The gradation of type A_n on Lie-Santilli admissibility

Pipina Nikolaidou, Thomas Vougiouklis
Democritus University of Thrace, School of Education,
68 100 Alexandroupolis, Greece
pnikolai@eled.duth.gr, tvougiou@eled.duth.gr

Abstract

The largest class of hyperstructures,called H_v-structures, is the one which satisfy the weak properties. In this paper we deal with the Lie-admissible hyperstrucuttures and we present a construction of the hyperstructures used in the Lie-Santilli admissible theory on square matrices of type A_n using the P-hyperstructures.

Key words: hyperstructures, H_v-structures, hopes, weak hopes, ∂-hopes, e-hyperstructures, admissible Lie-algebras

MSC2010: 20N20, 17B67, 17B70, 17D25.

1 Introduction

The main object of this paper is the class of hyperstructures called H_v-structures introduced in 1990 [15], which satisfy the weak axioms where the non-empty intersection replaces the equality. Some basic definitions are the following:

Algebraic hyperstructure is called any set H equipped with at least one hyperoperation (abbreviation: *hyperoperation=hope*) $\cdot : H \times H \to P(H) - \{\varnothing\}$. We abbreviate by *WASS* the *weak associativity*: $(xy)z \cap x(yz) \neq$

$\varnothing, \forall x, y, z \in H$ and by *COW* the *weak commutativity*: $xy \cap yx \neq \varnothing, \forall x, y \in H$. The hyperstructure $(H, \cdot)$ is called H_v-*semigroup* if it is *WASS*, it is called H_v-*group* if it is reproductive H_v-semigroup, i.e., $xH = Hx = H$, $\forall x \in H$.

Motivation. In the classical theory the quotient of a group with respect to an invariant subgroup is a group. F. Marty from 1934, states that, the quotient of a group with respect to any subgroup is a hypergroup. Finally, the quotient of a group with respect to any partition (or equivalently to any equivalence relation) is an H_v-group. This is the motivation to introduce the H_v-structures [15], [16].

In an H_v-semigroup the powers of an element $h \in H$ are defined as follows: $h^1 = \{h\}, h^2 = h \cdot h, ..., h^n = h \circ h \circ ... \circ h$, where $(\circ)$ denotes the *n-ary circle hope*, i.e. take the union of hyperproducts, n times, with all possible patterns of parentheses put on them. An H_v-semigroup $(H, \cdot)$ is called *cyclic of period s*, if there exists an element h, called *generator*, and a natural number s, the minimum one, such that $H = h^1 \cup h^2 ... \cup h^s$. Analogously the cyclicity for the infinite period is defined [16]. If there is an element h and a natural number s, the minimum one, such that $H = h^s$, then $(H, \cdot)$ is called *single-power cyclic of period s.*

In an a similar way more complicated hyperstructures can be defined:

$(R, +, \cdot)$ is called H_v-*ring* if $(+)$ and $(\cdot)$ are WASS, the reproduction axiom is valid for $(+)$ and $(\cdot)$ is *weak distributive* with respect to $(+)$:

$$x(y+z) \cap (xy+xz) \neq \varnothing,\ (x+y)z \cap (xz+yz) \neq \varnothing,\ \forall x, y, z \in R.$$

Let $(R, +, \cdot)$ be an H_v-ring, $(M, +)$ be a COW H_v-group and there exists an external hope

$$\cdot : R \times M \to P(M) : (a, x) \to ax$$

such that $\forall a, b \in R$ and $\forall x, y \in M$ we have

$$a(x+y) \cap (ax+ay) \neq \varnothing,\ (a+b)x \cap (ax+bx) \neq \varnothing,\ (ab)x \cap a(bx) \neq \varnothing,$$

then M is called an H_v-*module* over F. In the case of an H_v-field F, which is defined later, instead of an H_v-ring R, then the H_v-*vector space* is defined.

For more definitions and applications on H_v-structures one can see the books [1], [2], [16].

Let $(H,\cdot),(H,*)$ be H_v-semigroups defined on the same set H. The hope $(\cdot)$ is called *smaller* than the hope $(*)$, and $(*)$ *greater* than $(\cdot)$, iff there exists an

$$f \in Aut(H,*) \text{ such that } xy \subset f(x*y),\ \forall x,y \in H.$$

Then we write $\cdot \leq *$ and we say that $(H,*)$ *contains* $(H,\cdot)$. If $(H,\cdot)$ is a structure then it is called *basic structure* and $(H,*)$ is called H_b-*structure*.

Theorem 1.1. *(The Little Theorem). Greater hopes than the ones which are WASS or COW, are also WASS or COW, respectively.*

This Theorem leads to a partial order on H_v-structures and mainly to a correspondence between hyperstructures and posets.Therefore we can obtain an extreme large number of H_v-structures just putting more elements on any result. Using the partial ordering with the fundamental relations one can give several definitions to obtain constructions used in several applications [17]:

Let $(H,\cdot)$ be hypergroupoid. We *remove* $h \in H$, if we consider the restriction of $(\cdot)$ in the set $H-\{h\}$. $\underline{h} \in H$ *absorbs* $h \in H$ if we replace h by $\underline{h}$ and h does not appear in the structure. $\underline{h} \in H$ *merges* with $h \in H$, if we take as product of any $x \in H$ by $\underline{h}$, the union of the results of x with both h, $\underline{h}$, and consider h and $\underline{h}$ as one class with representative $\underline{h}$.

The main tool to study hyperstructures is the fundamental relation. In 1970 M. Koscas defined in hypergroups the relation β and its transitive closure β^*. This relation connects the hyperstructures with the corresponding classical structures and is defined in H_v-groups as well. T. Vougiouklis [15], [16] introduced the γ^* and ϵ^* relations, which are defined, in H_v-rings and H_v-vector spaces, respectively. He also named all these relations β^*, γ^* and ϵ^*, Fundamental Relations because they play very important role to the study of hyperstructures especially in the representation theory of them. For similar relations see [16], [17].

Definition 1.1. *The fundamental relations β^*, γ^* and ϵ^*, are defined, in H_v-groups, H_v-rings and H_v-vector space, respectively, as the smallest equivalences so that the quotient would be group, ring and vector space, respectively.*

Specifying the above motivation we remark the following: Let $(G,\cdot)$ be a group and R be an equivalence relation (or a partition) in G, then $(G/R,\cdot)$ is an H_v-group, therefore we have the quotient $(G/R,\cdot)/\beta^*$ which is a group, the fundamental one. Remark that the classes of the fundamental group $(G/R,\cdot)/\beta^*$ are a union of some of the R-classes. Otherwise, the $(G/R,\cdot)/\beta^*$ has elements classes of G where they form a partition which classes are larger than the classes of the original partition R.

Definition 1.2. *[14], [16] Let $(G,\cdot)$ be a groupoid then for every $P \subset G$, $P \neq \varnothing$, we define the following hopes called P-hopes: for all $x, y \in G$*

$$\underline{P} : x\underline{P}y = (xP)y \cup x(Py),$$

$$\underline{P}_r : x\underline{P}_r y = (xy)P \cup x(yP), \quad \underline{P}_l : x\underline{P}_l y = (Px)y \cup P(xy).$$

The $(G,\underline{P})$,$(G,\underline{P}_r)$ and $(G,\underline{P}_l)$ are called P-hyperstructures. The most usual case is if $(G,\cdot)$ is semigroup, then $x\underline{P}y = (xP)y \cup x(Py) = xPy$ and $(G,\underline{P})$ is a semihypergroup but we do not know about $(G,\underline{P}_r)$ and $(G,\underline{P}_l)$. In some cases, depending on the choice of P, the $(G,\underline{P}_r)$ and $(G,\underline{P}_l)$ can be associative or WASS.

A generalization of P-hopes is the following [4]:

Construction 1.1. *Let $(G,\cdot)$ be an abelian group and P any subset of G with more than one elements. We define the hope $\times_P$ as follows:*

$$x \times_P y = \begin{cases} x \cdot P \cdot y = \{x \cdot h \cdot y | h \in P\} & \text{if } x \neq e \text{ and } c \neq e \\ x \cdot y & \text{if } x = e \text{ and } y = e \end{cases}$$

we call this hope P_e-hope. The hyperstructure $(G,\times_p)$ is an abelian H_v-group. For the proof one can see in [4].

2 H_v-matrices and representations

H_v-structures are used in Representation Theory of H_v-groups which can be achieved either by generalized permutations or by H_v-matrices [13], [16], [17]. Representations by generalized permutations can be faced by

translations. H_v-matrix is called a matrix if has entries from an H_v-ring. The hyperproduct of H_v-matrices is defined in a usual manner. The problem of the H_v-matrix representations is the following:

Definition 2.1. *Let $(H, \cdot)$ be an H_v-group, find an H_v-ring R, a set*

$$M_R = \{(a_{ij}) | a_{ij} \in R\}$$

and a map

$T : H \to M_R : h \mapsto T(h)$ *such that* $T(h_1h_2) \cap T(h_1)T(h_2) \neq \varnothing, \forall h_1, h_2 \in H$.

Then the map T is called $\mathbf{H_v}$***-matrix representation***.
If the $T(h_1h_2) \subset T(h_1)(h_2)$,$\forall h_1, h_2 \in H$ is valid, then T is called ***inclusion representation***.
If $T(h_1h_2) = T(h_1)(h_2) = \{T(h) | h \in h_1h_2\}$, $\forall h_1, h_2 \in H$, then T is called ***good representation*** *and then an induced representation T^* for the hypergroup algebra is obtained.*
If T is one to one and good then it is a ***faithful representation.***

In representations of H_v-groups there are two difficulties: To find an H_v-ring or an H_v-field and an appropriate set of H_v-matrices. However more interesting are the small H_v-fields i.e. the results have one or few elements. The single elements, if any exist, are playing a crucial role.

Hopes on any type of ordinary matrices can be defined [5], they are called *helix hopes*. Using several classes of H_v-structures one can face several representations. Some of those classes are as follows [11]:

Definition 2.2. *Let $\mathbf{M} = \mathbf{M}_{m\times n}$ be a module of $m \times n$ matrices over a ring $\mathbf{R}$ and $\mathbf{P} = \{P_i : i \in I\} \subseteq \mathbf{M}$. We define, a kind of, a P-hope $\underline{P}$ on $\mathbf{M}$ as follows*

$$\underline{P} : \mathbf{M} \times \mathbf{M} \to \mathbf{P}(\mathbf{M}) : (A, B) \to A\underline{P}B = \{AP_i^tB : i \in I\} \subseteq \mathbf{M}$$

where P^t denotes the transpose of the matrix P.

The hope $\underline{\mathrm{P}}$, which is a bilinear map, is a generalization of Rees operation where, instead of one sandwich matrix, a set of sandwich matrices is used.

The hope $\underline{\mathrm{P}}$ is strong associative and the inclusion distributivity with respect to addition of matrices

$$A\underline{P}(B+C) \subseteq A\underline{P}B + A\underline{P}C \text{ for all } A, B, C \text{ in } \mathbf{M}$$

is valid. Therefore, $(\mathbf{M}, +, \underline{P})$ defines a multiplicative hyperring on non-square matrices. Multiplicative hyperring means that only the multiplication is a hope.

Definition 2.3. *Let $\mathbf{M} = \mathbf{M}_{m\times n}$ be a module of $m \times n$ matrices over R and let us take sets*

$$\mathbf{S} = \{s_k : k \in K\} \subseteq R, \quad \mathbf{Q} = \{Q_j : j \in J\} \subseteq \mathbf{M}, \quad \mathbf{P} = \{P_i : i \in I\} \subseteq \mathbf{M}.$$

Define three hopes as follows

$$\underline{S} : R \times \mathbf{M} \to \mathbf{P}(\mathbf{M}) : (r, A) \to r\underline{S}A = \{(rs_k)A : k \in K\} \subseteq \mathbf{M}$$

$$\underline{Q}_+ : \mathbf{M} \times \mathbf{M} \to \mathbf{P}(\mathbf{M}) : (A, B) \to A\underline{Q}_+B = \{A + Q_j + B : j \in J\} \subseteq \mathbf{M}$$

$$\underline{P} : \mathbf{M} \times \mathbf{M} \to \mathbf{P}(\mathbf{M}) : (A, B) \to A\underline{P}B = \{AP_i^tB : i \in I\} \subseteq \mathbf{M}$$

Then $(\mathbf{M}, \underline{S}, \underline{Q}_+, \underline{P})$ is a hyperalgebra over $\boldsymbol{R}$ called general matrix P-hyperalgebra.

In a similar way a generalization of this hyperalgebra can be defined if one considers an H_v-ring or an H_v-field instead of a ring and using H_v-matrices.

3 Lie-hyperalgebras

Theorem 3.1. *Let $(\boldsymbol{M}, +)$ be an H_v-module over the H_v-ring $\boldsymbol{R}$. Denote by $\boldsymbol{U}$ the set of all expressions consisting of finite hopes either on $\boldsymbol{R}$ and $\boldsymbol{M}$ or the external hope applied on finite sets of elements $\boldsymbol{R}$ and $\boldsymbol{M}$. We define the relation ϵ in $\boldsymbol{M}$ as follows:*

$$x\epsilon y \text{ iff } \{x, y\} \subset u \text{ where } u \in \boldsymbol{U}$$

Then the relation ϵ^* is the transitive closuer of the relation ϵ

The general definition of an H_v-Lie algebra was given in [17] as follows:

Definition 3.1. *[18] Let $(L,+)$ be an H_v-vector space over the H_v-field $(F,+,\cdot)$, $\phi : F \to F/\gamma^*$ the canonical map and $\omega_F = \{x \in F : \phi(x) = 0\}$, where 0 is the zero of the fundamental field F/γ^*. Similarly, let ω_L be the core of the canonical map $\phi' : L \to L/\epsilon^*$ and denote by the same symbol 0 the zero of L/ϵ^*. Consider the bracket (commutator) hope:*

$$[,] : L \times L \to P(L) : (x,y) \to [x,y]$$

*then **L** is an $\mathbf{H_v}$**-Lie algebra** over F if the following axioms are satisfied:*

(L1) The bracket hope is bilinear, i.e.
$[\lambda_1 x_1 + \lambda_2 x_2, y] \cap (\lambda_1[x_1,y] + \lambda_2[x_2,y]) \neq \varnothing$
$[x, \lambda_1 y_1 + \lambda_2 y_2] \cap (\lambda_1[x,y_1] + \lambda_2[x,y_2]) \neq \varnothing,$
$\forall x, x_1, x_2, y, y_1, y_2 \in L, \lambda_1, \lambda_2 \in F$

(L2) $[x,x] \cap \omega_L \neq \varnothing, \quad \forall x \in L$

(L3) $([x,[y,z]] + [y,[z,x]] + [z,[x,y]]) \cap \omega_L \neq \varnothing, \quad \forall x, y \in L$

This is a general definition thus one can use special cases in order to face problems in applied sciences.

4 Mathematical Realisation of type A_n

This paper presents the mathematical Realisation of Kac-Moody Lie Algebras of the type A_n [6], [12]. We present the problem and we give the basic definitions on the topic which cover the four following cases:

Algebraic structures can be easily handled using representation theory by matrices. This is the reason for which Lie-Santilli's admissibility is being studied using matrices or hypermatrices, in the case of multivalued/ hypercase. Lie-Santilli's admissibility is being extended into hyperstructure case, using the well know class of P-hyperstructures. We present the problem through the following cases:

Construction 4.1. *[11] Let R and S two sets of square matrices (or hypermatrices). The hyper-Lie bracket can be defined in one of the four following ways:*

1. $[x, y]_{RS} = xRy - ySx$ *(General Case)*
2. $[x, y]_R = xRy - yx$
3. $[x, y]_S = xy - ySx$
4. $[x, y]_{RR} = xRy - yRx$

The question is when the two following conditions of a hyper-Lie algebra, the Antisymmetry and the Jacobbi identity are satisfied , for all square matrices (or hypermatrices) x, y, z,

$$[x, x]_{RS} \ni 0$$

$$[x, [y, z]_{RS}]_{RS} + [y, [z, x]_{RS}]_{RS} + [z, [x, y]_{RS}]_{RS} \ni 0$$

We apply this generalization on the Lie algebras of the type A_n.

Lie-Algebra of type A_n, a graded algebra refering to traceless matrices or hypermatrices (Tr(M)=0), uses the principal realisation used in Infinite Dimensional Kac Moody Lie Algebras introduced in 1981 by Lepowsky, Wilson and Kac [6].

Let $E_{ij}(i, j = 1, ..., n)$ the $n \times n$ matrix which is 1 in the ij-entry and 0 all the other entries, and by $e_i = E_{ii} - E_{i+1,i+1}, i = 1, ..., n-1$. This graded algebra has n levels, $0, 1, ..., n-1$. Each level has dimension n excpet Level 0, whose dimension is $n-1$, because of the limitation of zero trace.

Simple Basis elements:

Level 0 : e_i, $i = 1, 2, ..., n-1$

Level 1 : $E_{i,i+1}$, $i = 1, 2, ..., n$

Level 2 : $E_{i,i+2}$, $i = 1, 2, ..., n$

...

Level n-1:$E_{i,i+(n-1)}$, $i = 1, 2, ..., n$

Denote that all the subscripts are $modn$. Therefore, the levels are:

Level 0 :

$$\begin{pmatrix} \boldsymbol{a_{11}} & 0 & 0 & \dots & 0 \\ 0 & \boldsymbol{a_{22}} & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \boldsymbol{a_{(n-1)(n-1)}} & 0 \\ 0 & 0 & 0 & \dots & \boldsymbol{b} \end{pmatrix}$$

where $\boldsymbol{b} = -a_{11} - a_{22} - ... - a_{(n-1)(n-1)}$

Level 1 :

$$\begin{pmatrix} 0 & \boldsymbol{a_{12}} & 0 & \dots & 0 \\ 0 & 0 & \boldsymbol{a_{23}} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \boldsymbol{a_{n-1,n}} \\ \boldsymbol{a_{n1}} & 0 & 0 & \dots & 0 \end{pmatrix}$$

Level 2 :

$$\begin{pmatrix} 0 & 0 & \mathbf{a_{13}} & \dots & 0 \\ 0 & 0 & 0 & \dots & \mathbf{a_{2n}} \\ \dots & \dots & \dots & \dots & \dots \\ \mathbf{a_{n-1,1}} & 0 & 0 & \dots & 0 \\ 0 & \mathbf{a_{n2}} & 0 & \dots & 0 \end{pmatrix}$$

. . . . . .

Level n-1 :

$$\begin{pmatrix} 0 & 0 & 0 & \dots & \mathbf{a_{1n}} \\ \mathbf{a_{21}} & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & \mathbf{a_{n,n-1}} & 0 \end{pmatrix}$$

Let E be the Konstant's Cyclic Element, as the sum of the elements of First Level's Simple Basis , $E = E_{12} + E_{23} + E_{34} + ... + E_{n-1,n} + E_{n1}$. The use of this element is to shift every element of each Level L to the next Level $L+1$, [6], [9], [12] creating an one-to-one correspondance. More precisely, an element of a Level is being represented to a different element

of the next Level. However, as already said, the First Level's Basis and all other Levels' Basis has n elements, except zero which has $n-1$ elements. Consequently, this one to one correspondance between the Levels is being corrupted on Level $n-1$, which is being represented to Level 0, which has less elements than the others.Therefore, in order to create a complete correspondance, we remove from every level, but zero, all the powers of E until $n-1$ $(E, E^2, ..., E^{n-1})$.

In this realisation we prove that the n power of the Lie bracket of E_{n1}, n times by E, where E_{n1} is an element of the first level, equals to a diagonal matrix, the elements of which are the coefficients of a binomial. Denote by :

$[E, E_{n1}]^1 = E \cdot E_{n1} - E_{n1} \cdot E = A_1$, the first power,
$[E, E_{n1}]^2 = [E, A_1] = A_2$, the second power,
. . .
$[E, E_{n1}]^n = [E, A_{n-1}] = A_n$, the n power,
So, the following can be proved [9]:

Theorem 4.1.

$$[E, E_{n1}]^n =$$
$$= diag(\tbinom{n-1}{0}, (-1)^1\tbinom{n-1}{1}, (-1)^2\tbinom{n-1}{2}, ..., (-1)^{n-2}\tbinom{n-1}{n-2}, (-1)^{n-1}\tbinom{n-1}{n-1})$$

This theorem can be used, to find the basic element of first Level's Basis and all the nth powers of the elements of the first Level.

Hint of the proof.
One can easily see that following the powers of E, they follow the coefficients of Laplace Binomial. ■

Theorem 4.2. *Let*

$$[E, E_{n1}]^n = diag(d_1, d_2, d_3, ..., d_n)$$

be the nth power by E of the last element of the base $E_{12}, E_{23}, ..., E_{n1}$ of the first level.Then the nth power of the rest elements of the base $E_{12}, E_{23}, ..., E_{n-1,n}$ are the following:
The first element is : $[E, E_{12}]^n = diag(d_n, d_1, d_2, d_3, ..., d_{n-1})$
The second element is: $[E, E_{23}]^n = diag(d_{n-1}, d_n, d_1, d_2, ..., d_{n-2})$

.....

so the k element $E_{k,k+1}$ is:

$$[E, E_{k,k+1}]^n = diag(d_{1-k}, d_{2-k}, d_{3-k}, ..., d_{n-k}), k = 1, ..., n$$

Remark. [9] Based on this theory and P-hyperstructures a set P with two elements can be used, either from zero or first level, but only with two elements. In this case the shift is depending on the level, so if we take P from Level-0, the result will not change, although the result will be multivalued. In case of different level insted, the shift will be analogous to the level of P.

In the general case in Construction 4.1(1), one can notice the possible cardinality of the result, checking the Jacoby identity is very big. Even in the small case when $|R| = |S| = |P| = 2$ in the anticommutativity $xPx - xPx$ could have cardinality 4 and the left side of the Jacoby identity is

$$(xP(yPz - zPy) - (yPz - zPy)Px) + (yP(zPx - xPz) -$$
$$-(zPx - xPz)Py) + (zP(xPy - yPx) - (xPy - yPx)Pz)$$

could have cardinality 2^{18}. The number is reduced in special cases.

Theorem 4.3. *[3] In the case of the Lie-algebra of type A_n, of traceless matrices M, we can define a hyper-Lie-Santilli-admissible bracket hope as follows:*

$$[xy]p = xPy - yPx$$

where $P = \{p, q\}$, with p,q elements of the zero level. Then we obtain a hyper-Lie-Santilli-algebra.

We need only to proof the anticommutativity and the Jacobi identity as in the hyperstructure case [9]. Therefore we have

(a) $[xx]p = xPx - xPx = \{0, xpx - xqx, xqx - xpx\} \ni 0$, so the "weak" anticommutativity is valid, and

(b) $[x, [y, z]p]p + [y, [z, x]p]p + [z, [x, y]p]p =$
$(xP(yPz - zPy) - (yPz - zPy)Px) + (yP(zPx - xPz) -$
$-(zPx - xPz)Py) + (zP(xPy - yPx) - (xPy - yPx)Pz).$

But this set contains the element

$$xpypz - xpzpy - ypzpx + zpypx + ypzpx - ypxpz-$$

$$-zpxpy + xpzpy + zpxpy - zpypx - xpypz + ypxpz = 0$$

So the "weak" Jacobi identity is valid.

Thus, zero belongs to the above results, as it has to be, but there are more elements because it is a multivalued operation.

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