$

$$\underline{Q}_+ : \mathbf{M} \times \mathbf{M} \rightarrow \mathbf{P}(\mathbf{M}) : (A, B) \rightarrow \underline{AQ}_+ B = \{A + Q_j + B : j \in J\} \subseteq \mathbf{M}$$

$$\underline{P} : \mathbf{M} \times \mathbf{M} \rightarrow \mathbf{P}(\mathbf{M}) : (A, B) \rightarrow \underline{APB} = \{AP_i^t B : i \in I\} \subseteq \mathbf{M}$$

Then $(\mathbf{M}, \underline{S}, \underline{Q}_+, \underline{P})$ is a hyperalgebra over \mathbf{R} which we shall call matrix P -hyperalgebra.

Remark 2.1. In a similar way a generalization of this hyperalgebra can be defined if one considers an H_v -ring or an H_v -field instead of a ring and using H_v -matrices instead of matrices.

Definition 2.4. Let $A = (a_{ij}), B = (b_{ij}) \in M_{m \times n}$, we call (A, B) a unitize pair of matrices if $A^t B = I_n$, where I_n denotes the $n \times n$ unit matrix. We prove the following theorem which can be applied in the classical theory as well [23].

Theorem 2.1. If $m < n$, then there is no unitize pair.

Proof. Suppose that $A^t B = (c_{ij})$, that is $c_{ij} = \sum_{k=1}^m a_{ik} b_{kj}$, and we denote by A_m the block of the matrix A such that $A_m = (a_{ij}) \in M_{m \times m}$, i.e. we consider the matrix of the first m columns. Then we suppose that we have $(A_m)^t B_m = I_m$, therefore we must have $\det(A_m) \neq 0$. Now, since $n > m$, we can consider the homogeneous system with respect to the "unknowns" $b_{1n}, b_{2n}, \dots, b_{mn}$:

$$c_{in} = \sum_{k=1}^m a_{ik} b_{kn}, i = 1, 2, \dots, m$$

From which, since $\det(A_m) \neq 0$, we obtain that $b_{1n} = b_{2n} = \dots = b_{mn} = 0$. Using this fact on the last equation, on the same unknowns, $c_{nn} = \sum_{k=1}^m a_{nk} b_{kn} = 1$ we have $0=1$, absurd. \square

Now we restrict ourselves on the minimal non-degenerate case of matrices for which a unitize pair is defined. This is the case of $M_{3 \times 2}$, of 3×2 matrices, because in order to have unitized pairs we must not to have $m < n$. We will denote by E_{ij} the matrix which has in ij entry 1 and 0 in the rest cases. A characteristic matrix which can be used in the helix hope [6], [25], in order to become an operation is the case of the matrix $\mathbf{E} = \mathbf{E}_{11} + \mathbf{E}_{22} + \mathbf{E}_{31}$.

Proposition 2.1. *All matrices of type $X = x\mathbf{E}_{11} + y\mathbf{E}_{12} + \mathbf{E}_{22} + (1-x)\mathbf{E}_{31} - y\mathbf{E}_{32}$, form a unitize pair with E .*

Proof. Straightforward we have $\mathbf{X}^t\mathbf{E} = \mathbf{I}_n$, for all \mathbf{X} . □

Theorem 2.2. *In the set $M_{3 \times 2}$, of matrices over \mathbf{R} , consider a set \mathbf{P} of matrices of type*

$$\mathbf{P} = x_i\mathbf{E}_{11} + y_i\mathbf{E}_{12} + \mathbf{E}_{22} + (1-x_i)\mathbf{E}_{31} - y_i\mathbf{E}_{32}, i \in \mathbf{I},$$

Then the \underline{P} hope, the hyperproduct of matrices,

$$\underline{\mathbf{P}} : \mathbf{M}_{3 \times 2} \times \mathbf{M}_{3 \times 2} \rightarrow P(\mathbf{M}_{3 \times 2}) : (\mathbf{A}, \mathbf{B}) \rightarrow \mathbf{A}\underline{\mathbf{P}}\mathbf{B} = \{\mathbf{A}\mathbf{P}_i^t\mathbf{B} : i \in \mathbf{I}\} \subseteq \mathbf{M}_{3 \times 2}$$

has the matrix $\mathbf{E} = \mathbf{E}_{11} + \mathbf{E}_{22} + \mathbf{E}_{31}$, as a right scalar unit.

Proof. Straightforward we have $\mathbf{P}^t\mathbf{E} = \mathbf{I}_n$, is singleton. □

The general definition of an H_v -Lie algebra was given in [10], [23] as follows:

Definition 2.5. *Let $(L, +)$ be an H_v -vector space over the field $(F, +, \cdot)$, $\phi : F \rightarrow F/\gamma^*$ the canonical map and $\omega_F = \{x \in F : \phi(x) = 0\}$, where 0 is the zero of the field F/γ^* . Similarly, let ω_L be the core of the canonical map $\phi' : L \rightarrow L/\epsilon^*$ and denote by the same symbol 0 the zero of L/ϵ^* . Consider the bracket (commutator) hope:*

$$[,] : L \times L \rightarrow P(L) : (x, y) \rightarrow [x, y]$$

then \mathbf{L} is an H_v -Lie algebra over \mathbf{F} if the following axioms are satisfied:

(L1) The bracket hope is bilinear, i.e. $\forall x, x_1, x_2, y, y_1, y_2 \in L, \lambda_1, \lambda_2 \in F$

$$[\lambda_1 x_1 + \lambda_2 x_2, y] \cap (\lambda_1 [x_1, y] + \lambda_2 [x_2, y]) \neq \emptyset$$

$$[x, \lambda_1 y_1 + \lambda_2 y_2] \cap (\lambda_1 [x, y_1] + \lambda_2 [x, y_2]) \neq \emptyset,$$

(L2) $[x, x] \cap \omega_L \neq \emptyset, \forall x \in L$

(L3) $([x, [y, z]] + [y, [z, x]] + [z, [x, y]]) \cap \omega_L \neq \emptyset, \forall x, y \in L$

This is a general definition thus one can use special cases to the problems in applied sciences.

3 Some applications

Hyperstructures have a variety of applications in other branches of mathematics and in many other sciences. This theory is closely related to fuzzy theory; consequently, hyperstructures can now be widely applicable in industry and production, too. In several books and papers [1], [3], [4], [5], [7], [8], [9], [10], [22] one can find numerous applications. The Lie-Santilli theory on isotopies was born in 1970's to solve Hadronic Mechanics problems. The isofields needed in this theory correspond into the hyperstructures were introduced by Santilli and Vougiouklis in 1996 [8] and they are called *e-hyperfields*.

Definition 3.1. *A hyperstructure (H, \cdot) which contain a unique scalar unit e , is called *e-hyperstructure*, if $\forall x$, there exists an inverse x^{-1} , i.e. $e \in x \cdot x^{-1} \cap x^{-1} \cdot x$. Inverses are not necessarily unique. $(F, +)$ is an abelian group with unit 0, (\cdot) is WASS, (\cdot) is weak distributive with respect to $(+)$, 0 is absorbing: $0 \cdot x = x \cdot 0 = 0, \forall x \in F$, there exist a scalar unit 1, i.e. $1 \cdot x = x \cdot 1 = x, \forall x \in F$, and for every $x \in F$ there exists a unique inverse x^{-1} , such that $1 \in x \cdot x^{-1} \cap x^{-1} \cdot x$. The elements of an *e-hyperfield* are called *e-hypernumbers*. In the case that: $1 = x \cdot x^{-1} = x^{-1} \cdot x$, then we have a strong *e-hyperfield*.*

A general construction is the following:

Definition 3.2. *The Main e-Construction. Given a group (G, \cdot) , where e is the unit, then we define in G , a large number of hopes (\otimes) as follows:*

$$x \otimes y = \{xy, g_1, g_2, \dots\}, \forall x, y \in G - \{e\}, \text{ and } g_1, g_2, \dots \in G - \{e\}$$

*g_1, g_2, \dots are not necessarily the same for each pair (x, y) . Then (G, \otimes) becomes an H_v -group, in fact is an H_b -group which contains the (G, \cdot) . The H_v -group (G, \otimes) is an *e-hypergroup*. Moreover, if for each x, y such that $xy = e$, so we have $x \otimes y = xy$, then (G, \otimes) becomes a strong *e-hypergroup**

The proof is immediate since for both cases we enlarge the results of the group by putting elements from G and applying the Little Theorem. Moreover one can see that the unit e is a unique scalar element and for

each x in G , there exists a unique inverse x^{-1} , such that $1 \in x \cdot x^{-1} \cap x^{-1} \cdot x$ and if this condition is valid then we have $1 = x \cdot x^{-1} = x^{-1} \cdot x$. So the hyperstructure (G, \otimes) is a strong e-hypergroup. The above main e-construction gives an extremely large number of e-hopes. These e-hopes can be used in the several more complicate cases to obtain appropriate e-hyperstructures. However, notice that the most useful are the ones where only few products are enlarged.

The Lie-Santilli admissibility [9], [23] on square matrices is not faced in this presentation. We can present this problem on the non-square case. The problem can be faced in two ways: first, using real or complex numbers, so using ordinary matrices and hopes, second, using hypernumbers (e-hypernumbers) as entries and the ordinary operations on non-square hypermatrices.

The general definition is the following:

Construction 3.1. *Let $\mathbf{L} = (\mathbf{M}_{m \times n}, +)$ be an H_v -vector space of $m \times n$ hyper-matrices over the H_v -field $(\mathbf{F}, +, \cdot)$, $\phi : \mathbf{F} \rightarrow \mathbf{F}/\gamma^*$, the canonical map and $\omega_F = \{x \in F : \phi(x) = 0\}$, where 0 is the zero of the fundamental field \mathbf{F}/γ^* . Similarly, let ω_L be the core of the canonical map $\phi' : \mathbf{L} \rightarrow \mathbf{L}/\epsilon^*$ and denote by the same symbol 0 the zero of \mathbf{L}/ϵ^* . Take any two subsets $\mathbf{R}, \mathbf{S} \subseteq \mathbf{L}$ then a **Santilli's Lie-admissible** hyperalgebra is obtained by taking the Lie bracket, which is a hope:*

$$[,]_{RS} : \mathbf{L} \times \mathbf{L} \rightarrow \mathbf{P}(\mathbf{L}) : [x, y]_{RS} = x\mathbf{R}^t y - y\mathbf{S}^t x.$$

*Notice that $[x, y]_{RS} = x\mathbf{R}^t y - y\mathbf{S}^t x = \{xr^t y - ys^t x / r \in \mathbf{R} \text{ and } s \in \mathbf{S}\}$
Special cases, but not degenerate, are the "small" and "strict" ones:*

- $\mathbf{R} = e$. Then, $[x, y]_{RS} = xy - y\mathbf{S}^t x = \{xy - ys^t x / s \in \mathbf{S}\}$
- $\mathbf{S} = e$. Then, $[x, y]_{RS} = x\mathbf{R}^t y - yx = \{xr^t y - yx / r \in \mathbf{R}\}$
- $\mathbf{R} = \{r_1, r_2\}$, and $\mathbf{S} = \{s_1, s_2\}$ then

$$[x, y]_{RS} = x\mathbf{R}^t y - y\mathbf{S}^t x = \{xr_1^t y - ys_1^t x, xr_1^t y - ys_2^t x, xr_2^t y - ys_1^t x, xr_2^t y - ys_2^t x\}$$

Remark 3.1. *In the above constructions whenever a "shift" of elements is needed, as in Santilli's isothory then the elements for the subsets \mathbf{S} and \mathbf{R} must belong to a set of hypermatrices where a reversibility could be applied.*

An important new application, which combines hyperstructure theory and fuzzy theory, is to replace in questionnaires the scale of Likert by the bar of Vougiouklis & Vougiouklis [24]. The suggestion is the following:

Definition 3.3. *In every question substitute the Likert scale with 'the bar' whose poles are defined with '0' on the left end, and '1' on the right end:*

$$0 \text{ ————— } 1$$

The subjects/participants are asked instead of deciding and checking a specific grade on the scale, to cut the bar at any point s/he feels expresses her/his answer to the specific question.

The use of the bar of Vougiouklis & Vougiouklis instead of a scale of Likert has several advantages during both the filling-in and the research processing [24]. The final suggested length of the bar, according to the Golden Ratio, is 6.2cm.

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