Elements of Santilli's Lie - isotopic time evolution theory

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Abstract

In this article we define Santilli's Lie isotopic powe series and investigate it for absolutely and uniformly convergence and differentiability.

Index Terms: evolution, isotopic, Lie.

1 Introduction

Numerous aspects help the bradening of the scattering theory to incorporate non - Hamiltonian effects, effects which can not be represented using the conventional Hamiltonian. Following decades of research this incorporated required the construction by various authors of a new mathematics, well known as isomathematics and proposed by Santilli [2] in 1978, subsequently studied by the same author and numerious pure and applied mathematicians as: S. Okubo, H. Myung, M. Tomber, Gr. Tsagas, D. Sourlas, J. Kadeisvili, A. Aringazin, A. Kirhukin, R. Ohemke, G. Wene, G. M. Benkart, J. Osborn, D. Britten, J. Lohmus, E. Paal, L. Sorgsepp, D. Lin, J. Voujouklis, P. Broadbridge, P. Chernooff, S. Guiasu, E. Prugovecki, A. Sagle, C. Jiang, R. Falcon Ganfornina, J. Nunez Valdes, A. Davvaz, and others.

As a result of these efforts, the new mathematics can be constructed via the systematic application of axiom - preservibg liftings, called isotopies, of the totality of the mathematics of quantum mechanics, inluding all its operators and all its operations, icluding the isotopic lifting of numbers, functional analysis, differential calculas, geometries, topologies, Lie theory and others [3], [4], [5]. In [6] is shown that the isotopies can be very easly constructed using the application of nonunitary transforms to the totality of the formalism of the conventional scattering theory.

The physical needs for isomathematics have been indicated in [6], and consists in the necessity for a representation of non-Hamiltonian scattering effects in a form that is invariant over time so as to admit the same numerical predictions under the same conditions at different times. Following the study of all possible alternatives, the latter condition required the representation of non-Hamiltonian scattering effects with an axiom − preserving generalization of the trivial (positive-de

nite) unit of quantum mechanics $\hat{h} = 1$ into the most general possible (positive-deffinite as a condition to characterize an isotopy), integro-differential operator $\hat{I}$ which is as positive - definite as $+1$, functional depending of local variables, that is assumed to be the inverse of the isotopic element $\hat{T}$

$$+1\hat{>}0 \longrightarrow \hat{I}(t, r, p, a, E, \dots) = \frac{1}{\hat{T}}\hat{>}0$$

and it is called Santilli isounit. Santilli introduced a generalization called lifting of the conventional associative product ab into the form

$$ab \longrightarrow a\hat{\times}b = a\hat{T}b$$

called isoproduct for which:

$$\hat{I}\hat{\times}a = \frac{1}{\hat{T}}\hat{T}a = a\hat{\times}\hat{I} = a\hat{T}\frac{1}{\hat{T}} = a$$

for every element a of the field of real numbers, complex numbers and quaternions.

The Santilli isonumbers are defined as follows: for given real number or complex number or quaternion a,

$$\hat{a} = a\hat{I},$$

with isoproduct

$$\hat{a}\hat{\times}\hat{b} = \hat{a}\hat{T}\hat{b} = a\frac{1}{\hat{T}}\hat{T}b\frac{1}{\hat{T}} = ab\frac{1}{\hat{T}} = \hat{ab}.$$

If $a \neq 0$ the corresponding isoelement of $\frac{1}{a}$ will be denoted with $\hat{a}^{-1}$ or $\hat{I} \Lsh \hat{a}$.

On more technical grounds, the nonconservative character of the events implies the inapplicability of Lie's theory with the familiar time evolution of a (Hermitean) operator A

$$i\frac{dA}{dt} = [A, H] = AH - HA$$

in favor of Santilli's Lie-isotopic time evolution, which is presented firstly in [7] of 1978)

$$i\frac{dA}{dt} = [A, H] = ATH - HTA,$$

where T is a second operator(generally independent from, and non - commutating with H), characterizing the nonunitarity of the theory. Let X and Y be complex Banach spaces. With $\mathcal{L}(X, y)$ we will denote the space of linear bounded operators $C : X \longrightarrow Y$.

Let A, T and $H \in \mathcal{L}(X, y)$ and

$$\frac{dA}{dt} = -i(ATH - HTA).$$

Our aim here is to be investigated the serties

$$A(0) + \frac{dA}{dt}(0)w + \frac{1}{2!}\frac{d^2A}{dt^2}(0)w^2 + \cdots . \quad (1.1)$$

The series (1.1) will be called Santilli's Lie isotopic power series.

2 Convergence of Santilli's Lie isotopic power series

Firstly we will deduct the general term of (991.1).

We have

$$\frac{d^2A}{dt^2} = \frac{d}{dt}\left(\frac{dA}{dt}\right)$$

$$= -i\left(\frac{dA}{dt}TH - HT\frac{dA}{dt}\right)$$

$$= -i\Big(-i(ATH - HTA)TH$$

$$-HT(-i)(ATH - HTA)\Big)$$

$$= (-i)^2((ATH - HTA)TH$$

$$-HT(ATH - HTA)),$$

$$\frac{d^3A}{dt^3} = \frac{d}{dt}\left(\frac{d^2A}{dt^2}\right)$$

$$= -i\left(\frac{d^2A}{dt^2}TH - HT\frac{d^2A}{dt^2}\right)$$

$$= -i((-i)^2((ATH - HTA)TH$$

$$-HT(ATH - HTA))TH$$

$$-HT(-i)^2((ATH - HTA)TH$$

$$-HT(ATH - HTA)))$$

$$= (-i)^3((ATH - HTA)(TH)^2$$

$$-HT(ATH - HTA)TH$$

$$-HT(ATH - HTA)TH$$

$$+(HT)^2(ATH - HTA))$$

$$= (-i)^3((ATH - HTA)(TH)^2$$

$$-2HT(ATH - HTA)TH$$

$$+(HT)^2(ATH - HTA)),$$

$\frac{d^4A}{dt^4} = \frac{d}{dt}\left(\frac{d^3A}{dt^3}\right)$

$= -i\left(\frac{d^3A}{dt^3}TH - HT\frac{d^3A}{dt^3}\right)$

$= -i((-i)^3((ATH - HTA)(TH)^2$

$-2(HT)(ATH - HTA)TH$

$+(HT)^2(ATH - HTA))TH$

$-(-i)^3(HT)((ATH - HTA)(TH)^2$

$-2(HT)(ATH - HTA)TH$

$+(HT)^2(ATH - HTA)))$

$= (-i)^4((ATH - HTA)(TH)^3$

$-2HT(ATH - HTA)(TH)^2$

$+(HT)^2(ATH - HTA)(TH)$

$-(HT)(ATH - HTA)(TH)^2$

$+2(HT)^2(ATH - HTA)TH$

$-(HT)^3(ATH - HTA))$

$= (-i)^4((ATH - HTA)(TH)^3$

$-3(HT)(ATH - HTA)(TH)^2$

$+3(HT)^2(ATH - HTA)(TH)$

$-(HT)^3(ATH - HTA))$

$= (-i)^4\sum_{k=0}^{3}\begin{pmatrix} 3 \\ 3-k \end{pmatrix}(HT)^k(ATH$

$-HTA)(TH)^{3-k}$.

We suppose that for some natural number n we have

$\frac{d^nA}{dt^n}$

$= (-i)^n\sum_{k=0}^{n-1}\begin{pmatrix} n-1 \\ n-1-k \end{pmatrix}(-1)^k(HT)^k(ATH$

$-HTA)(TH)^{n-1-k}$.

We will prove that

$\frac{d^{n+1}A}{dt^{n+1}}$

$= (-i)^{n+1}\sum_{k=0}^{n}\begin{pmatrix} n \\ n-k \end{pmatrix}(-1)^k(HT)^k(ATH$

$-HTA)(TH)^{n-k}$.

Really,

$\frac{d^{n+1}A}{dt^{n+1}} = \frac{d}{dt}\left(\frac{d^nA}{dt^n}\right)$

$= -i\left(\frac{d^nA}{dt^n}TH - HT\frac{d^nA}{dt^n}\right)$

$= -i\Big((-i)^n\sum_{k=0}^{n-1}(-1)^k\begin{pmatrix} n-1 \\ n-1-k \end{pmatrix}(HT)^k(ATH$

$-HTA)(TH)^{n-1-k}$

$-(-i)^nHT\sum_{k=0}^{n-1}(-1)^k\begin{pmatrix} n-1 \\ n-1-k \end{pmatrix}(HT)^k(ATH$

$-HTA)(TH)^{n-1-k}\Big)$

$= (-i)^{n+1}\Big(((ATH - HTA)(TH)^{n-1}$

$-(HT)(ATH - HTA)(TH)^{n-2}$

$+\cdots$

$+(-1)^{n-1}(HT)^{n-1}(ATH - HTA))TH$

$-HT((ATH - HTA)(TH)^{n-1}$

$-(HT)(ATH - HTA)(TH)^{n-2}$

$+\cdots+(-1)^{n-1}(HT)^{n-1}(ATH - HTA))\Big)$

$$= (-i)^{n+1}\Big((ATH - HTA)(TH)^n$$

$$-\left(\begin{array}{c} n \\ n-1 \end{array}\right)(HT)(ATH - HTA)(TH)^{n-1}$$

$$+\cdots$$

$$+(-1)^n(HT)^n(ATH - HTA)\Big)$$

$$= (-i)^n \sum_{k=0}^{n}(-1)^k \left(\begin{array}{c} n \\ n-k \end{array}\right)(HT)^k(ATH$$

$$-HTA)(TH)^{n-k}.$$

From here and induction principle follows that for every natural n we have

$$\frac{d^n A}{dt^n}$$

$$= (-i)^n \sum_{k=0}^{n-1}(-1)^k \left(\begin{array}{c} n-1 \\ n-1-k \end{array}\right)(HT)^k(ATH$$

$$-HTA)(TH)^{n-1-k}.$$

Let

$$A^n$$

$$= \frac{(-i)^n}{n!} \sum_{k=0}^{n-1}(-1)^k \left(\begin{array}{c} n-1 \\ n-1-k \end{array}\right)(HT)^k(ATH$$

$$-HTA)(TH)^{n-1-k}(0),$$

$$A^0 = A(0)..$$

We note that when we write $C(0)$ we have in mind that the operator C acts on the zero element of X.

Now we will investigate the series

$$g(w) = \sum_{n=0}^{\infty} A^n w^n, \tag{2.1}$$

where w is a complex variable. If $|w| > 1$ then we will make the change $w_1 = w - 1$ and therefore $|w_1| = |w - 1| \geq |w| - 1 > 0$.

Let Ω be the set of all w for which the series (2.1) is convergent. The set Ω is not empty because $0 \in \Omega$.

For $r > 0$ and $x_0 \in \Omega$ we will denote with $S_r(x_0)$ the ball

$$S_r(x_0) = \{x \in \mathbb{C} : |x - x_0| < r\}.$$

Theorem 2.1. *Let $w_0 \neq 0$ and $w_0 \in \Omega$. Then $S_{|w_0|}(0) \subset \Omega$ and in every ball $S_r(0)$, $0 < r < |w_0|$, the series (2.1) is absolutely and uniformly convergent.*

Proof. Since $w_0 \in \Omega$ then the series $\sum_{n=0}^{\infty} A^n w_0^n$ is convergent. From the properties of convergent series we have that $\lim_{n \longrightarrow \infty} A^n w_0^n = 0$. From here we conclude that the sequence $\{A^n w_0^n\}_{n=1}^{\infty}$ is bounded. Therefore there exists a constant $M > 0$ such that

$$||A^n w_0^n|| \leq M \quad \text{for} \quad \forall n \in \mathbb{N}.$$

Let $w \in S_{|w_0|}(0)$. Then $|w| < |w_0|$ and

$$||A^n w^n|| = \Big|\Big|A^n w_0^n \tfrac{w^n}{w_0^n}\Big|\Big|$$

$$= \Big|\tfrac{w}{w_0}\Big|^n ||A^n w_0^n|| \leq M \Big|\tfrac{w}{w_0}\Big|$$

and from here

$$\sum_{n=0}^{\infty} ||A^n w^n|| \leq M \sum_{n=0}^{\infty} \Big|\frac{w}{w_0}\Big|^n < \infty.$$

If $|w| < r$

$$||A^n w^n|| = \Big|\Big|A^n r^n \tfrac{w^n}{r^n}\Big|\Big|$$

$$= r^n ||A^n|| \Big|\tfrac{w}{r}\Big|^n$$

$$< |w_0|^n ||A^n|| \Big|\tfrac{w}{r}\Big|^n$$

$$= ||A^n w_0^n|| \Big|\tfrac{w}{r}\Big|^n$$

$$\leq M \Big|\tfrac{w}{r}\Big|^n.$$

Consequently

$$\sum_{n=0}^{\infty} ||A^n w^n|| \leq M \sum_{n=0}^{\infty} \Big|\frac{w}{r}\Big|^n < \infty,$$

i.e. the series (2.1) is absolutely and uniformly convergent. □

With R we will denotre the radius of convergence of (2.1).

From the definition of radius of convergence of power series we have

$$R = \sup_{w\in\Omega} |w|.$$

Also,

1. If $R = 0$ then $\Omega = \{0\}$.

2. If $R = \infty$ then the series (2.1) is convergent in all complex plane.

3. From Cauchy - Hadamard formula we have

$$R = \frac{1}{\overline{\lim}_{n\longrightarrow\infty}||A^n||^{\frac{1}{n}}}.$$

Theorem 2.2. *Let $A, T, H \in \mathcal{L}(X, y)$, $||T|| > 0$, $||H|| > 0$. Then*

$$R > \frac{1}{2||T||||H||}.$$

Proof. We have

$$||A^n||$$

$$= \left|\left|(-i)^n \sum_{k=0}^{n-1} \begin{pmatrix} n-1 \\ n-1-k \end{pmatrix} (-1)^k (HT)^k\right.\right.$$

$$\left.\left.\times(ATH - HTA)(TH)^{n-k-1}\right|\right|$$

$$\leq \sum_{k=0}^{\infty} \begin{pmatrix} n-1 \\ n-1-k \end{pmatrix} ||(HT)^k(ATH$$

$$-HTA)(TH)^{n-1-k}||$$

$$\leq \sum_{k=0}^{\infty} \begin{pmatrix} n-1 \\ n-1-k \end{pmatrix} ||(HT)^k(ATH$$

$$-HTA)||||(TH)^{n-1-k}||$$

$$\leq \sum_{k=0}^{\infty} \begin{pmatrix} n-1 \\ n-1-k \end{pmatrix} ||(HT)^k(ATH$$

$$-HTA)||||TH||^{n-1-k}$$

$$\leq \sum_{k=0}^{\infty} \begin{pmatrix} n-1 \\ n-1-k \end{pmatrix} ||(HT)^k||||ATH$$

$$-HTA||||TH||^{n-1-k}$$

$$\sum_{k=0}^{\infty} \begin{pmatrix} n-1 \\ n-1-k \end{pmatrix} ||HT||^k(||ATH||$$

$$+||THA||)||T||^{n-1-k}||H||^{n-1-k}$$

$$\sum_{k=0}^{\infty} \begin{pmatrix} n-1 \\ n-1-k \end{pmatrix} ||H||^k||T||^k(||A||||T||||H||$$

$$+||T||||H||||A||)||H||^{n-1-k}||T||^{n-1-k}$$

$$= 2||A||||T||^n||H||^n \sum_{k=0}^{\infty} \begin{pmatrix} n-1 \\ n-1-k \end{pmatrix}$$

$$= 2^n||A||||T||^n||H||^n,$$

i.e.

$$||A^n|| \leq 2^n||A||||T||^n||H||^n,$$

from here

$$||A^n||^{\frac{1}{n}} \leq 2||T||||H||||A||^{\frac{1}{n}},$$

therefore

$$\overline{\lim}_{n\longrightarrow\infty}||A^n||^{\frac{1}{n}}$$

$$\leq 2||T||||H||\overline{\lim}_{n\longrightarrow\infty}||A||^{\frac{1}{n}} = 2||T||||H||.$$

From here and Cauchy - Hadamard formulae we conclude that

$$R = \frac{1}{\overline{\lim}_{n\longrightarrow\infty}||A^n||^{\frac{1}{n}}} \geq \frac{1}{2||T||||H||}.$$

□

Theorem 2.3. *If there exist positive constants M_1 and l such that*

$$||A^n|| \leq M_1 l^n$$

then $R \geq \frac{1}{l}$.

Proof. It is enough to be proved that the series (2.1) is uniformly bounded for $|w| < \frac{1}{l}$.

Let

$$|w|l = q < 1.$$

Then

$$||A^n w^n|| = |w|^n ||A^n|| \leq |w|^n M_1 l^n = M_1 q^n,$$

therefore

$$\left|\left|\sum_{k=0}^{\infty} A^k w^k\right|\right| \leq \sum_{k=0}^{\infty} ||A^k w^k||$$

$$\leq M_1 \sum_{k=0}^{\infty} q^k = \frac{M_1}{1-q} < \infty.$$

□

Theorem 2.4. *Let*

$$\sum_{n=0}^{\infty} A^n w^n = \sum_{n=0}^{\infty} \tilde{A}^n w^n \quad \text{in} \quad S_R(0). \qquad (2.2)$$

Then

$$A^n = \tilde{A}^n \quad \text{for} \quad \forall n \in \mathbb{N} \cup \{0\}.$$

Proof. Since $w = 0 \in \Omega$ then, after we put $w = 0$ in (2.2) we get

$$A^0 = \tilde{A}^0.$$

From here and (2.2) we obtain

$$\sum_{k=1}^{\infty} A^k w^k = \sum_{k=1}^{\infty} \tilde{A}^k w^k \quad \text{in} \quad S_R(0),$$

from where

$$\sum_{k=1}^{\infty} A^k w^{k-1} = \sum_{k=1}^{\infty} \tilde{A}^k w^{k-1} \quad \text{in} \quad S_R(0). \qquad (2.3)$$

We put $w = 0$ in the last equality and we have

$$A^1 = \tilde{A}^1.$$

From here and (2.3)

$$\sum_{k=2}^{\infty} A^k w^{k-1} = \sum_{k=2}^{\infty} \tilde{A}^k w^{k-1} \quad \text{in} \quad S_R(0)$$

and etc. □

Theorem 2.5. *The function g is a continuous function in $S_R(0)$.*

Proof. Let $\rho \in (0, R)$ and $w, w_0 \in S_R(0)$. Then

$$g(w) - g(w_0) = \sum_{n=1}^{\infty} A^n w^n - \sum_{n=1}^{\infty} A^n w_0^n$$

$$= \sum_{n=1}^{\infty} A^n (w^n - w_0^n)$$

$$= \sum_{n=1}^{\infty} A^n (w - w_0)(w^{n-1} + w^{n-2} w_0$$

$$+ \cdots + w_0^{n-1}),$$

therefore

$$||g(w) - g(w_0)|| = \left|\left|\sum_{n=1}^{\infty} A^n (w - w_0)(w^{n-1} + w^{n-2} w_0\right.\right.$$

$$\left.\left.+ \cdots + w_0^{n-1})\right|\right|$$

$$\leq \sum_{n=1}^{\infty} ||A^n (w - w_0)(w^{n-1} + w^{n-2} w_0 + \cdots + w_0^{n-1})||$$

$$= \sum_{n=1}^{\infty} ||A^n|| |w - w_0| |w^{n-1} + w^{n-2} w_0 + \cdots + w_0^{n-1}|$$

$$\leq \sum_{n=1}^{\infty} ||A^n|| |w - w_0| (|w|^{n-1} + |w|^{n-2} |w_0|$$

$$+ \cdots + |w_0|^{n-1})$$

$$\leq \sum_{n=1}^{\infty} ||A^n|| |w - w_0| (\rho^{n-1} + \rho^{n-2} \rho$$

$$+ \cdots + \rho^{n-1})$$

$$= \sum_{n=1}^{\infty} n \rho^{n-1} ||A^n||,$$

i.e.

$$||g(w) - g(w_0)|| \leq \sum_{n=1}^{\infty} n \rho^{n-1} ||A^n|| |w - w_0|. \quad (2.4)$$

Now we will prove that the series $\sum_{n=1}^{\infty} n \rho^{n-1} A^n$ is uniformly convergent for every $\rho \in (0, R)$. Let $\rho \in (0, R)$ is arbitrary chosen and fixed. Let also $\tilde{\rho} \in (\rho, R)$. Then, since $\tilde{\rho} < R$ then the series $\sum_{n=1}^{\infty} A^n \tilde{\rho}^n$ is uniformly convergent, from where $\lim_{n \longrightarrow \infty} A^n \tilde{\rho}^n = 0$ and therefore the sequence $\{\tilde{\rho}^n A^n\}_{n=1}^{\infty}$ is a bounded sequence. Consequently there exists a positive constant M_2 such that

$$||A^n|| \tilde{\rho}^n \leq M_2 \quad \text{for} \quad \forall n \in \mathbb{N}.$$

From here

$\sum_{n=1}^{\infty} n||A^n||\rho^{n-1} = \sum_{n=1}^{\infty} n||A^n||\tilde{\rho}^n \frac{1}{\tilde{\rho}} \left(\frac{\rho}{\tilde{\rho}}\right)^{n-1}$

$\leq \frac{M_2}{\tilde{\rho}} \sum_{n=1}^{\infty} n \left(\frac{\rho}{\tilde{\rho}}\right)^{n-1}.$

Let $q_1 = \frac{\rho}{\tilde{\rho}}$. Then $q_1 < 1$ and

$\left|\left|\sum_{n=1}^{\infty} nA^n\rho^{n-1}\right|\right| \leq \sum_{n=1}^{\infty} ||nA^n\rho^{n-1}||$

$\leq \sum_{n=1}^{\infty} n||A^n||\rho^{n-1}$

$\leq \frac{M_2}{\tilde{\rho}} \sum_{n=1}^{\infty} n \left(\frac{\rho}{\tilde{\rho}}\right)^{n-1}$

$= \frac{M_2}{\tilde{\rho}} \sum_{n=1}^{\infty} nq_1^{n-1}.$

Let $b_k = kq_1^{k-1}$. Then

$\lim_{k\longrightarrow\infty} \frac{b_{k+1}}{b_k} = \lim_{k\longrightarrow\infty} \frac{(k+1)q_1^k}{kq_1^{k-1}}$

$= \lim_{k\longrightarrow\infty} \frac{k+1}{k} q_1 = q_1 < 1.$

Consequently the series $\sum_{n=1}^{\infty} nq_1^{n-1}$ is convergent and then

$$c(\rho) := \sum_{n=1}^{\infty} n||A^n||\rho^{n-1} < \infty.$$

Since $\rho \in S_R(0)$ was arbitrary chosen then the series $\sum_{n=1}^{\infty} nA^n\rho^{n-1}$ is uniformly convergent for every $\rho \in S_R(0)$.

From (2.4) we obtain

$$||g(w) - g(w_0)|| \leq c(\rho)|w - w_0|. \qquad (2.5)$$

Let $\epsilon > 0$ be arbitrary chosen and fixed. Let also $\delta = \frac{\epsilon}{1+c(\rho)}$. Then if $|w - w_0| < \delta$, from (2.5), we get

$$||g(w)-g(w_0)|| \leq c(\rho)|w-w_0| < c(\rho)\delta = c(\rho)\frac{\epsilon}{1 + c(\rho)} < \epsilon.$$

Since $\epsilon > 0$ was arbitrary chosen and for it we find $\delta = \delta(\epsilon) > 0$ such that form $|w - w_0| < \delta$ we have $||g(w) - g(w_0)|| < \epsilon$ we conclude that g is a continuous function at w_0.

Because $w_0 \in S_R(0)$ was arbitrary chosen then g is a continuous function in $S_R(0)$. □

Corollary 2.6. *The series*

$$\sum_{n=1}^{\infty} nA^n w^{n-1}$$

is a convergent series in $S_R(0)$.

Proof. Let $|w| < R$ and $\rho \in (|w|, R)$. Then $\frac{|w|}{\rho} < 1$ and from here and the proof of the previous Theorem we have

$\left|\left|\sum_{n=1}^{\infty} nA^nw^{n-1}\right|\right| \leq \sum_{n=1}^{\infty} ||nA^nw^{n-1}||$

$= \sum_{n=1}^{\infty} n||A^n|||w|^{n-1}$

$= \sum_{n=1}^{\infty} n||A^n||\rho^{n-1} \frac{|w|^{n-1}}{\rho^{n-1}}$

$\leq \sum_{n=1}^{\infty} n||A^n||\rho^{n-1} < \infty.$

Because $w \in S_R(0)$ was arbitrary chosen we conclude that $\sum_{n=1}^{\infty} nA^nw^{n-1}$ is convergent in $S_R(0)$. □

Theorem 2.7. *The function* g *is differentiable function in* $S_R(0)$.

Proof. For $w \in S_R(0)$ we define the function

$$u(w) = \sum_{n=2}^{\infty} nA^n w^{n-1}.$$

For $w, w_1 \in S_R(0)$ we have

$\frac{g(w)-g(w_1)}{w-w_1} - u(w_1)$

$= \frac{1}{w-w_1} \left(\sum_{n=2}^{\infty} A^nw^n - \sum_{n=2}^{\infty} A^nw^{n-1}\right)$

$- \sum_{n=2}^{\infty} nA^nw_1^{n-1}$

$= \frac{1}{w-w_1} \sum_{n=2}^{\infty} A^n(w^n - w_1^n) - \sum_{n=2}^{\infty} nA^nw_1^{n-1}$

$= \sum_{n=2}^{\infty} A^n \frac{w^n-w_1^n}{w-w_1} - \sum_{n=2}^{\infty} nA^nw_1^{n-1}$

$= \sum_{n=2}^{\infty} A^n \left(\frac{w^n-w_1^n}{w-w_1} - nw_1^{n-1}\right),$

i.e.

$$\frac{g(w) - g(w_1)}{w - w_1} - u(w_1) = \sum_{n=2}^{\infty} A^n \Big(\frac{w^n - w_1^n}{w - w_1} - nw_1^{n-1}\Big). \qquad (2.6)$$

We will note that

$$\frac{w^n-w_1^n}{w-w_1} - nw_1^{n-1} = n(n-1)(w-w_1) \times \int_0^1 (1-\theta)((1-\theta)w_1+\theta w)^{n-2} d\theta. \tag{2.7}$$

Really,

$$\begin{aligned}
& n(n-1)(w-w_1) \\
& \times \int_0^1 (1-\theta)((1-\theta)w_1+\theta w)^{n-2} d\theta \\
& = n(n-1)(w-w_1) \\
& \times \int_0^1 (1-\theta)(w_1+\theta(w-w_1))^{n-2} d\theta \\
& = n(n-1) \\
& \times \int_0^1 (1-\theta)(w_1+\theta(w-w_1))^{n-2} d(w_1+\theta(w-w_1)) \\
& = n\int_0^1 (1-\theta) d(w_1+\theta(w-w_1))^{n-1} \\
& = n(1-\theta)(w_1+\theta(w-w_1))^{n-1}\Big|_{\theta=0}^{\theta=1} \\
& +n\int_0^1 (w_1+\theta(w-w_1))^{n-1} d\theta \\
& = -nw_1^{n-1} \\
& +\frac{n}{w-w_1}\int_0^1 (w_1+\theta(w-w_1))^{n-1} d(w_1+\theta(w-w_1)) \\
& = -nw_1^{n-1} \\
& +\frac{1}{w-w_1}(w_1+\theta(w-w_1))^{n}\Big|_{\theta=0}^{\theta=1} \\
& = \frac{w^n-w_1^n}{w-w_1} - nw_1^{n-1}.
\end{aligned}$$

Now we apply (2.7) in (2.6) and we obtain

$$\begin{aligned}
& \frac{g(w)-g(w_1)}{w-w_1} - u(w_1) \\
& = (w-w_1)\sum_{n=1}^{\infty} n(n-1)A^n \\
& \times \int_0^1 (1-\theta)((1-\theta)w_1+\theta w)^{n-2} d\theta.
\end{aligned} \tag{2.8}$$

Let $\rho \in (0,R)$ is arbitrary chosen and fixed. Then for $w, w_1 \in S_\rho(0)$ and from (2.8) we have

$$\begin{aligned}
& \left|\left|\frac{g(w)-g(w_1)}{w-w_1} - u(w_1)\right|\right| \\
& = \left|\left|(w-w_1)\sum_{n=2}^{\infty} n(n-1)A^n \right.\right. \\
& \left.\left.\times \int_0^1 (1-\theta)((1-\theta)w_1+\theta w)^{n-2} d\theta\right|\right| \\
& \le |w-w_1|\sum_{n=2}^{\infty} n(n-1)\left|\left|A^n\right.\right. \\
& \left.\left.\times \int_0^1 (1-\theta)((1-\theta)w_1+\theta w)^{n-2} d\theta\right|\right| \\
& = |w-w_1|\sum_{n=2}^{\infty} n(n-1)||A^n|| \\
& \times\left|\int_0^1 (1-\theta)((1-\theta)w_1+\theta w)^{n-2} d\theta\right| \\
& \le |w-w_1|\sum_{n=2}^{\infty} n(n-1)||A^n|| \\
& \times \int_0^1 (1-\theta)|((1-\theta)w_1+\theta w)^{n-2}| d\theta \\
& = |w-w_1|\sum_{n=2}^{\infty} n(n-1)||A^n|| \\
& \times \int_0^1 (1-\theta)|(1-\theta)w_1+\theta w|^{n-2} d\theta \\
& \le |w-w_1|\sum_{n=2}^{\infty} n(n-1)||A^n|| \\
& \times \int_0^1 (1-\theta)((1-\theta)|w_1|+\theta|w|)^{n-2} d\theta \\
& \le |w-w_1|\sum_{n=2}^{\infty} n(n-1)||A^n|| \\
& \times \int_0^1 (1-\theta)((1-\theta)\rho+\theta\rho)^{n-2} d\theta \\
& = |w-w_1|\sum_{n=1}^{\infty} n(n-1)||A^n|| \\
& \times \int_0^1 (1-\theta)\rho^{n-2} d\theta \\
& \le |w-w_1|\sum_{n=1}^{\infty} n(n-1)||A^n||\rho^{n-2},
\end{aligned}$$

i.e.

$$\left|\left|\frac{g(w)-g(w_1)}{w-w_1} - u(w_1)\right|\right| \le |w-w_1|\sum_{n=1}^{\infty} n(n-1)||A^n||\rho^{n-2}. \tag{2.9}$$

Now we will prove that for every $\rho \in (0,R)$ the series $\sum_{n=2}^{\infty} n(n-1)A^n\rho^{n-2}$ is a convergent series. Really, let $\rho \in (0,R)$ is arbitrary chosen and fixed. and let also $\tilde{\rho} \in (\rho, R)$. Since $0 < \tilde{\rho} < R$ then the series $\sum_{n=2}^{\infty} A^n \tilde{\rho}^n$ is a convergent series. Therefore $\lim_{n\longrightarrow\infty} A^n \tilde{\rho}^n = 0$ and from here the sequence $\{A^n \tilde{\rho}^n\}_{n=1}^{\infty}$ is a convergent sequence. Consequently, there exists a positive constant M_3 so that

$$||A^n||\tilde{\rho}^n \leq M_3 \quad \text{for} \quad \forall n \in \mathbb{N}.$$

Therefore

$$\left|\left|\sum_{n=1}^{\infty} n(n-1)A^n\rho^{n-2}\right|\right|$$

$$\leq \sum_{n=2}^{\infty} ||n(n-1)A^n\rho^{n-2}||$$

$$= \sum_{n=2}^{\infty} n(n-1)||A^n|||\rho^{n-2}$$

$$= \sum_{n=2}^{\infty} n(n-1)||A^n||\tilde{\rho}^n \left(\frac{\rho}{\tilde{\rho}}\right)^{n-2} \tilde{\rho}^2$$

$$\leq M_3 R^2 \sum_{n=2}^{\infty} n(n-1)\left(\frac{\rho}{\tilde{\rho}}\right)^{n-2},$$

i.e.

$$\left|\left|\sum_{n=2}^{\infty} n(n-1)A^n\rho^{n-2}\right|\right| \leq M_3 R^2 \sum_{n=2}^{\infty} n(n-1)\left(\frac{\rho}{\tilde{\rho}}\right)^{n-2}. \tag{2.10}$$

We put

$$q_2 = \frac{\rho}{\tilde{\rho}}.$$

Then, using (2.10), $q_2 < 1$ and

$$\left|\left|\sum_{n=2}^{\infty} n(n-1)A^n\rho^{n-2}\right|\right| \leq M_3 R^2 \sum_{n=2}^{\infty} n(n-1)q_2^{n-2}. \tag{2.11}$$

Let

$$d_n = n(n-1)q_2^{n-2}.$$

Then

$$\lim_{n\longrightarrow\infty} \frac{d_{n+1}}{d_n} = \lim_{n\longrightarrow\infty} \frac{(n+1)nq_2^{n-1}}{n(n-1)q_2^{n-2}}$$

$$= q_2 \lim_{n\longrightarrow\infty} \frac{n+1}{n-1} = q_2 < 1.$$

Consequently $\sum_{n=2}^{\infty} n(n-1)q_2^{n-1}$ is convergent and from (2.10) the series $\sum_{n=2}^{\infty} n(n-1)A^n\rho^{n-2}$ is convergent. Because $\rho \in (0,R)$ was arbitrary chosen then the series $\sum_{n=2}^{\infty} n(n-1)A^n\rho^{n-2}$ is convergent for every $\rho \in (0,R)$. Therefore

$$c_1(\rho) = \sum_{n=2}^{\infty} n(n-1)||A^n||\rho^{n-2} < \infty$$

for every $\rho \in (0,R)$. From (2.9) follows that

$$\left|\left|\frac{g(w)-g(w_1)}{w-w_1} - u(w_1)\right|\right| \leq c_1(\rho)|w-w_1| \tag{2.12}$$

for every $\rho \in (0,R)$. Let $\epsilon > 0$ be arbitrary chosen and fixed. Let also $\delta = \frac{\epsilon}{1+c_1(\rho)}$. Then from $|w-w_1| < \delta$ we have

$$\left|\left|\frac{g(w)-g(w_1)}{w-w_1} - u(w_1)\right|\right| \leq c_1(\rho)|w-w_1| < c_1(\rho)\delta$$

$$= c_1(\rho)\frac{\epsilon}{1+c_1(\rho)} < \epsilon$$

for every $\rho \in (0,R)$. Because $\epsilon > 0$ was arbitrary chosen and for it we found $\delta = \delta(\epsilon) > 0$ so that from $|w-w_1| < \delta$ we have $\left|\left|\frac{g(w)-g(w_1)}{w-w_1} - u(w_1)\right|\right| < \epsilon$ then the function g is a differentiable function at w_1 and $g'(w_1) = u(w_1)$. Since $w_1 \in S_R(0)$ was arbitrary chosen then the function g is a differentiable function in $S_R(0)$ and for every $w \in S_R(0)$ we have $g'(w) = u(w)$. □

Using induction one can prove

Corollary 2.8. $g \in \mathcal{C}^{\infty}(S_R(0))$.

Theorem 2.9. *Let $\sum_{n=0}^{\infty} A^n$ be absolutely convergent series to* 0. *Then*

$$\lim_{w\longrightarrow 1, w: \left|\frac{|1-w|}{1-|w|}\right| < \infty} g(w) = 0.$$

Proof. Without loss of generality we will consider the case when $w \longrightarrow 1$ and $0 < \frac{|1-w|}{1-|w|} < \infty$.

Then $|w| < 1$ and there exists a positive constant M_4 such that

$$0 < \frac{|1-w|}{1-|w|} \leq M_4. \tag{2.13}$$

Let

$$P^n = \sum_{k=0}^{n} A^k, \quad n = 0, 1, 2, \ldots.$$

Then the sequence $\{P^n\}_{n=1}^{\infty}$ is a convergent sequence.

$$A^0 = P^0, A^k = P^k - P^{k-1}, \quad k = 1, 2, \dots.$$

We put

$$s_n(w) = \sum_{k=0}^{n} A^k w^k.$$

Then

$$s_n(w) = A^0 + A^1 w + A^2 w^2 + \dots + A^n w^n$$

$$= P^0 + (P^1 - P^0)w + (P^2 - P^1)w^2$$

$$+ \dots + (P^n - P^{n-1})w^n$$

$$= P^0(1-w) + P^1(w - w^2) + P^2(w^2 - w^3)$$

$$+ \dots + P^{n-1}(w^{n-1} - w^{n-2}) + P^n w^n$$

$$= P^0(1-w) + P^1 w(1-w) + P^2 w^2(1-w)$$

$$+ \dots + P^{n-1}w^{n-1}(1-w) + P^n w^n,$$

i.e.

$$s_n(w) = (1-w)\sum_{k=0}^{n-1} P^k w^k + P^n w^n. \quad (2.14)$$

Since $\sum_{k=0}^{\infty} A^n$ is an absolutely convergent series to 0 then for $|w| < 1$ we have $\lim_{n\longrightarrow\infty} P^n w^n = 0$ and from (2.14)

$$g(w) = \lim_{n\longrightarrow\infty} s_n(w) = (1-w)\sum_{k=0}^{\infty} P^k w^k. \quad (2.15)$$

Let $\epsilon > 0$. Then there exists $m \in \mathbb{N}$ such that $||P^n|| < \epsilon$ for every $n \geq m$.

We choose w so that to satisfy (2.13) and $|1-w| < \epsilon$. From here

$$\left|\left|\sum_{n=m}^{\infty} P^n w^n\right|\right| \leq \sum_{n=m}^{\infty} ||P^n|| |w|^n$$

$$< \epsilon \sum_{n=m}^{\infty} |w|^n = \epsilon \tfrac{|w|^m}{1-|w|}. \quad (2.16)$$

From (2.13) we have

$$|w| - 1 \leq |1-w| \leq M_4(1-|w|),$$

from where

$$|w| \leq M_4(1-|w|) + 1.$$

Since $|w| < 1$ then

$$|w|^m \leq |w| \leq M_4(1-|w|) + 1$$

and from (2.16) we obtain

$$\left|\left|\sum_{n=m}^{\infty} P^n w^n\right|\right| < \epsilon \tfrac{M_4(1-|w|)+1}{1-|w|}$$

$$= \epsilon M_4 + \tfrac{\epsilon}{1-|w|}.$$

From the last inequality we get

$$\left|\left|(1-w)\sum_{n=m}^{\infty} P^n w^n\right|\right|$$

$$\leq |1-w|\left(\epsilon M_4 + \tfrac{\epsilon}{1-|w|}\right)$$

$$= M_4\epsilon|1-w| + \epsilon\tfrac{|1-w|}{1-|w|}$$

$$\leq M_4\epsilon|1-w| + M_4\epsilon$$

$$= M_4\epsilon(1 + |1-w|)$$

and from (2.15)

$$||g(w)|| = \left|\left|(1-w)\sum_{n=0}^{\infty} P^n w^n\right|\right|$$

$$= \left|\left|(1-w)\sum_{n=0}^{m-1} P^n w^n\right.\right.$$

$$\left.\left.+(1-w)\sum_{n=m}^{\infty} P6nw^n\right|\right|$$

$$\leq |1-w|\left|\left|\sum_{n=0}^{m-1} P^n w^n\right|\right|$$

$$+\left|\left|(1-w)\sum_{n=m}^{\infty} P^n w^n\right|\right|$$

$$\leq |1-w|\left|\left|\sum_{n=0}^{m-1} P^n w^n\right|\right|$$

$$+M_4\epsilon(1 + |1-w|).$$

Because $\epsilon > 0$ was arbitrary chosen

$$\lim_{w\longrightarrow 1, w: 0 < \frac{|1-w|}{1-|w|} < \infty} g(w) = 0.$$

$\square$

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