INTRODUCTION to the BASIC CATEGORY THEORY*

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Abstract

The great revolution of the 20th century started with the theory of special and general relativity and culminated in quantum theory. However, up to date, there are still some fundamental issues with quantum theory that are, yet, to be solved. A possible way to overcome critical issues in present-day quantum physics is through a reformulation of astroparticle physics and quantum theory in terms of a different mathematical framework called category theory.

Parts 1 and 2 introduce the basic notions of category theory, functors and natural transformations. Examples of Representations of categories are presented in Part 3. In Part 4 there is introduced the very important concept of Multiplicative Structures on Categories. In Part 5 there are formulated the basic definitions of the Method of Additional Structures on Objects of a Category, on the one hand, and Topological Quantum Field Theory, on the other hand.

This is one of a text to address all of basic aspects of category theory at the graduate student level.

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1 Categories, monoids and groupoids

Category theory groups together in categories the mathematical objects with some common structure (e.g., sets, partially ordered sets, groups, rings, and so forth) and the appropriate morphisms between such objects [1–7]. These morphisms are required to satisfy certain properties which make the set of all such relations coherent. Given a category, it is not the case that every two objects have a relation between them, some do and others dont. For the ones that do, the number of relations can vary depending on which category we are considering.

Definition 1.1. *A category is a quadruple* $(\mathrm{Ob}\,, \mathrm{Hom}, \mathrm{id}\,, \circ)$ *consisting of:*

(Cl) a class Ob *of objects;*

(C2) for each ordered pair (A, B) *of objects a set* $\mathrm{Hom}(A, B)$ *of morphisms;*

(C3) for each object A *a morphism* $\mathrm{id}\,_A \in \mathrm{Hom}(A, A)$*, the identity of* A*;*

(C4) a composition law associating to each pair of morphisms $f \in \mathrm{Hom}(A, B)$ *and* $g \in \mathrm{Hom}(B, C)$ *a morphism* $g \circ f \in \mathrm{Hom}(A, C)$*;*

which is such that:

(M1) $h \circ (g \circ f) = (h \circ g) \circ f$ *for all* $f \in \mathrm{Hom}(A, B)$*,* $g \in \mathrm{Hom}(B, C)$ *and* $h \in \mathrm{Hom}(C, D)$*;*

(M2) $\mathrm{id}\,_B \circ f = f \circ \mathrm{id}\,_A = f$ *for all* $f \in Hom(A, B)$*;*

(M3) the sets $\mathrm{Hom}(A, B)$ *are pairwise disjoint.*

This last axiom is necessary so that given a morphism we can identify its domain A and codomain B, however it can always be satisfied by replacing $\mathrm{Hom}(A, B)$ by the set $\mathrm{Hom}(A, B) \times (\{A\}, \{B\})$.

Example 1.1. *The classic example is* **Set***, the category with sets as objects and functions as morphisms, and the usual composition of functions as composition.*

To understand this definition, we should ask first: what is a category with one object? It is a — *monoid.* We shall also give the definition of a monoid in [7] in more details like this: a set M with an associative binary product and a unit element 1 such that $a1 = 1a = a$ for all a in M. Monoids abound in mathematics; they are in a sense the most primitive interesting algebraic structures.

To check that a category with one object is "essentially just a monoid", note that if our category $\mathbf{C}$ has one object x, the set $\mathrm{Hom}(x, x)$ of all morphisms from x to x is indeed a set with an associative binary product, namely composition, and a unit element, namely id_x.

How about categories in which every morphism is invertible? We say a morphism $f : x \to y$ in a category has inverse $g : y \to x$ if $f \circ g = \mathrm{id}_y$ and $g \circ f = \mathrm{id}_x$. Well, a category in which every morphism is invertible is called a *groupoid*.

Finally, a group is a category with one object in which every morphism is invertible. It's both a monoid and a groupoid!

When we use groups in physics to describe symmetry, we think of each element g of the group G as a "process". The element 1 corresponds to the "process of doing nothing at all". We can compose processes g and h — do h and then g — and get the product $g \circ h$. Crucially, every process g can be "undone" using its inverse g^{-1}.

So: a monoid is like a group, but the "symmetries" no longer need be invertible; a category is like a monoid, but the "symmetries" no longer need to be composable.

A morphism $a\colon A \to B$ is called *isomorphism* if there exists a morphism $b\colon B \to A$ such that $a \circ b = i_B$ and $b \circ a = i_A$. In this case objects A and B are called *isomorphic.*

Morphisms $a\colon D \to A$ and $b\colon D \to B$ are called *isomorphic* if there exists an isomorphism $c\colon A \to B$ such that $c \circ a = b$.

An object Z is called *terminal* object if for any object A there exists a unique morphism from A to Z, which is denoted $z_A\colon A \to Z$ in what follows.

It is said that a category has (pairwise) products if for every pair of objects A and B there exists their *product*, that is, an object $A \times B$ and a pair of morphisms $\pi_{A,B}\colon A \times B \to A$ and $\nu_{A,B}\colon A \times B \to B$, called projections, such that for any object D and for any pair of morphisms $a\colon D \to A$ and $b\colon D \to B$ there exists a unique morphism $c\colon D \to A \times B$, satisfying the following conditions:

$$\pi_{A,B} \circ c = a, \qquad \nu_{A,B} \circ c = b. \tag{1}$$

We call such morphism c the *product of morphisms* a and b and denote it $a * b$.

It is easily seen that existence of products in a category implies the following equality:

$$(a * b) \circ d = (a \circ d) * (b \circ d). \tag{2}$$

In a category with products, for two arbitrary morphisms $a\colon A \to C$ and $b\colon B \to D$ one can define the morphism $a \times b$:

$$a \times b\colon A \times B \to C \times D, \qquad a \times b \stackrel{\text{def}}{=} (a \circ \pi_{A,B}) * (b \circ \nu_{A,B}). \tag{3}$$

This definition and (1) obviously imply that the morphism $c = a \times b$ satisfy the following conditions:

$$\pi_{C,D} \circ c = a \circ \pi_{A,B}, \qquad \nu_{C,D} \circ c = b \circ \nu_{A,B}. \tag{4}$$

Moreover, $c = a \times b$ is the only morphism satisfying conditions (4).

It is also easily seen that (2) and (3) imply the following equality:

$$(a \times b) \circ (c * d) = (a \circ c) * (b \circ d). \tag{5}$$

Suppose $A \times B$ and $B \times A$ are two products of objects A and B taken in different order. By the properties of products, the objects $A \times B$ and $B \times A$ are isomorphic and the natural isomorphism is

$$\sigma_{A,B}\colon A \times B \to B \times A, \qquad \sigma_{A,B} \stackrel{\text{def}}{=} \nu_{A,B} * \pi_{A,B}. \tag{6}$$

Moreover, for any object D and for any morphisms $a\colon D \to A$ and $b\colon D \to B$, the morphisms $a * b$ and $b * a$ are isomorphic, that is,

$$\sigma_{A,B} \circ (a * b) = b * a. \tag{7}$$

Similarly, by the properties of products, the objects $(A \times B) \times C$ and $A \times (B \times C)$ are isomorphic. Let

$$\alpha_{A,B,C} \colon (A \times B) \times C \to A \times (B \times C)$$

be the corresponding natural isomorphism. Its "explicit" form is:

$$\alpha_{A,B,C} \stackrel{\text{def}}{=} (\pi_{A,B} \circ \pi_{A \times B,C}) * \Big((\nu_{A,B} \circ \pi_{A \times B,C}) * \nu_{A \times B,C} \Big) . \tag{8}$$

Then for any object D and for any morphisms $a \colon D \to A$, $b \colon D \to B$, and $c \colon D \to C$ we have

$$\alpha_{A,B,C} \circ ((a * b) * c) = a * (b * c). \tag{9}$$

REMARK 1.1. *If* $\mathbf{A} = (\mathrm{Ob}\,, \mathrm{Hom}, \mathrm{id}\,, \circ)$ *is a category, then*
1) The class Ob *of* $\mathbf{A}$*-objects is usually denoted by* $\mathrm{Ob}\,(\mathbf{A})$.
2) The class of all $\mathbf{A}$*-morphisms (denoted by* $\mathrm{Mor}\,(\mathbf{A})$*) is defined to be the union of all the sets* $\mathrm{Hom}(A, B)$ *in* $\mathbf{A}$.
3) If $A \xrightarrow{f} B$ *is an* $\mathbf{A}$*-morphism, we call* A *the* **domain** *of* f *[and denote it by dom(*f*)] and call* B *the* **codomain** *of* f *[and denote it by cod(*f*)]. Observe that condition (M3) guarantees that each* $\mathbf{A}$*-morphism has a* **unique** *domain and a* **unique** *codomain. However, this condition is given for technical convenience only, because whenever all other conditions are satisfied, it is easy to "force" condition (M3) by simply replacing each morphism* f *in* $\mathrm{Hom}(A, B)$ *by a triple* (A, f, B)*. For this reason, when verifying that an entity is a category, we will disregard condition (M3).*
4)The composition, $\circ$*, is a partial binary operation on the class* $\mathrm{Mor}\,(\mathbf{A})$*. For a pair* (f, g) *of morphisms,* $f \circ g$ *is defined if and only if the domain of* f *and the codomain of* g *coincide.*
5) If more than one category is involved, subscripts may be used for clarification (as in $\mathrm{Hom}_{\mathbf{A}}(A, B)$*).*

2 Functors and natural transformations

DEFINITION 2.1. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two categories. A* **covariant functor** *from a category* $\mathbf{X}$ *to a category* $\mathbf{Y}$ *is a family of functions* $\mathcal{F}$ *which associates to each object* A *in* $\mathbf{X}$ *an object* $\mathcal{F}A$ *in* $\mathbf{Y}$ *and to each morphism* $f \in \mathrm{Hom}_{\mathbf{X}}(A, B)$ *a morphism* $\mathcal{F}f \in \mathrm{Hom}_{\mathbf{Y}}(\mathcal{F}A, \mathcal{F}B)$*, and which is such that:*

(FI) $\mathcal{F}(g \circ f) = \mathcal{F}g \circ \mathcal{F}f$ *for all* $f \in \mathrm{Hom}_{\mathbf{X}}(A, B)$ *and* $g \in \mathrm{Hom}_{\mathbf{Y}}(B, C)$*;*
(F2) $\mathcal{F}\,\mathrm{id}_A = \mathrm{id}_{\mathcal{F}A}$ *for all* $A \in \mathrm{Ob}\,(\mathbf{X})$.

It is clear from the above that a covariant functor is a transformation that preserves both:

- The domains and the codomains identities.
- The composition of arrows, in particular it preserves the direction of the arrows.

DEFINITION 2.2. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two categories. A* **contravariant functor** *from a category* $\mathbf{X}$ *to a category* $\mathbf{Y}$ *is a family of functions* $\mathcal{F}$ *which associates to each object* A *in* $\mathbf{X}$ *an object* $\mathcal{F}A$ *in* $\mathbf{Y}$ *and to each morphism* $f \in \mathrm{Hom}_{\mathbf{X}}(A, B)$ *a morphism* $\mathcal{F}f \in \mathrm{Hom}_{\mathbf{Y}}(\mathcal{F}A, \mathcal{F}B)$*, and which is such that:*

(FI) $\mathcal{F}(g \circ f) = \mathcal{F}f \circ \mathcal{F}g$ *for all* $f \in \mathrm{Hom}_{\mathbf{X}}(A, B)$ *and* $g \in \mathrm{Hom}_{\mathbf{Y}}(B, C)$*;*
(F2) $\mathcal{F}\,\mathrm{id}_A = \mathrm{id}_{\mathcal{F}A}$ *for all* $A \in \mathrm{Ob}\,(\mathbf{X})$.

Thus, a contravariant functor in mapping arrows from one category to the next reverses the directions of the arrows, by mapping domains to codomains and vice versa. A contravariant functor is also called a presheaf.

So far we have defined categories and maps between them called functors. We will now abstract one step more and define maps between functors. These are called *natural transformations.*

DEFINITION 2.3. *Let* $\mathcal{F} : \mathbf{X} \to \mathbf{Y}$ *and* $\mathcal{G} : \mathbf{X} \to \mathbf{Y}$ *be two functors. A* **natural transformation** $\alpha : \mathcal{F} \to \mathcal{G}$ *is given by the following data:*

For every object A in $\mathbf{X}$ there is a morphism $\alpha_A : \mathcal{F}(A) \to \mathcal{G}(A)$ in $\mathbf{Y}$ such that for every morphism $f : A \to B$ in $\mathbf{X}$ the following diagram is commutative

$$\begin{array}{ccc} \mathcal{F}(A) & \xrightarrow{\alpha_A} & \mathcal{G}(A) \\ \mathcal{F}(f)\downarrow & & \downarrow \mathcal{G}(f) \\ \mathcal{F}(B) & \xrightarrow{\alpha_B} & \mathcal{G}(B). \end{array}$$

Commutativity means (in terms of equations) that the following compositions of morphisms are equal: $\mathcal{G}(f) \circ \alpha_A = \alpha_B \circ \mathcal{F}(f)$.

The morphisms α_A, $A \in \mathrm{Ob}(\mathbf{X})$, are called the *components of the natural transformation* α.

So, we can certainly speak, as before, of the "equality" of categories. We can also speak of the *isomorphism of categories*: an isomorphism between $\mathbf{C}$ and $\mathbf{D}$ is a functor $\mathcal{F} : \mathbf{C} \to \mathbf{D}$ for which there is an inverse functor $\mathcal{G} : \mathbf{D} \to \mathbf{C}$. I.e., $\mathcal{FG}$ is the identity functor on $\mathbf{C}$ and $\mathcal{GF}$ the identity on $\mathbf{D}$, where we define the composition of functors in the obvious way. But because we also have natural transformations, we can also define a subtler notion, the *equivalence of categories*. An equivalence is a functor $\mathcal{F} : \mathbf{C} \to \mathbf{D}$ together with a functor $\mathcal{G} : \mathbf{D} \to \mathbf{C}$ and natural isomorphisms $a : \mathcal{FG} \to 1_{\mathbf{C}}$ and $b : \mathcal{GF} \to 1_{\mathbf{D}}$. A *natural isomorphism* is a natural transformation which has an inverse.

As we can "relax" the notion of equality to the notion of isomorphism when we pass from sets to categories, we can relax the condition that $\mathcal{FG}$ and $\mathcal{GF}$ equal identity functors to the condition that they be isomorphic to the identity functor when we pass from categories to the 2-category **Cat**.

We need to have the natural transformations to be able to speak of functors being isomorphic, just as we needed functions to be able to speak of sets being isomorphic. In fact, with each extra level in the theory of n-categories, we will be able to come up with a still more refined notion of "n-equivalence" in this way.

Thus, in contrast to a set, which consists of a static collection of "things", a category consists not only of objects or "things" but also morphisms which can viewed as "processes" transforming one thing into another. Similarly, in a 2-category, the 2-morphisms can be regarded as "processes between processes", and so on. The eventual goal of basing mathematics upon omega-categories is thus to allow us the freedom to think of any process as the sort of thing higher-level processes can go between. By the way, it should also be very interesting to consider "$\mathbb{Z}$-categories" (where $\mathbb{Z}$ denotes the integers), having j-morphisms not only for $j = 0, 1, 2, ...$ but also for negative j. Then we may also think of any thing as a kind of process.

It is also possible to combine all these definitions in a coherent way and define the category of functors. In particular, given two categories $\mathbf{C}$ and $\mathbf{D}$, the collection of all covariant (or contravariant) functors $F : \mathbf{C} \to \mathbf{D}$ is actually a category which will be denoted as $\mathbf{D}^{\mathbf{C}}$. This is called the *category of functors* or *functor category* and has as objects covariant (or contravariant) functors and as map natural transformations between functors.

3 Representations of categories

As it was shown in §1 to define a category $\mathbf{K}$ we require the following data:

(a) a set $\mathrm{Ob}\,(\mathbf{K})$ of elements called the objects of the category $\mathbf{K}$;

(b) for any two objects $Z, W \in \mathrm{Ob}\,(\mathbf{K})$ a set $\mathrm{Mor}_{\mathbf{K}}(Z, W)$ is defined, called the *morphisms* from Z to W (when it is clear what the category in question is, we omit the index $\mathbf{K}$ and merely write $\mathrm{Mor}\,(Z, W)$);

(c) for any $P \in \mathrm{Mor}\,(Z, Z')$ and $Q \in \mathrm{Mor}\,(Z', Z'')$ their product is defined $QP \in \mathrm{Mor}\,(Z, Z'')$. The product must be associative: the formula

$$R(QP) = (RQ)P$$

holds for any $P \in \mathrm{Mor}\,(Z, Z')$, $Q \in \mathrm{Mor}\,(Z', Z'')$, and $R \in \mathrm{Mor}\,(Z'', Z''')$;

(d) it is usually assumed that the set $\mathrm{Mor}\,(Z, Z)$ contains an element 1_Z called the *identity* such that, for any $P \in \mathrm{Mor}\,(Y, Z)$, we have $P \cdot 1_Z = P$ and, for any $Q \in \mathrm{Mor}\,(Z, W)$, we have $1_Z \cdot Q = Q$.

Example 3.1. *The objects of the category are the finite-dimensional complex linear spaces, and the morphisms are the linear operators.*

EXAMPLE 3.2. *The objects of the category are the groups. The morphisms are the group homomorphisms. The categories of rings, algebras, fields, semigroups, and so on, are introduced in a similar fashion.*

EXAMPLE 3.3. *Let G be a fixed group. The objects of the category are the representations of G, and the morphisms are the intertwining operators (see [7]).*

The set Mor (Z, Z) is also denoted by $\mathrm{End}(Z)$ (or $\mathrm{End}_{\mathbf{K}}(Z)$). It is a semigroup and the elements of $\mathrm{End}(Z)$ are called *endomorphisms*. The invertible endomorphisms of an object Z form a group $\mathrm{Aut}_A(Z)$ and its elements are called the *automorphisms* of the object Z. Finally, the set $\mathrm{Mor}_{\mathbf{K}}$ of all morphisms of the category $\mathbf{K}$ forms a so-called *groupoid*. We recall that by a *groupoid* we mean a set with a partially defined operation: if the products ab and $(ab)c$ are defined, then bc and $a(bc)$ are also defined and $(ab)c = a(bc)$.

It is customary to think of categories as belonging to a different level in the mathematical hierarchy from groups, rings, algebras, and so on; see, for example, Shafarevich [8]. (Even from the point of view of formal logic, it is usual to regard the set of objects of a category as a proper class rather than a set.) However, in this book we shall make no distinction, dealing with categories in the same way as with groups, rings, and so on [6, 9, 10].

3.1 Category of linear relations

The objects of this category are linear spaces over a field $\mathbb{F}$ (and we will suppose them to be finite-dimensional). The morphisms $P : Z \rightrightarrows W$ are the *linear relations*, that is, the subspaces P of $Z \oplus W$.

Sometimes such subspaces are the graphs of linear operators from Z into W, but in general this is not the case.

If $P : Z \rightrightarrows W$ and $Q : W \rightrightarrows Y$ are linear relations, then their product $QP : Z \rightrightarrows Y$ is defined as follows: $(z, y) \in Z \oplus Y$ is contained in the subspace QP if there exists $\omega \in W$ such that $(z, \omega) \in P$ and $(\omega, y) \in Q$ (and this is how one would want to define the product of "multivalued maps").

The following are defined for a linear relation $P : Z \rightrightarrows W$ in the same way as for an operator:

(a) the *kernel* ker P — the set of all $z \in Z$ such that $(z, 0) \in P$;

(b) the *image* im P — the projection of P onto W;

(c) the *domain of definition* $D(P)$ — the projection of P onto Z.
In addition we define

(d) the *indefiniteness* $\mathrm{Indef}(P)$; this is the set of $\omega \in W$ such that $(0, \omega) \in P$;
if P is the graph of an operator then $\mathrm{Indef}(P) = 0$;

(e) the *rank* $\mathrm{rk}(P)$:
$\mathrm{rk}(P) = \dim D(P) - \dim \ker P = \dim \mathrm{im} P - \dim \mathrm{Indef} P =$
$= \dim P - \dim \ker P - \dim \mathrm{Indef} P$.

3.2 Representations of categories

A *covariant functor* $(\mathcal{F}, \phi)$ from a category $\mathbf{K}$ to a category $\mathbf{L}$ is determined by the following data:

(i) a map $\mathcal{F} : \mathrm{Ob}(\mathbf{K}) \to \mathrm{Ob}(\mathbf{L})$;

(ii) the collection of maps $(\phi_{Z,W} : \mathrm{Mor}(Z, W) \to \mathrm{Mor}(\mathcal{F}(Z), \mathcal{F}(W))$ is defined for all $Z, W \in \mathrm{Ob}(\mathbf{K})$, and these maps must satisfy the condition that

$$\phi_{Z,Y}(PQ) = \phi_{W,Y}(P)\phi_{Z,W}(Q), \qquad \phi(1_Z) = 1_{\mathcal{F}(Z)}.$$

Similarly, by a *contravariant functor* from a category $\mathbf{K}$ to a category $\mathbf{L}$, we mean a map $\mathcal{F} : \mathrm{Ob}(\mathbf{K}) \to \mathrm{Ob}(\mathbf{L})$ and a collection of maps

$$\phi_{Z,W} : \mathrm{Mor}(Z, W) \to \mathrm{Mor}(\mathcal{F}(W), \mathcal{F}(Z)),$$

defined for all $Z, W \in \mathrm{Ob}\,(\mathbf{K})$, such that

$$\phi_{Z,Y}(PQ) = \phi_{Z,W}(Q)\phi_{W,Y}(P).$$

EXAMPLE 3.4. *Let* $\mathbf{A}$ *be the category of linear spaces and operators (see Example 3.1), and let* $\mathbf{K}$ *be the category of associative algebras and homomorphisms. We construct a functor* $\mathcal{F} : \mathbf{A} \to \mathbf{K}$. *We associate with each* $Z \in \mathrm{Ob}\,(\mathbf{A})$ *the exterior algebra* $\mathcal{F}(Z)$ *and with each operator* $L : Z \to W$ *the corresponding natural map of exterior algebras.*

By a *representation of a category* $\mathbf{K}$ we mean a covariant functor $(\mathcal{T}, \tau)$ from $\mathbf{K}$ to the category $\mathbf{A}$. In other words, we associate with each $Z \in \mathrm{Ob}\,(\mathbf{K})$ a linear space $\mathcal{T}(Z)$, and with each morphism $P : Z \to W$ an operator $\tau(P)$ from $\mathcal{T}(Z)$ to $\mathcal{T}(W)$, such that for each triple $(Z, W, Y) \in \mathrm{Ob}\,(\mathbf{K})$, and for each $P \in \mathrm{Mor}\,(Z, W)$ and $Q \in \mathrm{Mor}\,(W, Y)$, we have

$$\tau(QP) = \tau(Q)\tau(P).$$

Contravariant functions from $\mathbf{K}$ to Op are called *antirepresentations*.

EXAMPLE 3.5. *Let* $\mathbf{X}$ *be the category of finite sets and maps. We define* $\mathcal{T}(X)$ *as the space of functions on* X *and we define* $\tau(p)f(x) = f(p(x))$ *for each map* $p : X \to Y$. *Then* $(\mathcal{T}, \tau)$ *is an antirepresentation.*

Any mathematician, if he delves into his memory, can recall many examples of representations of categories. As a reminder we give three examples (although they are not of great significance as far as this book is concerned).

EXAMPLE 3.6. *Let* $\mathbf{K}$ *be the category of smooth* n*-dimensional manifolds and smooth maps. Let* $\mathcal{T}_k(M)$ *be the space of differential forms of degree* k *on* M, *and let* $\tau_k(P)$ *be the natural maps of differential forms.*

EXAMPLE 3.7. *Let* $\mathbf{K}$ *be the same category as above, and let* $\mathcal{T}_k(M)$ *be the space of* k*-th (de Rham or any other) cohomologies of the manifold, and let* $\tau_k(P)$ *be the natural map of cohomologies.*

EXAMPLE 3.8. *We define the category* $\mathbf{K}$ *as follows. The set* $\mathrm{Ob}\,(\mathbf{K})$ *consists of two elements* Z, W, *and* $\mathrm{End}(Z)$ *and* $\mathrm{End}(W)$ *consist of the identity element. The set* $\mathrm{Mor}\,(Z, W)$ *consists of two elements, and* $\mathrm{Mor}\,(W, Z)$ *is empty. The problem of the classification of representations of* $\mathbf{K}$ *is precisely Kronecker's problem on the reduction of a pair of operators from* Z *to* W *to canonical form.*

3.3 Projective representations of categories

By a *projective representation of a category* $\mathbf{K}$ we mean the following. Associated with each $Z \in \mathrm{Ob}\,(\mathbf{K})$ is a linear space $\mathcal{T}(Z)$, and with each $P \in \mathrm{Mor}\,(Z, W)$ a linear operator $\tau(P) : \mathcal{T}(Z) \to \mathcal{T}(W)$ such that, for any $P \in \mathrm{Mor}\,(Z, W)$ and $Q \in \mathrm{Mor}\,(W, Y)$, we have

$$\tau(QP) = \lambda(Q, P)\tau(Q)\tau(P),$$

where $\lambda(Q, P) \in \mathbb{C}^*$. (We emphasize that $\lambda(Q, P)$ is not equal to 0; this is important.)

In [7] there are many examples of projective representations of categories. Meanwhile we make the following obvious remark concerning the category $\mathbf{A}^*$, defined as follows.

The *category* $\mathbf{A}^*$ has the linear spaces as its objects, and the linear operators defined to within multiplication by a non-zero constant as its morphisms. In essence, a *projective representation of a category* $\mathbf{K}$ is the same as a functor from $\mathbf{K}$ to $\mathbf{A}^*$ [6, 9].

4 Multiplicative Structures on Categories

4.1 Multiplicative categories

The prototype of a category is the category **Sets** of sets and functions. The prototype of a 2-category is the category **Cat** of small categories and functors. **Cat** has more structure than a simple category because we have natural transformations between functors. This can be viewed in the following way: The extra structure implies that every morphism set $\mathrm{Hom}(C, D)$ in **Cat** is actually not only a set but a category itself where composition and identities in **Cat** are compatible with this categorical structure on the Hom-sets (i.e. composition and identities are functorial with respect to the structure on the Hom-sets). A general category with this kind of extra structure is called a *2-category*.

The definition of a 2-category can be put in a more general setting (which will be convenient below) by using the language of enriched categories. A category **C** is enriched over a category **V** if every Hom-set in **C** has the structure of an object in **V** and if composition and identities in **C** are compatible with this extra structure on the Hom-sets. So, a 2-category is a category enriched over **Cat**. Now, the (small) 2-categories again form a category **2-Cat** and a *3-category* can be defined as a category enriched over **2-Cat** (indeed, **2-Cat** turns out to be a 3-category itself). In this way we can proceed iteratively to define *n-categories* and then *ω-categories* as categories involving n-categorical structures of all levels.

A concrete recipe to obtain monoidal (braided etc.) 2-categories via *Hopf categories* has been proposed by Crane and Frenkel [11]. Namely, it is supposed that the 2-category of module-categories over a Hopf category now plays an important role in 4-dimensional topology and topological quantum field theory (TQFT) [12]. Although the theory of Hopf categories is formulated, in general, by Neuchl [13], interesting examples are still missing. In particular the Hopf category, underlying Lusztig's canonical basis [14] of a quantized universal enveloping algebra, is not constructed yet. We propose to define it as a family of abelian categories of perverse l-adic sheaves equipped with some functors of multiplication and comultiplication. These perverse sheaves are equivariant in the sense of Bernstein and Lunts [15].

It turns out that the notions of n-category and ω-category are not general enough for several interesting applications. What one gets there are weak versions of these concepts (instead of weak n-category sometimes the notions *bicategory*, *tricategory*, etc. are used). Let us shortly explain what this means: In a category it does not make sense to ask for equality of objects but the appropriate notion is isomorphism. In the same way, in a 2-category we should not ask for equality of morphisms but only for equality up to an invertible 2-morphism (the morphisms between the morphisms, e. g. the *natural transformations* in **Cat**). Applying this to the categorical structure itself (i.e. requiring associativity and identity properties only up to natural equivalence) leads to the notion of *weak 2-category* (or *bicategory*). In the same way, we can weaken the structure of an n-category up to the $(n-1)$-th level to obtain a *weak n-category.*

The point making this weakening an involved matter is that in general we need so called coherence conditions in addition to the weakened laws in order to assure that some properties, known from the strict case, hold. E.g., to assure that associativity is iteratively applicable (i.e. that we can up to a 2-isomorphism rebracket composites involving more than three factors), we need a coherence condition stating that even four factors can be rebracketed (and the other cases follow then). See the literature [11–15,18] for the details.

A satisfactory version of a weak n-category for higher n and of a *weak ω-category* was not available for a long time but now there are several approaches at hand [16, 17]. The relationship between these approaches and a universal understanding of these structures has still to be achieved [1, 2].

DEFINITION 4.1. *A* **multiplication** *in the category* **C** *is an associative functor*

$$*: \mathbf{C} \times \mathbf{C} \to \mathbf{C} : (X, Y) \mapsto X * Y.$$

An **associativity morphism** *for $*$ is a functor isomorphism*

$$\varphi_{X,Y,Z} : X * (Y * Z) \to (X * Y) * Z$$

such that for any four objects X, Y, Z, T the following diagram is commutative:

$$\begin{array}{ccccc} X*(Y*(Z*T)) & \xrightarrow{\varphi_{X,Y,Z*T}} & (X*Y)*(Z*T) & \xrightarrow{\varphi_{X*Y,Z,T}} & X*Y*Z*T \\ {\scriptstyle id_X*\varphi_{Y,Z,T}}\downarrow & & & & \uparrow{\scriptstyle \varphi_{X,Y,Z}*id_T} \\ X*((Y*Z)*T) & & \xrightarrow{\varphi_{X,Y*Z,T}} & & (X*(Y*Z))*T \end{array}$$

An **commutativity morphism** *for $*$ is a functor isomorphism*

$$\psi_{X,Y} : X*Y \to Y*X$$

such that for any two objects X, Y we have

$$\varphi_{X,Y} \circ \varphi_{Y,X} = id_{X*Y} : X*Y \to X*Y.$$

Morphisms associativity φ and commutativity ψ are compatible if for any three objects X, Y, Z the following diagram is commutative:

$$\begin{array}{ccccc} X*(Y*Z) & \xrightarrow{\varphi_{X,Y,Z}} & (X*Y)*Z & \xrightarrow{\psi_{X*Y,Z}} & Z*(X*Y) \\ {\scriptstyle id_X*\psi_{Y,Z}}\downarrow & & & & \uparrow{\scriptstyle \varphi_{Z,X,Y}} \\ X*(Z*Y) & \xrightarrow{\varphi_{X,Z,Y}} & (X*Z)*Y & \xrightarrow{\psi_{X,Z}*id_Y} & (Z*X)*Y \end{array}$$

A pair (U, u) where $U \in Obj(\mathbf{C})$ and an isomorphism $u : U \to U*U$ is called a **unit object** *for $\mathbf{C}, *$ if the functor*

$$X \mapsto U*X : \mathbf{C} \to \mathbf{C}$$

is equivalence of categories.

Definition 4.2. *A* **multiplicative category** *is a collection $(\mathbf{C}, *, \varphi, \psi, U, u)$.*

If there are some additional structures on category, then it is usually assumed that product $*$ and others elements of the collection are compatible with these structures.

4.2 $\mathcal{C}$-monoids or multiplicative objects

Let $\mathcal{C} = (\mathbf{C}, *, \varphi, \psi, U, u)$ be a multiplicative category. An *multiplicative object* in $\mathcal{C}$ or $\mathcal{C}$*-monoid* is an object $M \in Ob(\mathbf{C})$ with multiplication $\mu : M * M \to M : (m, m') \mapsto \mu(m, m')$ and an unit $\varepsilon : U \to M$ such that the following axioms are faithful:

(1) Associativity: the following diagram is commutative

$$\begin{array}{ccccc} M * (M * M) & & \xrightarrow{\varphi_{M,M,M}} & & (M * M) * M \\ {\scriptstyle id_M * \mu}\downarrow & & & & \downarrow{\scriptstyle \mu * id_M} \\ M * M & \xrightarrow{\mu} & M & \xleftarrow{\mu} & M * M \end{array}$$

(2) Unit: the following diagram is commutative

$$\begin{array}{ccccc} M & \longrightarrow & U * M & \xrightarrow{\psi_{U*M}} & M * U \\ || & & {\scriptstyle \varepsilon * id_M}\downarrow & & \downarrow{\scriptstyle id_M * \varepsilon} \\ M & \xrightarrow{\mu} & M * M & = & M * M \end{array}$$

EXAMPLE 4.1. *Let $\mathcal{R}$ be a commutative ring. The category $\mathcal{R}$-**mod** of $\mathcal{R}$ -modules is a multiplicative category under the tensor product $\otimes_{\mathcal{R}}$ with the unit object is the left $\mathcal{R}$-module $\mathcal{R}$. Multiplicative objects in the category is $\mathcal{R}-$algebras with units.*

EXAMPLE 4.2. *A small multiplicative category $\mathcal{C}$ is a multiplicative object of the multiplicative category $Sets//Ob(\mathbf{C})$.*

4.3 Monoidal categories and comonoids

Multiplicative structures may be described in categories as monoids in a monoidal category.

A *monoidal category* $(\mathbf{C}, \otimes, K, \varphi, \ldots)$ consists of:
$\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$, $K \in \mathrm{Ob}(\mathbf{C})$ – the unit object,

and the functor-isomorphisms:

$$\varphi_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$$

$\psi_A : A \otimes K \to A, \ldots,$
where $\otimes$ is symmetrical, if there exists a functor-isomorphism

$$\theta_{A,B} : A \otimes B \to B \otimes A.$$

A *monoidal functor* (a *morphism of monoidal categories*) of two monoidal categories is defined by $\mathcal{F} : (\mathbf{C}, \otimes, K) \to (\mathbf{C}', \otimes', K')$ if

$$\mathcal{F}(A \otimes B) \cong \mathcal{F}(A) \otimes' \mathcal{F}(B)$$

and $\mathcal{F}(K) \cong K'$.

EXAMPLE 4.3. *A monoidal category is a monoid in the monoidal category* $(\mathcal{C}\dashv\sqcup, \times)$ *of categories with Cartesian product.*

EXAMPLE 4.4. *A monoidal category is the category* $\mathcal{S}ymm$ *with objects* $[n]$ *for* $n = 0, 1, \ldots$ *and morphisms*

$$\mathcal{S}ymm([n],[m]) = \begin{cases} \emptyset, & \textit{if } n \neq m, \\ \Sigma_n, & \textit{if } n = m. \end{cases}$$

where Σ_n *is the group of permutations of* $(1, \ldots, n)$. *with the multiplication*

$$* : \mathcal{S}ymm \times \mathcal{S}ymm \to \mathcal{S}ymm$$

such that $[n] * [m] \cong [n+m-1]$ *with the folowing identification of the inputs*

$$(1, \ldots, n) * (\overline{1}, \ldots, \overline{m}) = (1, \ldots, n, \overline{2}, \ldots, \overline{m})$$

which explaines the action of $*$ *on morphisms.*

EXAMPLE 4.5. *Let* $(\mathcal{C}, \otimes, K)$ *and* $(\mathcal{C}', \otimes', K')$ *be two monoidal categories,* $\mathcal{F} \in Ob(\mathcal{C}\mathcal{C}')$ *and* $\mathcal{F}(K) = K'$. *Then for such functors* $\mathcal{F}$ *on the category there is a monoidal structure and a monoid is defined by a functor morphism*

$$\mu_{A,B} : \mathcal{F}(A) \otimes' \mathcal{F}(B) \to \mathcal{F}(A \otimes B)$$

with natural associativity axioms and unit.

5 Method of Additional Structures on Objects of a Category

5.1 Functors as additional structure on categories

5.1.1 A. Forgetful functor

In a category, two objects x and y can be equal or not equal, but they can be *isomorphic* or not, and if they are isomorphic, they can be isomorphic in many different ways. An isomorphism between x and y is simply a morphism $f : x \to y$ which has an inverse $g : y \to x$, such that $f \circ g = \mathrm{id}_y$ and $g \circ f = \mathrm{id}_x$.

In the category **Sets** an isomorphism is just a one-to-one and onto function, i.e. a bijection. If we know two sets x and y are isomorphic we know that they are "the same in a way", even if they are not equal. But specifying an isomorphism $f : x \to y$ does more than say x and y are the same in a way; it specifies a *particular way* to regard x and y as the same.

In short, while equality is a yes-or-no matter, a mere *property*, an isomorphism is a *structure*. It is quite typical, as we climb the categorical ladder (here from elements of a set to objects of a category) for properties to be reinterpreted as structures [1–5].

Definition 5.1. *We tell that a functor $\mathcal{F} : \mathbf{C} \to \mathbf{C}'$ define a additional $\mathscr{C}-$structure on objects of the category $\mathbf{C}'$ if*

1. *$\forall X, Y \in Ob(\mathbf{C})$ the map $F : \mathbf{C}(X, Y) \to \mathbf{C}'(\mathcal{F})X), \mathcal{F}(Y))$ is injective,*
2. *$\forall X \in Ob(\mathbf{C}), Y \in Ob(\mathbf{C}')$ and an isomorphism $u : Y \to \mathcal{F}(X)$ there is an object $\tilde{Y} \in Ob$ and an isomorphism $\tilde{u} : \tilde{Y} \to X$ such that $\mathcal{F}(\tilde{Y}) = Y$ and $\mathcal{F}(\tilde{u}) = u$.*

Such functor is called a **forgetful** *functor.*

Let the category $\mathbf{X}$ be concrete over some category $\mathbf{A}$ in the sense that there exists a faithful functor $\mathcal{U}$ from $\mathbf{X}$ to $\mathbf{A}$, usually called the forgetful functor. The left adjoint to this functor (if it exists) is then called the free functor. A standard example is the forgetful functor from complete metric spaces to metric spaces, whose left adjoint in the completion functor.

Almost all usual mathematical structures are structures on sets in this sense and there are corresponding forgetful functors to the category **Set** of sets.

A forgetful functor $\mathcal{F} : \mathbf{C} \to \mathbf{C}'$ defines a $\mathscr{C}$-structure *on morphisms of the category* $\mathbf{C}'$.

For our general structures we can define the usual construction:

- inverse and direct images of structures;
- restrictions on subobjects,
- different products of structures.

We can define the category $\mathbf{Str}(\mathscr{C})$ of forgetful functors to the category $\mathbf{C}$. It is a full subcategory of $\mathbf{Cat}/\mathscr{C}$ of all categories over $\mathscr{C}$.

Some properties of structures (= forgetful functors):

- In the category $\mathbf{Str}(\mathscr{C})$ the (bundle) product always exist. It gives a "union" structure.
- Any functor $\mathcal{A} : \mathbf{C} \to \mathbf{C}'$ transfers structures to inverse direction, i.e. it defines the functor

$$\mathcal{H}^* : \mathbf{Str}(\mathscr{C}') \to \mathbf{Str}(\mathscr{C}) : \mathcal{F} \mapsto \mathcal{H}^*\mathcal{F}.$$

- For a forgetful functor $\mathcal{F} : \mathbf{C} \to \mathbf{C}'$ the functors

$$(\mathcal{F}\circ) = Funct(id, \mathcal{F}) : Funct(\mathbf{B}, \mathbf{C}) \to Funct(\mathbf{B}, \mathbf{C}')$$

$$(\circ\mathcal{F}) = Funct(\mathcal{F}, id) : Funct(\mathbf{C}', \mathbf{B}) \to Funct(\mathbf{C}, \mathbf{B})$$

  are forgetful functors.

- One of the constructions which transfers the structure $\mathcal{F} : \mathbf{C} \to \mathbf{Set}$ defined on sets to objects of any category $\mathbf{B}$ is the functor

$$\mathcal{G} : \mathbf{B} \to Funct(\mathbf{B}', \mathbf{Set}) : \mathbf{B} \mapsto \mathcal{G}_{\mathbf{B}}.$$

Thus we have

$$\begin{array}{ccc} {\mathcal{G}_{\mathbf{B}}}^{*}\mathbf{C} & \longrightarrow & \mathbf{C} \\ \downarrow & & \downarrow \mathcal{F} \\ \mathbf{B}' & \xrightarrow{\mathcal{G}_{\mathbf{B}}} & \mathbf{Set} \end{array}$$

- If a functor $\mathcal{A} : \mathbf{B} \to \mathbf{C}$ is injective on morphisms (the condition (1) of the definition of the forgetful functor) then a forgetful functor $\mathcal{F} : \mathbf{B}' \to \mathbf{C}$ and an equivalence $\mathcal{I} : \mathbf{B} \to \mathbf{B}'$ exist, such that the following diagram is commutative

$$\begin{array}{ccc} \mathbf{B} & \xrightarrow{\mathcal{A}} & \mathbf{C} \\ \mathcal{I}\downarrow & \nearrow \mathcal{F} & \\ \mathbf{B}' & & \end{array}$$

5.1.2 B. Bundle of categories

Let $\mathcal{E} : \mathbf{F} \to \mathbf{C}$ be a functor and for all objects $U \in Ob(\mathbf{C})$ and we denote by $\mathbf{F}_U = \mathcal{E}^{-1}(U, id_U)$ the subcategory of $\mathbf{F}$ with

$$Ob(\mathbf{F}_U) = \{u \in Ob(\mathbf{F}) |\ \mathcal{E}(u) = U\},$$

$$Mor(\mathbf{F}_U) = \{f \in Mor(\mathbf{F}) |\ \mathcal{E}(f) = id_U\}.$$

Let $(f : w \to u) \in Mor(\mathbf{F}),\quad \mathcal{E}(f : w \to u) = (g : W \to U)$. Then one tells that f is *Descartes's morphism* , or that w is the *inverse image* $g^*(u)$ *of the object* u, if $\forall w' \in Ob(\mathbf{F}_W)$ the map

$$f_* : \mathbf{F}_W(w', w) \to \mathbf{F}_g(w', u) : h \mapsto f \circ h$$

is a bijection. Here we define

$$\mathbf{F}_g(w, u) \stackrel{def}{=} \{h \in \mathbf{F}(w, u) \mid \mathcal{E}(h) = g.\}$$

So we arrive at the diagram

$$\begin{array}{ccc}
\forall\, w' & & \\
\downarrow^{h} & \searrow^{f\circ h} & \\
w & \underset{f}{\longrightarrow} & u \\
 & & \\
V & \overset{g}{\longrightarrow} & U
\end{array}$$

A functor $\mathcal{P} : \mathbf{F} \to \mathbf{C}$ is called a *bundle of categories* if inverse images are allowed to exist and a composition of two Descartes morphism is a Descartes morphism too. Then g^* may be transfered to the functor $\mathcal{G} : \mathbf{F}(U) \to \mathbf{F}(W)$, and $(g_1 \circ g_2)^*$ will be canonical isomorphic to $g_2^* \circ g_1^*$.

EXAMPLE 5.1. *The projection*

$$\Pi_1 : \mathbf{Mor}(\mathbf{Top}) \to \mathbf{Top} : (f : X \to Y) \mapsto X$$

is a bundle of categories. For different structures on topological spaces it is not always true for the category of all morphisms, but may be true for a subcategory.

EXAMPLE 5.2. *Let* $\mathbf{Sub}$ *be a subcategory in* $\mathbf{Mor}(\mathbf{Man})$ *which consists from submersions. Then the projection*

$$\Pi_2 : \mathbf{Sub} \to \mathbf{Man} : (f : X \to Y) \mapsto Y$$

is a bundle of categories and for each morphism $h \in \mathbf{Man}(B', B)$ *we have the functor of the* **inverse image***:*

$$\mathcal{H}^* : \mathbf{Sub}_B \to \mathbf{Sub}_{B'} : (f : M \to B) \mapsto (B' \times_B M \to B').$$

The set $\Gamma(\pi)$ *of sections of an submersion* $\pi : M \to B$ *is the set of morphisms* $\mathbf{Sub}(id_B, \pi)$*.*

EXAMPLE 5.3. *Let* **Mod** *be the category of pairs* $(\mathcal{R}, M)$ *where* $\mathcal{R}$ *is a ring and* M *is a left* $\mathcal{R}$*-module. Let* **Ring** *be the category of rings. Then the functor*

$$\mathcal{H} : \mathbf{Mod} \to \mathbf{Ring} : (\mathcal{R}, M) \mapsto \mathcal{R}$$

is a bundle of categories and for each morphism $h \in \mathbf{Ring}(\mathcal{R}', \mathcal{R})$ *we have the functor of the* **inverse image***:*

$$\mathcal{H}^* : \mathcal{R}\text{-}\mathbf{mod} \to \mathcal{R}'\text{-}\mathbf{mod} : M \mapsto \mathcal{R}' \otimes_{\mathcal{R}} M.$$

5.1.3 C. Fibers of functor morphisms

Grothendieck's definition of a fiber of a functor morphism is applicable to morphisms of functors from any category to the category **Set** of sets. Let $\mathcal{F}, \mathcal{G} : \mathbf{C} \to \mathbf{Set}$, and $\varphi : \mathcal{F} \to \mathcal{G}$ be their morphism. For each object $S \in Ob(\mathbf{C})$ and an element $\alpha \in \mathcal{G}(S)$ the *fiber* $\mathcal{H}_\alpha$ *of* φ *over* α is the following functor

$$\mathcal{H}_\alpha : \mathbf{C}/S \to \mathbf{Set} : f \mapsto \mathcal{H}_\alpha(f),$$

where for a morphism $f : T \to S$

$$\mathcal{H}_\alpha(f) = \{\beta \in \mathcal{F}(T) \mid \mathcal{G}(f) \circ \mathcal{H}_T(\beta) = \alpha\}.$$

So we have the following diagram

$$\begin{array}{ccccc} \mathcal{H}^\alpha(f) \subset & \mathcal{F}(T) & & \mathcal{F}(S) & \\ & \mathcal{H}_T \downarrow & & \downarrow \mathcal{H}_S & \\ & \mathcal{G}(T) & \xrightarrow{\mathcal{G}(f)} & \mathcal{G}(S) & \ni \alpha\,. \end{array}$$

5.2 Categorical structures on topological spaces

Among the structures on topological spaces we can select that one, which is compatible with the topology. Let **Top** be a category of some topological spaces with a forgetful functor $\mathcal{F} : \mathbf{Top} \to \mathbf{Set}$.

The categories associated with a topological space $T \in Ob(\mathbf{Top})$ are as follows:

– The category $\mathbf{T}(T)$, where $Ob(\mathbf{T})$ is the set of all open subsets of T, and $Mor(T', T''))$ are all their inclusions.

– The category (pseudogroup) **DE**, where $Ob(\mathbf{DE})$ is the set of all open subsets of T, and $Mor(T', T'')$ are all their homeomorphisms.

Functors $\mathcal{PRESH} : \mathbf{T} \to \mathbf{Set}$ are called *presheaves* of sets on T. Some of them are called sheaves. Thus we have the inclusions

$$\mathcal{SH}(T) \subset \mathcal{PRESH}(T) \subset \mathcal{FUNCT}(\mathbf{T}, \mathbf{Set}).$$

A *Grothendieck topology* on a category is defined by saying which families of maps into an object constitute a *covering* of the object when certain axioms are fulfilled. A category together with a Grothendieck topology on it is called a *site*. For a site $\mathfrak{C}$ one define the full subcategories $\mathcal{SH}(\mathfrak{C}) \subset \mathcal{PRESH}(\mathfrak{C}) \subset \mathcal{FUNCT}(\mathfrak{C}^\circ, \mathbf{Set})$. The objects of $\mathcal{FUNCT}(\mathfrak{C}^\circ, \mathbf{Set})$ are called *presheaves* on the site $\mathfrak{C}$, and the objects of $\mathcal{SH}(\mathfrak{C})$ are called *sheaves* on $\mathfrak{C}$.

For any category there exists the finest topology such that all representable presheaves are sheaves. It is called the *canonical* Grothendieck topology. *Topos* is a category which is equivalent to the category of sheaves for the canonical topology on them.

Hence, the topology is already transfered on a category. So now it is natural to consider on language of toposes and sheaves in all questions connected to local properties.

Here we shall not consider local structures on toposes in general, and we shall restrict ourselves to the consideration of the elementary case of the category **Top**.

Definition 5.2. *A structure defined by a forgetful functor* $\mathcal{F} : \mathbf{C} \to \mathbf{Top}$ *is called a* **local structure** *if*

$\forall C \in Ob(\mathbf{C})$ *and any inclusion map* $i : U \to \mathcal{F}(C)$ *of the open subset* U *an object* $\tilde{U} \in Ob(\mathbf{C})$ *and a morphism* $\tilde{i} \in \mathbf{Mor}(\tilde{U}, C)$ *exist such that* $\mathcal{F}(\tilde{U}) = U$ $\mathcal{F}(\tilde{i}) = i$. *This* $C-$*structure* $\tilde{U}$ *is denoted by* $C|U$ *and called a* **restriction** *of* C *on* U.

In other words we can restrict ourselves to local structures on open subsets.

For a local structure $\mathcal{F} : \mathbf{C} \to \mathbf{Top}$ and each object $X \in Ob(\mathbf{Top})$ there is the presheaf of categories

$$\mathbf{T}(X)^\circ \to \mathbf{Cat} : U \mapsto \mathcal{F}^{-1}(U, id_U).$$

Often this presheaf is a sheaf.

5.3 Categorical structures on smooth manifolds

Let $\mathbf{M}$ be the category of smooth (∞-differentiable) manifolds with forgetful functor $\mathcal{F} : \mathbf{M} \to \mathbf{Top}$, which defines a local structure and the presheaves of these structures are sheaves. On the category $\mathbf{M}$ there is the *tangent* functor $\mathcal{T} : \mathbf{M} \to \mathbf{M} : M \mapsto \mathcal{T}(M)$.

Its iterations give us almost all interesting functors on $\mathbf{M}$. Among them we note the following:

- The cotangent functor $\mathcal{T}^* : \mathbf{M} \to \mathbf{M} : M \mapsto \mathcal{T}^*(M)$.
- For a manifold M and natural number $k = 0, 1, \ldots$ the functor of k–jets $\mathcal{J}^k : \mathbf{M} \to \mathbf{M} : N \mapsto \mathcal{J}^k(M, N)$.
- For a manifold M, $x \in M$, and natural number $k = 0, 1, \ldots$ the functor of k–jets at the point x $\mathcal{J}^k_x : \mathbf{M} \to \mathbf{M} : N \mapsto J^k_x(M, N)$.

Any category $\mathbf{C}$ of structures on smooth manifolds (or on $M/$) has an additional structure, which give us a possibility to define "smooth families of morphisms".

Definition 5.3. *Let $M, M', M'' \in \mathbf{M}$. A map*

$$\Phi : M \to \mathbf{M}(M', M'') : x \mapsto \Phi_x$$

is called a **smooth family** *of morphisms if there exists a smooth map $\phi : M \times M' \to M''$ such that*

$$\forall x \in M,\ x' \in M' \quad \Phi_x(x') = \phi(x, x').$$

Thus we get the class of categories with smooth families and it appears as the natural condition on functors.

Definition 5.4. *A functor is called a smooth functor if it maps each smooth family to a smooth family.*

Of course all functors $\mathcal{T}, \mathcal{T}^*, \mathcal{J}^k, \mathcal{J}^k_x$ are smooth.

5.4 Double categories as additional structure on categories

5.4.1 A. Definition and examples of double categories

In any category **C** with bundle products for some morphisms we can define so-called intern categories. This is a monoid in the multiplicative category $\mathbf{C}/O$ of pairs of (special) morphisms $D, R : M \to O$ with the bundle product:

for $\xi = (D, R : M \to O)$ and $\xi' = (D', R' : M \to O)$ we get $\xi \star \xi' = (D \circ \pi_1, R' \circ \pi_2 : M \times_O M' \to O)$ where the unit objects $\mathrm{id}_M : 0 \to M$ and $\mathrm{id}_{M'} : 0 \to M'$ and the following diagram is commutative

$$\begin{array}{ccc} M \times_O M' & \xrightarrow{\pi_2} & M' \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle R'} . \\ M & \xrightarrow{R} & O \end{array}$$

So an intern category is an object $\xi = (D, R : M- > O)$ with a multiplication $\mu : \xi \star \xi' \to \xi$ and the unit $\mathrm{id}_M : O \to M$.

Now we consider such an intern category as the category **Cat** of categories and will call it *double category* [2].

Definition 5.5. *A* **double category D** *consists of the following:*

(1) A category $\mathbf{D}_0$ *of objects* $Ob(\mathbf{D}_0)$ *and morphisms* $Mor(\mathbf{D}_0)$ *of* 0*-level.*

(2) A category $\mathbf{D}_1$ *of morphisms* $Ob(\mathbf{D}_1)$ *of* 1*-level and morphisms* $Mor(\mathbf{D}_1)$ *of* 2*-level.*

(3) Two functors $\mathcal{D}, \mathcal{R} : \mathbf{D}_1 \rightrightarrows \mathbf{D}_0$.

(4) A composition functor

$$* : \mathbf{D}_1 \times_{\mathbf{D}_0} \mathbf{D}_1 \to \mathbf{D}_1$$

where the bundle product is defined by commutative diagram

$$\begin{array}{ccc} \mathbf{D}_1 \times_{\mathbf{D}_0} \mathbf{D}_1 & \overset{\pi_2}{\to} & \mathbf{D}_1 \\ \pi_1 \downarrow & & \downarrow \mathcal{D} \\ \mathbf{D}_1 & \overset{\mathcal{R}}{\to} & \mathbf{D}_0 \end{array}$$

(5) A unit functor $\mathcal{ID} : D_0 \to D_1$*, which is a section of* $\mathcal{D}, \mathcal{R}$.

The above data are subject to **Associativity Axiom** and **Unit Axiom**. If both of them are fulfilled only up to equivalence, then the double category is called a **weak double category**, but if they are fulfilled strictly, then it is called a **strong double category**.

Now we see that for two objects $A, B \in Ob(\mathbf{D}_0)$ there are 0-level morphisms $\mathbf{D}_0(A, B)$ which we denote by ordinary arrows $f : A \to B$, and 1-level morphisms $D_{(1)}(A, B)$, which we denote by the arrows $\xi : A \Rrightarrow B$ for $A = \mathcal{D}(\xi)$ and $B = \mathcal{R}(\xi)$. So with a 2-level morphism $\alpha : \xi \to \xi'$, where $\xi : A \Rrightarrow B$ and $\xi' : A' \Rrightarrow B'$ we can associate the following diagram

$$\begin{array}{ccccc} A & \overset{\xi}{\Rrightarrow} & B & & \xi \\ \mathcal{D}(\alpha) \downarrow & & \downarrow \mathcal{R}(\alpha) & \longmapsto & \downarrow \alpha \\ A' & \overset{\xi'}{\Rrightarrow} & B' & & \xi' \end{array}$$

and arrow $\alpha : \mathcal{D}(\alpha) \Rrightarrow \mathcal{R}(\alpha)$.

On each level we have the corresponding compositions:

$$\begin{array}{llcl} \text{0-level} & (A \overset{f}{\to} B \overset{g}{\to} C) & \mapsto & g \circ f : A \to C \\ & \xi \overset{\alpha}{\to} \eta \overset{\beta}{\to} \varsigma & \mapsto & \beta \circ \alpha : \xi \to \varsigma \\ \text{1-level} & (A \overset{\xi}{\Rrightarrow} B \overset{\eta}{\Rrightarrow} C) & \mapsto & \eta * \xi : A \Rrightarrow C \\ \text{2-level} & (f \overset{\alpha}{\Rrightarrow} g \overset{\beta}{\Rrightarrow} h) & \mapsto & \beta * \alpha : f \Rrightarrow h \end{array}$$

The composition on 2-level is associated with the diagram

$$\begin{array}{ccccc} A & \overset{\xi}{\Rrightarrow} & B & & \xi \\ \mathcal{D}(\alpha)\downarrow & & \downarrow\mathcal{R}(\alpha) & & \downarrow\alpha \\ A' & \overset{\xi'}{\Rrightarrow} & B' & \longmapsto & \xi' \\ \mathcal{D}(\alpha')\downarrow & & \downarrow\mathcal{R}(\alpha') & & \downarrow\alpha' \\ A'' & \overset{\xi''}{\Rrightarrow} & B'' & & \xi'' \end{array}$$

Thus, a double category $\mathbf{D}$ consists of

- four sets $Ob(\mathbf{D}_0), Mor(\mathbf{D}_0), Ob(\mathbf{D}_1), Mor(\mathbf{D}_1)$, and eight maps of type $\mathcal{D}, \mathcal{R}$
$$\begin{array}{ccc} Ob(\mathbf{D}_1) & \leftleftarrows & Mor(\mathbf{D}_1) \\ \downarrow\downarrow & & \downarrow\downarrow \\ Ob(\mathbf{D}_0) & \leftleftarrows & Mor(\mathbf{D}_0) \end{array}$$
- two associated categories $\mathbf{D}_0$, $\mathbf{D}_1$, and **almost categories**: $\mathbf{D}_{(2)}$ with the set of objects $Ob(\mathbf{D}_0)$ and the set of morphisms $Ob(\mathbf{D}_1)$, $\mathbf{D}_{(3)}$ with the set of objects $Mor(\mathbf{D}_0)$ and the set of morphisms $Mor(\mathbf{D}_1)$,
- $\mathcal{R}, \mathcal{D} : \mathbf{D}_{(3)} \to \mathbf{D}_{(2)}$ are almost functors.

Now we can define for double categories **double (category) functors** and their **morphisms**, **double subcategories** , the category **DCat** of double categories, **equivalence** of double categories, **dual double categories** (changed direction of 1-level morphisms, i.e. d, r are transposed), and so on.

Definition 5.6. *A* **double category functor** $\mathcal{F} : \mathbf{D} \to \mathbf{D}'$ *is a pair* $\mathcal{F}_0 : \mathbf{D}_0 \to \mathbf{D}'_0, \mathcal{F}_1 : \mathbf{D}_1 \to \mathbf{D}'_1$ *of usual functors such that*

$$\mathcal{D}' \circ \mathcal{F}_1 = \mathcal{F}_0 \circ \mathcal{D}, \quad \mathcal{R}' \circ \mathcal{F}_1 = \mathcal{F}_0 \circ \mathcal{R},$$
$$\forall\, \xi, \xi' \in Ob(\mathbf{D}_1) \qquad \varphi_{\xi,\xi'} : \mathcal{F}_1(\xi * \xi') \widetilde{\rightrightarrows} \mathcal{F}_1(\xi) *' \mathcal{F}_1(\xi'),$$
$$\forall\, A \in Ob(\mathbf{D}_0) \qquad \varphi_A : \mathcal{F}_1(\mathcal{ID}_A) \widetilde{\rightrightarrows} \mathcal{ID}_{\mathcal{F}_0(A)}.$$

Example 5.4. **Bicategories** *are the partial case of double category* $\mathbf{D}$ *when the category* $\mathbf{D}_0$ *is trivial, i.e. has only identical morphisms and the composition of 1-level and 2-level morphisms are associative.*

Example 5.5. *For each category* $\mathbf{C}$ *we have the canonical double category* $\mathbf{Mor}(\mathbf{C})$ *of morphisms. Let* $\mathbf{C}$ *be a category,* T *be the diagram* $\bullet \to \bullet$, $T\mathbf{C}$ *be the category of diagrams in* $\mathbf{C}$ *of type* T, *and let* $\mathbf{D}_0 = C$ *and* $\mathbf{D}_1 = TC$. *The functor* $\mathcal{D}$ *maps the diagram* $f : A \to B$ *into the object* A, *the functor* $\mathcal{R}$ *maps this diagram into the object* B, *and so on. It is easy to see that we get a double category* $\mathbf{D}$ *which is noted by* $\mathbf{Mor}(\mathbf{C})$. *Here* $Ob(\mathbf{D}_1) = Mor(\mathbf{D}_0)$, *where a 2-level morphism* $f \Rrightarrow g$ *is a pair* (u, v) *of morphisms* $u, v \in Mor(\mathbf{C})$ *from the commutative diagram*

$$\begin{array}{ccc} A & \stackrel{u}{\to} & A' \\ f\downarrow & & \downarrow f' \\ B & \stackrel{v}{\to} & B' \end{array}$$

with usual composition

Example 5.6. *Let* $\mathbf{C}$ *be a category with bundle products, i.e. for all morphisms* $u, v \to Y$, *and the universal square*

$$\begin{array}{ccc} X \times_Z Y & \longrightarrow & Y \\ \downarrow & & \downarrow v \\ X & \stackrel{u}{\longrightarrow} & Z \end{array}$$

exists. And let T *be the following diagram*

$$\bullet \leftarrow \bullet \to \bullet.$$

Then $T\mathbf{C}$ *be the category of diagrams in* $\mathbf{C}$ *of type* T. *Now we define the double category* $\mathbf{D}$ *with* $\mathbf{D}_0 = \mathbf{D}$ *and* $\mathbf{D}_1 = T\mathbf{C}$. *Two functors*

$$\mathcal{D}, \mathcal{R} : T\mathbf{C} \to \mathbf{C},$$

where the functor $\mathcal{D}$ *maps the diagram* $A \leftarrow M \to B$ *into the object* A, *the functor* $\mathcal{R}$ *maps this diagram into the object* B. *For two* 1*-level morphisms*

$\xi = (A \stackrel{\pi}{\leftarrow} M \stackrel{f}{\rightarrow} B) : A \Rrightarrow B$ and $\xi' = (B \stackrel{\pi'}{\leftarrow} M' \stackrel{f'}{\rightarrow} C) : B \Rrightarrow C$ we define their composition $\xi' \circ \xi = (A \stackrel{\pi \circ \pi_1}{\leftarrow} M \times_B M' \stackrel{f \circ \pi_2}{\rightarrow} C)$ where the bundle product is defined by the universal diagram

$$\begin{array}{ccc} M \times_B M' & \stackrel{\pi_2}{\rightarrow} & M' \\ \pi_1 \downarrow & & \downarrow \pi' \\ M & \stackrel{f}{\rightarrow} & B \end{array}$$

A 2-level morphism is a triple $\alpha = (u, v, w) : \xi \to \xi'$ from the following commutative diagram

$$\begin{array}{ccccc} M & \stackrel{f}{\rightarrow} & B & & \\ \pi \downarrow & \searrow v & & \searrow w & \\ A & & M' & \stackrel{f'}{\rightarrow} & B' \\ & \searrow u & \pi' \downarrow & & \\ & & A' & & \end{array}$$

with the evident composition.

Example 5.7. *Let us consider a multiplicative (tensor) category $(\mathbf{C}, \otimes, U, u)$ (see §4.1). Then we have the double category with $\mathbf{D}_1 = \mathbf{C}$, and $\mathbf{D}_0 = (*, *)$, i.e., a trivial category with one object and one morphism. The composition is*

$$\mathbf{D}_1 \times_{\mathbf{D}_0} \mathbf{D}_1 = \mathbf{C} \times \mathbf{C} \stackrel{\otimes}{\rightarrow} \mathbf{C}.$$

Let us consider it in more details. Let $(\mathbf{C}, \otimes, U, u)$ be a multiplicative (tensor) category with multiplication

$$\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C} : (X, Y) \mapsto X \otimes Y,$$

for the functor isomorphism of associativity

$$\varphi : \otimes \circ (id, \otimes) \to \otimes \circ (\otimes, id)$$

we write

$$\varphi_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$$

so the pentagon is commutative

$$\begin{array}{ccccc}
 & X \otimes (Y \otimes (V \otimes W)) & \stackrel{\varphi_{X,Y,V\otimes W}}{\longrightarrow} & & \\
 & id_X \otimes \varphi_{Y,V,W} \downarrow & & & \\
 & X \otimes ((Y \otimes V) \otimes W) & & & \\
\stackrel{\varphi_{X,Y,V\otimes W}}{\longrightarrow} & (X \otimes Y) \otimes (V \otimes W) & \stackrel{\varphi_{X\otimes Y,V,W}}{\longrightarrow} & & ((X \otimes Y) \otimes V) \otimes W \\
 & & & & \varphi_{X,Y,V} \otimes id_W \uparrow \\
 & \stackrel{\varphi_{X,Y,V\otimes W}}{\longrightarrow} & & & (X \otimes (Y \otimes V)) \otimes W
\end{array}$$

Then we have the double category $\mathbf{D}$ *with* $\mathbf{D}_0 = \mathbf{C}$ *and* $\mathbf{D}_1$ *such that*

$$Ob(\mathbf{D}_1) = \{(X, x) | A, B, X \in Ob(\mathbf{C}), \quad x : X \otimes A \to B\}.$$

So, we write $\xi = (X, x) : A \Rrightarrow B$ *and for* $\xi \in Ob(\mathbf{D}_1)$ *we denote* $\xi = (X_\xi, x_\xi)$, $\mathcal{D}(\xi) = A_\xi$, $\quad \mathcal{R}(\xi) = B_\xi$. *2-level morphisms*

$$\mathbf{D}_1(\xi, \xi') = \{(f_1, f_2, f_3) \mid \textit{commutative diagram} \begin{array}{ccc} X \otimes A & \stackrel{x}{\longrightarrow} & B \\ f_3 \otimes f_1 \downarrow & & \downarrow f_2' \\ X' \otimes A' & \stackrel{x'}{\longrightarrow} & B' \end{array} \}$$

and $\mathcal{D}(f_1, f_2, f_3) = f_1$, $\quad \mathcal{R}(f_1, f_2, f_3) = f_2$.

Composition $\mathbf{D}_1 \times_{\mathbf{D}_0} D_1 \to \mathbf{D}_1$ *is defined as follows:*

for $A \stackrel{\xi}{\Rrightarrow} B \stackrel{\xi'}{\Rrightarrow} B' \quad \xi \circ \xi' = (A, B', X'X, x'')$, *where* x'' *is the following composition*

$$(X' \otimes X) \otimes A \stackrel{\varphi^{-1}_{X',X,A}}{\longrightarrow} X' \otimes (X \otimes A) \stackrel{id_{X'} \otimes x}{\longrightarrow} X' \otimes B \stackrel{x'}{\longrightarrow} B'.$$

Associativity: For $A \stackrel{\xi}{\Rrightarrow} B \stackrel{\xi'}{\Rrightarrow} B' \stackrel{\xi''}{\Rrightarrow} B''$ *the left column gives* $x_{\xi''\circ(\xi'\circ\xi)}$, *the right column gives* $x_{(\xi''\circ\xi')\circ\xi}$

$$
\begin{array}{rcl}
(X'' \otimes (X' \otimes X)) \otimes A & & ((X'' \otimes X') \otimes X) \otimes A \\
\varphi^{-1}_{X'',X'\otimes X,A} \downarrow & & \varphi^{-1}_{X''\otimes X',X,A} \downarrow \\
X'' \otimes ((X' \otimes X) \otimes A) & & (X'' \otimes X') \otimes (X \otimes A) \\
id_{X''} \otimes \varphi^{-1}_{X',X,A} \downarrow & & id_{X''\otimes X'} \otimes x \downarrow \\
X'' \otimes (X' \otimes (X \otimes A)) & & (X'' \otimes X') \otimes B \\
id_{X''} \otimes (id_{X'} \otimes x) \downarrow & & \varphi^{-1}_{X'',X',B} \downarrow \\
X'' \otimes (X' \otimes B) & = & X'' \otimes (X' \otimes B) \\
id_{X''} \otimes x' \downarrow & & id_{X''} \otimes x' \downarrow \\
X'' \otimes B'' & = & X'' \otimes B' \\
x'' \downarrow & & x'' \downarrow \\
B'' & & B''
\end{array}
$$

So we have isomorphism

$(\varphi_{X'',X',X}, id_{A'}, id_{B'}) : \xi'' \circ (\xi' \circ \xi) \to (\xi'' \circ \xi') \circ \xi.$

5.4.2 B. Action of a double category

Double categories are categorical variants of usual monoids (and groups), and thus we have the corresponding variant for their actions. Below the definition of action of a double category $\mathcal{D}, \mathcal{R} : \mathbf{D}_1 \to \mathbf{D}_0$ on categories over $\mathbf{D}_0$ is given. Thus we get an analog of group-theoretical methods in categorical frames.

DEFINITION 5.7. *(Left)* **action of a double category** $\mathcal{D}, \mathcal{R} : \mathbf{D}_1 \to \mathbf{D}_0$ *on a category* $\mathcal{P} : \mathbf{M} \to \mathbf{D}_0$ *over* $\mathbf{D}_0$ *is a functor* φ *such that*

(1) The diagram

$$\begin{array}{ccc} \mathbf{D}_1 \times_{\mathbf{D}_0} M & \stackrel{\varphi}{\to} & \mathbf{M} \\ \mathcal{R}\circ\pi_1 & \searrow & \downarrow \mathcal{P} \\ & & \mathbf{D}_0 \end{array}$$

is commutative, where the bundle product $\mathbf{D}_1 \times_{\mathbf{D}_0} M$ *is defined by the diagram*

$$\begin{array}{ccc} \mathbf{D}_1 \times_{\mathbf{D}_0} M & \stackrel{\pi_2}{\to} & \mathbf{M} \\ \pi_1 \downarrow & & \downarrow \mathcal{P} \\ \mathbf{D}_1 & \stackrel{\mathcal{R}}{\to} & \mathbf{D}_0 \end{array}$$

(2) The diagram

$$\begin{array}{ccccc} (\mathbf{D}_1 \times_{\mathbf{D}_0} \mathbf{D}_1) \times_{\mathbf{D}_0} \mathbf{M} & \stackrel{\cong}{\to} & \mathbf{D}_1 \times_{\mathbf{D}_0} (\mathbf{D}_1 \times_{\mathbf{D}_0} \mathbf{M}) & \stackrel{id_{\mathbf{D}_1} \times_{\mathbf{D}_0} \varphi}{\longrightarrow} & \mathbf{D}_1 \times_{\mathbf{D}_0} \mathbf{M} \\ \otimes \times_{\mathbf{D}_0} id_{\mathbf{M}} \downarrow & & & & \downarrow \varphi \\ \mathbf{D}_1 \times_{\mathbf{D}_0} \mathbf{M} & & \stackrel{\varphi}{\longrightarrow} & & \mathbf{M} \end{array}$$

is commutative to an isomorphism and there exists a functor isomorphism φ *such that*

$$\forall\, \xi, \xi' \in Ob(\mathbf{D}_1),\ m \in Ob(\mathbf{M}_1) \quad \varphi_{\xi,\xi',m} : (\xi * \xi') * m \to \xi * (\xi' * m)$$

(3) For the unit functor we have a functor isomorphism $\chi : \varphi \circ (\mathcal{ID} \times id_{\mathbf{M}}) \widetilde{\longrightarrow} id_{\mathbf{M}}$ *or for objects*

$$\forall\, A \in Ob(\mathbf{D}_0),\ m \in Ob(\mathbf{M}_1) \quad \chi_{A,m} : \mathcal{ID}_A * m \widetilde{\longrightarrow} m$$

So we have the map of pair of objects $\xi \in Ob(\mathbf{D}_1)$, $m \in Ob(\mathbf{M})$ $(A \stackrel{\xi}{\Rightarrow} \mathcal{P}(m), m) \mapsto \varphi(\xi, m)$ such that $\mathcal{P}(\varphi(\xi, m)) = A$, and of morphisms $\alpha \in \mathbf{D}_1(\xi, \xi'), u \in \mathbf{M}(m, m')$

$$\begin{array}{cccccc} \xi & A & \stackrel{\xi}{\Rightarrow} & \mathcal{P}(m) & & \varphi(\xi, m) \\ \alpha \downarrow & \mathcal{F} = \mathcal{D}(\alpha) \downarrow & & \downarrow \mathcal{R}(\alpha) = \mathcal{P}(u) & \longmapsto & \downarrow \varphi(\alpha, u) \\ \xi' & A' & \stackrel{\xi'}{\Rightarrow} & \mathcal{P}(m') & & \varphi(\xi', m') \end{array}$$

and here $\mathcal{P}(\varphi(\alpha, u)) = \mathcal{F}$.

The definition of a **right action** is evident.

EXAMPLE 5.8. *Each double category* $\mathbf{D}$ *acts on itself one from left and from right by the composition* $*$.

EXAMPLE 5.9. *Let* $\mathbf{SC}$ *be a subcategory of morphisms of the category* $\mathbf{C}$ *such that for all* $(\pi : M \to B) \in Ob(\mathbf{SC})$ *and all* $(f : B' \to B) \in Mor(\mathbf{C})$ *there exists the universal square*

$$\begin{array}{cccc} f^*M = & B' \times_B M & \stackrel{f'}{\longrightarrow} & M \\ & \pi_1 \downarrow & & \downarrow \pi \\ & B' & \stackrel{f}{\longrightarrow} & B \end{array}$$

and $\Pi : \mathbf{SC} \to \mathbf{C}$ *is the projection on base, i.e.* $\Pi : (\pi : M \to B) \longmapsto B$. *Then we have the natural action of* $Mor(\mathbf{C})$ *on* $\mathbf{SC}$

$$Mor(\mathbf{C}) \times_C \mathbf{SC} \longrightarrow \mathbf{SC}$$

which maps the pair $((f : B' \to B), (\pi : M \to B))$ *to* $(\pi_1 : f^*M \to B')$.

EXAMPLE 5.10. *Let* $\mathbf{mod}_k$ *be the category of left modules over* k*-algebras and* $\Pi : \mathbf{mod}_k \to \mathbf{Alg}_k$ *be the natural projection, which an object* (A, M) *maps to* A. *There is natural action of* $\mathbf{Alg}_k$ *on* $\mathbf{mod}_k$

$$(\mathbf{Alg}_k)_1 \times_{(\mathbf{Alg}_k)_0} \mathbf{mod}_k \to \mathbf{mod}_k$$

such that for $\xi = N : A \Rightarrow B$ *and* B*-module* M

$$\xi^*(B, M) = (A, N \otimes_B M).$$

EXAMPLE 5.11. **Characteristic classes**. *Let a double category* $\mathbf{G}$ *act on* $\mathcal{P} : \mathbf{M} \to \mathbf{G}_0$, *and there is a contravariant functor* $\mathcal{H} : \mathbf{G}^\circ \to \mathbf{Mor}(\mathbf{M})$. *A characteristic class of* $m \in Ob(\mathbf{M})$ *is* $c(m) \in \mathcal{H}_0(\mathcal{P}(m))$, *such that for all* $\xi : \mathcal{P}(m') \Rightarrow \mathcal{P}(m)$ *we have*

$$c(\xi^* m) = \mathcal{H}_1(\xi)(c(m)).$$

Example 5.12. **Equivariant functors**. *Let* $\mathbf{M}$ *be a category of manifolds (topological or smooth),* $\mathbf{L}$ *be a category of locally trivial bundles over objects of* $\mathbf{M}$. *Then* $\mathbf{Mor}(\mathbf{M})$ *acts on* $\mathbf{L}$. *Let* G *be a topological group,* $\mathbf{M}_G$ *be the category of* G*-manifolds,* $\mathbf{P}$ *is the category of principal bundles with structure group* G *over objects of* $\mathbf{M}$. *The functor*

$$\mathbf{P} \times \mathbf{M}_G \to \mathbf{L}$$

which maps (η, F) *to the fiber bundle* $\eta[F]$ *with fiber* F. *This functor is equivariant relatively of action* $\mathbf{Mor}(\mathbf{M})$ *on* $\mathbf{P}$ *and* $\mathbf{L}$.

Example 5.13. *Let* $\mathbf{ISO}(\mathbf{C})$ *be a double subcategory of* $\mathbf{Mor}(\mathbf{C})$ *such that* $(\mathbf{ISO}(\mathbf{C}))_0 = \mathbf{C}$ *and* $(\mathbf{ISO}(\mathbf{C}))_1$ *is the full subcategory of* $(\mathbf{Mor}(\mathbf{C}))_1$ *with*

$$Ob((\mathbf{ISO}(\mathbf{C}))_1) = \{f \in Mor(\mathbf{C}) \quad | \quad f \text{ is an isomorphism } \}.$$

For any forgetful functor $\mathcal{F} : \mathbf{C}' \to \mathbf{C}$ *(see the next subsection) the double category* $\mathbf{ISO}(\mathbf{C})$ *acts on* $\mathbf{C}'$ *from the left*

$$\mathbf{ISO}(\mathbf{C}) \times_{\mathbf{C}} \mathbf{C}' \to \mathbf{C}' : (u : B \to F(C), C) \mapsto u^* C$$

and $w^*(u^*C) \cong (u \circ w)^* C$.

5.4.3 C. Cobordism and double categories

Let $\mathbf{M}_d$ be the category of oriented compact d-dimensional smooth manifolds (with boundary) and piecewise smooth maps (the sense of the condition we do not define more exactly here; this may be such continuous maps $f : M \to Y$ that are smooth on a dense open subset $U_f \subset M$) , let $\mathbf{CM}_d$ be its subcategory of closed (with empty boundary) manifolds and smooth maps, $\mathbf{CM}_d \subset \mathbf{M}_d$.

There are the following functors:

(1) Disjoint union

$$\cup : \mathbf{M}_d \times \mathbf{M}_d \to \mathbf{M}_d : (X, Y) \mapsto X \cup Y.$$

(2) Changing of the orientation of manifolds on opposite

$$(-) : \mathbf{M}_d \to \mathbf{M}_d : X \mapsto -X.$$

(3) Boundary operator

$$\partial : \mathbf{M}_{d+1} \to \mathbf{CM}_d : X \mapsto \partial X.$$

(4) Multiplication on the unit segment $I = [0, 1]$

$$I\times \ : \mathbf{CM}_d \to \mathbf{M}_{d+1} : X \mapsto I \times X.$$

Now we define a double category $\mathbf{C}(d)$ with

(1) $\mathbf{C}(d)_0 = \mathbf{CM}_d$.

(2) The 1-level morphisms $\mathbf{C}(d)_{(1)}(X, X')$ is a set of pairs (Y, f) where Z is oriented compact $(d+1)$-dimensional smooth manifold with the boundary ∂Y and f is an diffeomorphism

$$f : (-X) \cup X' \to \partial Y,$$

where $\cup$ notes the disjoint union of $-X$ and X'. Thus we write $(Y, f) : X \Rightarrow X'$.

(3) The composition of $(Y, f) : X \Rightarrow X'$ and $(Y', f') : X' \Rightarrow X''$ is the morphism

$$(Y \cup_{X'} Y', (f|_X) \cup (f'|_{X'})) : X \Rightarrow X'',$$

where $(Y \cup_{X'} Y')$ denotes the union $(Y \cup Y')$ after identification of each point $f(y) \in f(Y)$ with the point $f'(y) \in f'(Y)$ for all $y \in Y$ and smoothing this topological manifold.

(4) The 1-level identical morphism $\mathcal{ID}_X$ is $(X \times [0; 1], id_{(-X)\cup X})$, because $\partial(X \times [0; 1]) = (-X) \cup X$.

(5) The 2-level morphisms of $\mathbf{C}(d)_1(\xi, \xi')$ from $\xi = (Y, f : X' \cup (-X) \to \partial Y) : X \Rightarrow X'$ to $\xi' = (Y', f' : X'' \cup (-X') \to \ \partial Y') : X' \Rightarrow X''$ are such triples of smooth maps (f_1, f_2, f_3) that the following diagram is commutative

$$\begin{array}{ccccc} (-X) \cup X' & \xrightarrow{f} & \partial Y & \subset Y \\ \downarrow f_1 \cup f_2 & & & \downarrow f_3 \\ (-X') \cup X'' & \xrightarrow{f'} & \partial Y' & \subset Y' \end{array}$$

It easy to see that functors $\cup$ and $(-)$ may be expanded to double category functors

$$\cup : \mathbf{C}(d) \to \mathbf{C}(d),$$
$$(-) : \mathbf{C}(d) \to \mathbf{C}(d)^\circ$$

and $(-)$ is an equivalence of the double categories.

Remark. The appearance of the following two formulas for 1-level morphisms is interesting in algebras and cobordisms

$$f : A \otimes_k B^\circ \to End_k(N) \qquad f : (-X) \cup Y \to \partial Z,$$

where we have correspondence between the functors

$$\begin{array}{ccc} (\_)^\circ & \longleftrightarrow & -(\_), \\ \otimes_k & \longleftrightarrow & \cup, \\ End_k & \longleftrightarrow & \partial. \end{array}$$

5.5 Topological quantum field theory

Topological quantum field theory (TQFT) is a recent development in the interface between physics and mathematics. The mathematical interest in them comes from the hope that they will disclose new phenomena, or at least offer some efficient organization of previously studied invariants like the Jones invariants of links, or the Donaldson invariants of 4-manifolds.

The physical interest comes from their value as examples of categories derived from the representation category of a Hopf algebra in which extensive calculations are possible (see, for example, [7]).

In this case TQFT is a functor $\mathcal{Z}$ from the category $\mathbf{CM}(d)$ of d-dimensional manifolds to the category $\mathbf{H}$ of (usually Hermitian) finite dimensional vector spaces and some axioms are satisfied (see [7,19]). Actualy, the **functor** $\mathcal{Z}$ is a functor between double categories.

Thus, topological quantum field theory in dimension d is a functor

$$\mathcal{Z} : \mathbf{C}(d) \to \mathbf{Mor}(\mathbf{H}),$$

between double categories such that:

(1) the disjoint union in $\mathbf{C}(d)$ go to the tensor product

$$\cup \quad \mapsto \otimes,$$

where $(\_)^* : \mathbf{H} \to \mathbf{H}^\circ$ is the dualization of vector spaces.

(2) changing of orientation in $\mathbf{C}(d)_0$ go to dualization

$$(-) \mapsto (.)^*$$

Thus, as consequence of double categorical functorial properties, we get

(1) for each compact closed oriented smooth d-dimensional manifold $X \in Ob(\mathbf{C}(d)_0)$ the value of the functor $\mathcal{Z}(X)$ is a finite dimensional vector space over the field $\mathbb{C}$ of the complex numbers (usually with Hermitian metric),

(2) for each $(Y, f) : X \Rightarrow X'$ from $Ob(\mathbf{C}(d)_1)$ the value of the functor $\mathcal{Z}(Y, f)$ is a homomorphism $\mathcal{Z}(X) \to \mathcal{Z}(X')$ of (Hermitian) vector spaces,

and the following well known axioms of topological quantum field theory are satisfied:

A(1) (*involutivity*). $\mathcal{Z}(-X) = \mathcal{Z}(X)^*$, where $-X$ denotes the manifold with opposite orientation, and $*$ denotes the dual vector space.

A(2) (*multiplicativity*). $\mathcal{Z}(X \cup X') = \mathcal{Z}(X) \otimes \mathcal{Z}(X')$, where $\cup$ denotes disconnected union of manifolds.

A(3) (*associativity*). For the composition $(Y'', f'') = (Y, f) * (Y', f')$ of cobordisms have to be

$$\mathcal{Z}(Y'', f'') = \mathcal{Z}(Y', f') \circ \mathcal{Z}(Y, f) \in Hom_{\mathbf{C}}(\mathcal{Z}(X), \mathcal{Z}(X'')).$$

(Usually the identifications

$$\mathcal{Z}(X' - X) \cong \mathcal{Z}(X)^* \otimes \mathcal{Z}(X') \cong Hom_{\mathbf{C}}(\mathcal{Z}(X), \mathcal{Z}(X'))$$

allow us to identify $\mathcal{Z}(Y, f)$ with the element $\mathcal{Z}(Y, f) \in \mathcal{Z}(\partial Y)$.

A(4) For the initial object $\emptyset \in Ob(\mathbf{C}(d)_0)$ $\quad \mathcal{Z}(\emptyset) = \mathbf{C}$.

A(5) (*trivial homotopy condition*). $\mathcal{Z}(X \times [0, 1]) = id_{\mathcal{Z}(X)}$.

TQFTs are interesting as mathematical structures, but it is hard to see them playing direct roles as models of serious physical theories. The application of TQFT ideas to the theory of modern physics has not been carried very far, as yet (see, for example, [20–23]), but it would be interesting to see what can be achieved in this area).

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