Algebra, Hyperalgebra and Lie-santilli Theory

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Abstract

The theory of hyperstructures can offer to the Lie-Santilli Theory a variety of models to specify the mathematical representation of the related theory. In this paper we focus on the appropriate general hyperstructures, especially on hyperstructures with hyperunits. We define a Lie hyperalgebra over a hyperfield as well as a Jordan hyperalgebra, and we obtain some results in this respect. Finally, by using the concept of fundamental relations we connect hyper algebras to Lie algebras and Lie-Santilli-addmissible algebras.

Keywords: Algebra; Hyperring; Hyperfield; Hypervector space; Hyper algebra; Lie hyperalgebra; Lie admissible hyperalgebra; Fundamental relation

Introduction

The structure of the laws of physics is largely based on symmetries. The objects in Lie theory are fundamental, interesting and innovating in both mathematics and physics. It has many applications to the spectroscopy of molecules, atoms, nuclei and hadrons. The central role of Lie algebra in particle physics is well known. A Lie-admissible algebra, introduced by Albert [1], is a (possibly non-associative) algebra that becomes a Lie algebra under the bracket \([a,b] = ab - ba\). Examples include associative algebras, Lie algebras and Okubo algebras. Lie admissible algebras arise in various topics, including geometry of invariant affine connections on Lie groups and classical and quantum mechanics.

For an algebra \(A\) over a field \(F\), the commutator algebra \(A^\circ\) of \(A\) is the anti-commutative algebra with multiplication \([a,b] = ab - ba\) defined on the vector space \(A\). If \(A^\circ\) is a Lie algebra, i.e., satisfies the Jacobi identity, then \(A^\circ\) is called Lie-admissible. Much of the structure theory of Lie-admissible algebras has been carried out initially under additional conditions such as the flexible identity or power-associativity.

Santilli obtained Lie admissible algebras (brackets) from a modified form of Hamilton's equations with external terms which represent a general non-self-adjoint Newtonian system in classical mechanics. In 1967, Santilli introduced the product

\[
(A,B) = \lambda AB - \mu BA = \alpha(AB - BA) + \beta(AB + BA),
\]

where \(\lambda = \alpha + \beta, \mu = \alpha\), which is jointly Lie admissible and Jordan admissible while admitting Lie algebras in their classification. Then, he introduced the following infinitesimal and finite generalizations of Heisenberg equations

\[
\frac{dA}{dt} = (A, H) = \lambda AH - \mu HA,
\]

where \(H(t) = \sum_{\alpha} s_{\alpha} s_{\alpha}^* = e^{i\beta} \omega_{\alpha} \omega_{\alpha}^* \), \(U = e^{i\beta} \), \(V = e^{i\beta} \), and \(H\) is the Hamiltonian. In 1978, Santilli introduced the following most general known realization of products that are jointly Lie admissible and Jordan admissible

\[
(A,B) = AB - BA = (ATB - BTA) + \{AWB + BWA\}
\]

\[
= [A, B^*] + \{A, B\}^*
\]

\[
= (ATH - HTA) + \{AWH + HWA\},
\]

where \(R = T + W, S = W - T\) and \(R_S, R_T S\) are non-singular operators [2-25].

Algebraic hyperstructures are a natural generalization of the ordinary algebraic structures which was first initiated by Marty [11]. After the pioneered work of Marty, algebraic hyperstructures have been developed by many researchers. A review of hyperstructures can be found in studies of Corsini [3, 4, 7, 22, 24]. This generalization offers a lot of models to express their problems in an algebraic way. Several applications appeared already as in Hadronic Mechanics, Biology, Conchology, Chemistry, and so on. Davvaz, Santilli and Vougiouklis studied multi-valued hyperstructures following the apparent existence in nature of a realization of two-valued hyperstructures with hyperunits characterized by matter-antimatter systems and their extensions where matter is represented with conventional mathematics and antimatter is represented with isodual mathematics [6, 9, 10]. On the other hand, the main tools connecting the class of algebraic hyperstructures with the classical algebraic structures are the fundamental relations [5, 8, 22, 24]. In this paper, we study the notion of algebra, hyperalgebra and their connections by using the concept of fundamental relation. We introduce a special class of Lie hyperalgebra. By this class of Lie hyperalgebra, we are able to generalize the concept of Lie-Santilli theory to hyperstructure case.

Hyperrings, Hyperfields and Hypervector Spaces

Let \(\cal H\) be a non-empty set and \(\psi: \cal H \times \cal H \to \psi(H)\) be a hyperoperation, where \(\psi(H)\) is the family of all non-empty subsets of \(H\). The couple \((H, \omega)\) is called a hypergroupoid. For any two non-empty subsets \(A\) and \(B\) of \(H\) and \(x \in H\), we define \(A \cdot B = \bigcup_{a \in A, b \in B} a \cdot b\), \(A \omega x = A \omega x\), and \(A \cdot x = A \cdot x\). A hypergroupoid \((H, \omega)\) is called a semihypergroup if for all \(a, b, c \in H\) we have \((ab, c) = a \cdot \omega c = a \cdot c\). In addition, if for every \(a \in H, a \cdot H = H = H \cdot a\), then \((H, \omega)\) is called a hypergroup. A non-empty subset \(K\) of a semihypergroup \((H, \omega)\) is called a sub-semihypergroup.
We say, and, where general types of hyperrings. In what follows we shall consider one of the most hyperring theory appears in \[8\].

A \textit{hyperring} \((R,\cdot,+)\) is an operation such that 0 is a bilaterally absorbing element; the operation \(\cdot\) is commutative; it has a scalar identity (also called scalar unit), which means that there exists \(e \in H\), for all \(x \in H\), \(x \circ e = x\); every element has a unique inverse, which means that for all \(x \in H\), there exists a unique \(x' \in H\) such that \(e \circ x \circ x' = 1\); it is reversible, which means that if \(x \in y \circ z\), then \(z \in x' \circ y \circ x\); it is commutative; \(\circ\) is a sub-semihypergroup if it is a semihypergroup. In other words, a non-empty subset \(H\) of a \textit{hyper}group \((G,\circ,+)\) is a \textit{hyper}group if \(H\) is a sub-semihypergroup of \(G\). \(H\) is a \textit{hy}perfield if \((G,\circ,+)\) is a canonical \textit{hyper}group.

In literature of Davvaz, there are several types of hyperrings and hyperfields \[8\]. In what follows we shall consider one of the most general types of hyperrings.

The triple \((R,\cdot,+)\) is a \textit{hyperring} if \((R,\cdot)\) is a canonical \textit{hyper}group; \((R,\cdot)\) is a semihypergroup such that \(\bullet 0 = 0 \cdot x = 0\) for all \(x \in R\), i.e. 0 is a bilaterally absorbing element; the hyperoperation "\(\cdot\)" is distributive over the hyperoperation "\(+\)", which means that for all \(x, y, z\) of \(R\) we have:

\[
x \cdot (y + z) = x \cdot y + x \cdot z, \quad (x + y) \cdot z = x \cdot z + y \cdot z.
\]

\textbf{Example 1.} Let \(R=\{x,y,z,t\}\) be a set with the following hyperoperations:

\[
\begin{align*}
x & \cdot y = z + t \cdot x = y y \cdot z & = x t + x, & x \cdot z = x \cdot t + y t, & x + y = y x + x y + z, & x + z = z + y t, \\
x & \cdot t = y t + x x, & y \cdot x + x + x t z & = x, & y + z = z + x t, & x + z = z + y t.
\end{align*}
\]

Then, \((R,\cdot,+)\) is a hyperring.

We call \((R,\cdot,+)\) a \textit{hyperring} if \((R,\cdot,\cdot)\) and \((R-0,\cdot)\) is a hypergroup.

\textbf{Example 2.} Let \(F=\{x,y\}\) be a set with the following hyperoperations:

\[
\begin{align*}
+ & 0 = 0 \cdot 0 = 0 = 0 \\
\cdot & 0 = 0 \cdot 0 = 0 = 0
\end{align*}
\]

Then, \((R,\cdot,+)\) is a hyperfield.

A \textit{Krasner hyperring} is a hyperring such that \((R,\cdot)\) is a canonical hypergroup with identity 0 and \(\cdot\) is an operation such that 0 is a bilaterally absorbing element. An exhaustive review updated to 2007 of hyperring theory appears in \[8\].

\textbf{Definition 2.1.} Let \((R,\cdot,+)\) be a hyperring. We define the relation \(y\) as follows:

\[
x \cdot y \not\in 3n \in N, \exists(k_1,\ldots,k_n) \in N^n \quad \text{and} \quad [x(t_1,\ldots,t_n)] \in R^k, (i = 1,\ldots,n)]
\]

such that

\[
x \cdot y \in \sum_{i=1}^n \prod x_i^i.
\]

\textbf{Theorem 2.2.} \[24, 25\] Let \((R,\cdot,+)\) be a hypering \(y\) be the transitive closure of \(y\).

- \(y\) is a strongly regular relation both on \((R,\cdot)\) and \((R,\cdot)\).
- The quotient \(R/y\) be a ring.

- The relation \(y\) is the smallest equivalence relation such that the quotient \(R/y\) be a ring.

\textbf{Theorem 2.3.} \[12\] The relation \(y\) on every hyperfield is an equivalence relation and \(y = y\).

\textbf{Remark 1.} Let \((F,+,\cdot)\) be a hyperfield. Then, \(F/y\) is a field. If \(\phi: F/y\) is the canonical map, then \(\gamma(y) = \{x \in F | \phi(x) = 0\}\), where 0 is the zero of the fundamental field \(F/y\).

Let \((R,\cdot,+)\) be a hyperring, \((M,\cdot)\) be a canonical hypergroup and there exists an external map

\[
\gamma : R \times M \to y(M), (a, x) \to ax
\]

such that for all \(a, b \in R\) and for all \(x, y \in M\) have

\[
(a + x) y = ax + ay, (a + b) x = ax + bx, (ab) x = a(bx),
\]

then \(M\) is a hypermodule over \(R\). If we consider a hyperfield \(F\) instead of a hyperring \(R\), then \(M\) is called a hypervector space

\textbf{Remark 2.} Note that it is possible in a hypervector space one or more of hyperoperations be ordinary operations.

\textbf{Example 3.} Let \(F\) be a field and \(V\) be a vector space on \(F\). If \(S\) is a subspace of \(V\), we consider the following external hyperoperation:

\[
a \circ x = ax + S, \quad \text{for all } a \in F \text{ and x} \in V, \text{ then } V \text{ is a hypervector space.}
\]

\textbf{Algebra and Hyperalgebra}

\textbf{Definition 3.1.} Let \((L,\cdot,+)\) be a hypervector space over the hyperfield \((F,+,\cdot)\). Consider the bracket (commutator) hope:

\[
[\cdot,\cdot] : L \times L \to F/(\cdot,\cdot). (x,y) \to [x,y]
\]

then \(L\) is a Lie hyperalgebra over \(F\) if the following axioms are satisfied:

\[
\begin{align*}
(1) \quad [\cdot,\cdot] & \text{ is bilinear, i.e.} \quad [x, y + z] = [x, y] + [x, z], \quad [x, y] = [y, x], \quad [x, a y + b z] = [x, a y] + [x, b z], \\
(2) \quad 0 \in [x, x], \quad \text{for all } x \in L, \\
(3) \quad 0 \in ([x, y + z] + [y, z] + [z, x]), \quad \text{for all } x, y, z \in L
\end{align*}
\]

\textbf{Definition 3.2.} Let \(A\) be a hypervector space over a hyperfield \(F\). Then, \(A\) is a hyperalgebra over \(F\) if there exists a mapping \(\cdot : A \times A \to \gamma(y)(A)\) (images to be denoted by \(x\cdot y\) for \(x, y \in A\)) such that the following conditions hold:

\[
\begin{align*}
(x + y) \cdot z & = x \cdot z + y \cdot z \quad \text{and} \quad (x + y) \cdot z = y \cdot (x + y) + x \cdot y + x \cdot z, \\
(x \cdot y) \cdot z & = x \cdot y \cdot z \quad \text{and} \quad (x \cdot y) \cdot z = (x \cdot y) \cdot z, \\
0 \cdot y & = 0 = 0 \quad \text{for all } x, y, z \in A \text{ and } c \in F.
\end{align*}
\]

In the above definition, if all hyperoperations are ordinary operations, then we have an algebra.

\textbf{Example 4.} Let \(F\) be a hyperfield and

\[
A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, a, b, c, d \in F.
\]

We define the following hyperoperations on \(A\):
For every $a, b$, we have
\[ a \cdot b = a \cdot b - b \cdot a. \]  
\[ (7) \]

A hyperalgebra $A$ is called **Lie-admissible** if the hyperalgebra $A^*$ is a Lie hyperalgebra.

If $A$ is an associative hyperalgebra, then the hyperproduct $a \cdot b$ coincides with $a + b$. Thus, the associative hyperalgebras constitute a basic class of Lie-admissible hyperalgebras.

A Jordan algebra is a (non-associative) algebra over a field whose multiplication satisfies the following axioms:

1. $xy = yx$ (commutative law);
2. $(xy)(xz) = (x(yz))$ (Jordan identity).

**Definition 3.6.** A Jordan hyperalgebra is a (non-associative) hyperalgebra over a hyperfield such that multiplication satisfies the following axioms:

\[ [(I1)] x \cdot y = y \cdot x \] (commutative law);
\[ [(I2)] (xy)(zx) = x(yz) \] (Jordan identity).

Let $A$ be an associative hyperalgebra over a hyperfield. It is possible to construct a new hyperalgebra, denoted by $A^*$, by means of the commutative hyperproduct
\[ \{a, b\} = a \cdot b + b \cdot a = \bigcup_{xy} x + y. \]  
\[ (6) \]

**Proposition 3.7.** $A^*$ is a Jordan hyperalgebra.

**Proof.** It is straightforward.

**Definition 3.8.** Corresponding to any hyperalgebra $A$ with hyperproduct $a \cdot b$, it is possible to define, as for $A^*$, a commutative hyperalgebra $A^* \cdot b$ which is the same hypervector space as $A$ but with the new hyperproduct
\[ \{a, b\}_{\cdot} = a \cdot b + b \cdot a. \]  
\[ (7) \]

In this connection, the most interesting case occurs when $A^*$ is (commutative) Jordan hyperalgebra.

A hyperalgebra $A$ is said to be **Jordan admissible** if $A^*$ is a (commutative) Jordan hyperalgebra.

If $A$ is an associative hyperalgebra, then the hyperproduct (7) reduces to (6) and $A^*$ is a special Jordan hyperalgebra. Thus, associative hyperalgebras constitute a basic class of Jordan-admissible hyperalgebras.

**Definition 3.9.** The fundamental relation $\varepsilon$ is defined in a hyperalgebra as the smallest equivalence relation such that the quotient is an algebra.

By using strongly regular relations, we can connect hyperalgebras to algebras. More exactly, starting with a hyperalgebra and using a strongly regular relation, we can construct an algebra structure on the quotient. An equivalence relation $\varepsilon$ on a hyperalgebra is called right (resp. left) strongly regular if and only if $x\varepsilon y$ implies that $(x \cdot z)\varepsilon (y \cdot z)$ for every $z \in A$ (resp. $(z \cdot x)\varepsilon (z \cdot y)$ and $(z \cdot y)\varepsilon (z \cdot x)$), and $\varepsilon$ is strongly regular if it is both left and right strongly regular.

**Theorem 3.10.** Let $A$ be a hyperalgebra over the hyperfield $F$. Denote by $U$ the set of all finite polynomials of elements of $A$ over $F$. We define the relation $\varepsilon$ on $A$ as follows:

\[ [a, b] = a \cdot b - b \cdot a. \]  
\[ (5) \]
\[ x \not\in A \quad \text{if and only if} \quad \{x, y\} \not\subseteq u, \text{where} \ u \in U. \]

Then, the \( e' \) is the transitive closure of \( e \) and is called the fundamental equivalence relation on \( A \).

**Proof.** The proof is similar to the proof of Theorem 3.1 [18].

**Remark 3.** Note that the relation \( e' \) is a strongly regular relation.

**Remark 4.** In \( A^e/e' \), the binary operations and external operation are defined in the usual manner:
\[
\begin{align*}
\bar{e}'(x) & \in e'(x) = e'(x), \text{for} \ z \in e'(x) + e'(y), \\
\bar{e}'(x) & \in e'(y) = e'(y), \text{for} \ z \in e'(y) - e'(y), \\
\bar{e}'((e'(x)(e'(y))) & \in e'(x) + e'(y), \text{for} \ z \in e'(x) - e'(y), \text{for} \ all \ y \in e'(x). \\
\end{align*}
\]

**Theorem 3.11.** Let \( A \) be an associative hyperalgebra over a hyperfield \( F \). Then, \( A/e' \) is an \( A \)-Lie-admissible algebra with the following product:
\[
\langle e'(x), e'(y) \rangle = e'(x) \circ e'(y) \circ e'(x). 
\] (8)

**Proof.** By Definition 3.9 and Theorem 3.10, \( A/e' \) is an ordinary associative algebra. So, it is enough to show that it is a Lie algebra with the hyperproduct (8). By Proposition 3.4, \( A^e \) is a \( A \)-Lie hyperalgebra with the hyperproduct \([a, b] = a - b - a \).

(1) (By (L1), for all \( x, x_1, y, y_1 \in A \), \( b, b_1 \in F \), we have \([b_1, b_1] = b \cdot b_1 = b_1 b_1 = b_1 b_1 \). Hence,
\[
(\bar{b}_1(b_1 + b_2)) - y - y - (\bar{b}_1(b_1 + b_2)) = e'(x, y) \circ e'(x, y).
\]

This implies that
\[
\langle y'(b_1(b_1 + b_2)) - y - y - (\bar{b}_1(b_1 + b_2)) \rangle = \langle e'(x, y) \circ e'(x, y) \rangle.
\]

Therefore,
\[
\langle y'(b_1(b_1 + b_2)) - y - y - (\bar{b}_1(b_1 + b_2)) \rangle = y'(b_1(b_1 + b_2)) - y - y - (\bar{b}_1(b_1 + b_2)).
\]

Similarly, for all \( x, x_1, y, y_1 \in A \), \( b, b_1, b_2 \in F \), we obtain
\[
\langle x', e'(x, y) \rangle = e'(x, y) \circ e'(x, y).
\]

By (L2), \( 0 \in x \cdot x \cdot x \cdot x \), so
\[
\bar{e}'(0) = e'(x, x, x, x) = e'(x) \circ e'(x) \circ e'(x) = (\bar{e}'(x), e'(x)) = (x, x, x, x).
\]

By (L3), we have \( 0 \in \{x, y, z\} + \{z, y, x\} + \{x, z, y\} \), for all \( x, y \in A \). Thus,
\[
\bar{e}'(0) = (e'(x), e'(y), e'(z)) + (e'(x), e'(y), e'(z)) + (e'(x), e'(y), e'(z)) = (\bar{e}'(x), e'(y), e'(z)).
\]

**Theorem 3.12.** Let \( A \) be an associative hyperalgebra over a hyperfield \( F \). Then, \( A/e' \) is a Jordan-admissible algebra with the following product:
\[
\langle e'(x), e'(y) \rangle = e'(x) \circ e'(y) \circ e'(x).
\] (9)

**Proof.** By Definition 3.9 and Theorem 1, \( A/e' \) is an ordinary associative algebra. So, it is enough to show that it is a Jordan algebra with the hyperproduct (9). By Proposition 3.7, \( A^e \) is a Jordan hyperalgebra with the hyperproduct \([a, b] = a - b - a \).

Thus, \( \langle e'(x), e'(y) \rangle = e'(x) \circ e'(y) \circ e'(x) \).

**Definition 3.14.** Corresponding to any hyperalgebra \( A \) with hyperproduct \( a \cdot b \) it is possible to define a \( A' \), which is the same

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hypervector space as $A$ with the new hyperproduct

$$[a,b]_{x'} = a \cdot T \cdot b \cdot b \cdot T \cdot a.$$  \hspace{1cm} (13)

A hyperalgebra $A$ is called Lie-Santilli-admissible if the hyperalgebra $A^*$ is a Lie hyperalgebra.

**Corollary 3.15.** If $A$ is an associative hyperalgebra, then the hyperproduct (13) coincide with (12) and $A^*$ is a Lie hyperalgebra.

**Theorem 3.16.** Let $A$ be an associative hyperalgebra over a hyperfield $F$. Then, $A^*/e'$ is a Lie-Santilli-admissible algebra with the following product:

$$(e'(x), e'(y))_{x'} = e'(x) \circ e'(T) \circ e'(y) \circ e'(T) \circ e'(x).$$  \hspace{1cm} (14)

**Proof.** By Definition 3.9 and Theorem 3.10, $A/e'$ is an ordinary associative algebra. So, it is enough to show that it is a Lie algebra with the hyperproduct (4). By corollary 3.15, $A^*$ is a Lie hyperalgebra with the hyperproduct $[a,b]_{x'} = a \cdot T \cdot b \cdot b \cdot T \cdot a$.

(1) By (L1), for all $x,y \in A$, $\lambda_1, \lambda_2 \in F$, we have $[\lambda_1 x + \lambda_2 y, z]_{x'} = \lambda_1 [x,y]_{x'} + \lambda_2 [x,y]_{x'}$, Hence,

$$[\lambda_1 x + \lambda_2 y, z]_{x'} = \lambda_1 (x \cdot T \cdot y \cdot T \cdot x) + \lambda_2 (x \cdot T \cdot y \cdot T \cdot x),$$

and so

$$e'(\lambda_1 x + \lambda_2 y)_{x'} = e'(x)_{x'} \circ e'(T) \circ e'(y)_{x'} \circ e'(T) \circ e'(x).$$

Therefore,

$$(y' \lambda_1 x + y' \lambda_2 y)_{x'} \circ y' \lambda_1 x + y' \lambda_2 y)_{x'} = y' (\lambda_1 x + \lambda_2 y)_{x'} \circ y' (\lambda_1 x + \lambda_2 y)_{x'}.$$

Similarly, for all $x,y \in A$, $\lambda_1, \lambda_2 \in F$, we obtain

$$(x' y, x')_{x'} \circ y' \lambda_1 x + y' \lambda_2 y)_{x'} = y' (x' y)_{x'} \circ y' (\lambda_1 x + \lambda_2 y)_{x'}.$$

(2) By (L2), $0 \in x \cdot T \cdot x \cdot x \cdot T \cdot x$, so

$$e'(0) = e'(x \cdot T \cdot x \cdot x \cdot T \cdot x) = e'(x' \circ e'(T) \circ e'(x') \circ e'(T) \circ e'(x')) = (e'(x'), e'(x')).$$

(3) By (L3), we have $0 \in [x,[y,z]], [y,[z,x]], [z,[x,y]]$, for all $x,y \in A$. Thus,

$$e'(0) = e'(x' (y' z), e'(x' z) y')_{x'} = (x' (y' z), e'(x' z) y')_{x'} = (y' z, x')_{x'}.$$  \hspace{1cm} (15)

where $(T_i)_{x'}$ denotes the transpose of the matrix $(T_i)_{x'}.$

**References.**