CATEGORIES of the BASIC SANTILLI'S ISOSTRUCTURES*

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Abstract

The objects of modern iso–Geometry, iso-spaces, like all objects of modern iso-Mathematics, are sets of elements of arbitrary nature endowed with some mathematical iso-structure, for example, iso-symmetry etc.

The typical way to think about iso-symmetry is with the concept of a "iso-group". But to get a concept of iso-symmetry that's really up to the demands put on it by modern iso-Mathematics, we need — at the very least — to work with a "category" of iso-symmetries, rather than a iso-group of iso-symmetries. In this article we construct Santilli isofunctor for mathematical iso-structures. The compositions of morphisms between Santilli isofunctors are built in terms of diagrams and equations. There are obtained categories of iso-groups, iso-rings, iso-fields, vector iso-spaces, iso-algebras, etc.

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1 Santilli iso-functor

The most important mathematical iso-structures are geometric, algebraic and topological ones [1–5].

Santilli's central idea [6, 7] is the generalization of the fundamental unit of the number theory, from its trivial n-dimensional form $I = \mathrm{diag}(1, 1, \ldots)$ to an n-dimensional matrix $\hat{I}$ with the general dependence of all essential variables

$$I = \mathrm{diag}(1, 1, \ldots) \Rightarrow \hat{I} = \hat{I}(s, x, \dot{x}, \ddot{x}, \psi, \partial\psi, \partial\partial\psi, \mu, \tau, \ldots) \tag{1}$$

under the condition of preserving the original axioms of the unit (nondegeneracy, hermiticity, and positive-definiteness).

The "lifting" $I \Rightarrow \hat{I}$ requires, naturally, for necessary compatibility, a generalization of the conventional associative multiplication $x \circ y$ into the so-called isomultiplication

$$x \circ y \Rightarrow x \,\hat{\circ}\, y := xTy\,, \quad T = \text{fixed}\,, \tag{2}$$

where the quantity T is called the *isotopic* element. Then $\hat{I} = T^{-1}$ is a correct left and right unit element of the theory with respect the new multiplication $\hat{\circ}$ and it is called the *isounit*.

Definition 1.1. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two categories [8]. A covariant* **Santilli iso-functor** *from* $\mathbf{X}$ *to* $\mathbf{Y}$ *is a family of isofunctions* $\hat{I}$ *[9] which associates to each object* A *in* $\mathbf{X}$ *an object* $\hat{I}A$ *in* $\mathbf{Y}$ *and to each morphism* $f \in \mathrm{Hom}_{\mathbf{X}}(A, B)$ *a morphism* $\hat{I}f \in \mathrm{Hom}_{\mathbf{Y}}(\hat{I}A, \hat{I}B)$*, and which is such that:*

(FI) $\hat{I}(g \circ f) = \hat{I}g \circ \hat{I}f$ *for all* $f \in \mathrm{Hom}_{\mathbf{X}}(A, B)$ *and* $g \in \mathrm{Hom}_{\mathbf{Y}}(B, C)$*;*

(F2) $\hat{I}\,\mathrm{id}_A = \mathrm{id}_{\hat{I}A}$ *for all* $A \in \mathrm{Ob}\,(\mathbf{X})$*.*

It is clear from the above that a covariant functor is a transformation that preserves both:

- The domains and the codomains identities.

- The composition of arrows, in particular it preserves the direction of the arrows.

DEFINITION 1.2. *Let* **X** *and* **Y** *be two categories. A contravariant* **Santilli iso-functor** *from* **X** *to* **Y** *is a family of isofunctions* $\hat{F}$ *which associates to each object* A *in* **X** *an object* $\hat{F}A$ *in* **Y** *and to each morphism* $f \in \mathrm{Hom}_{\mathbf{X}}(A, B)$ *a morphism* $\hat{F}f \in \mathrm{Hom}_{\mathbf{Y}}(\hat{F}A, \hat{F}B)$*, and which is such that:*

(FI) $\hat{F}(g \circ f) = \hat{F}f \circ \hat{F}g$ *for all* $f \in \mathrm{Hom}_{\mathbf{X}}(A, B)$ *and* $g \in \mathrm{Hom}_{\mathbf{Y}}(B, C)$*;*
(F2) $\hat{F}\,\mathrm{id}_A = \mathrm{id}_{\hat{F}}A$ *for all* $A \in \mathrm{Ob}\,(\mathbf{X})$.

Thus, a contravariant Santilli functor in mapping arrows from one category to the next reverses the directions of the arrows, by mapping domains to codomains and vice versa. A contravariant Santilli functor is also called a Santilli presheaf. These types of Santilli functors will be the principal objects which we will study when discussing Santilli quantum isotheory in the language of topos theory.

DEFINITION 1.3. *Given two categories* C *and* D*, the collection of all covariant (or contravariant) Santilli functors* $F : C \to D$ *is actually a category which will be denoted as* D^C*. This is called the category of Santilli functors and has as objects covariant (or contravariant) Santilli functors and as map natural transformations between Santilli functors.*

2 Categories of Groups and iso-Groups

One of the simplest algebraic structures is the structure of a group [7,8]. A set G of elements of any kind is said to be a *group* if a group operation $a \circ b$ is defined in it satisfying the following axioms:

$G.1^\circ$ For any two elements a and b there exists an element

$$c = a \circ b \tag{3}$$

$G.2^\circ$ This operation is associative, that is, for any three elements a, b, c,

$$(a \circ b) \circ c = a \circ (b \circ c). \tag{4}$$

$G.3^\circ$ There exists a *neutral element* e, i. e. an element such that for every element a,

$$a \circ e = e \circ a = a. \tag{5}$$

$G.4^\circ$ For each element a there exists a *symmetric element* $\bar{a}$ such that

$$a \circ \bar{a} = \bar{a} \circ a = e. \tag{6}$$

If the group operation $a \circ b$ is called *addition*, we write $c = a + b$ and the element c is called the *sum*, the neutral element is called *zero* and is written as 0, the symmetric element is called the *opposite* and is written as $-a$, and the group is called *additive*.

If the group operation $a \circ b$ is called *multiplication*, we write $c = a \cdot b$, or $c = ab$, the element c is called the *product*, the neutral element is called *unit* and is written as 1, the symmetric element is called the *inverse* and is written as a^{-1}, and the group is called *multiplicative*.

If the group satisfies in addition the axiom

$G.5^\circ$. For any two elements a and b

$$a \circ b = b \circ a, \tag{7}$$

then the group is called *commutative* or *Abelian*.

A set of elements endowed with an operation $a \circ b$ without the properties $G.2^\circ$, $G.3^\circ$ and $G.4^\circ$ is called a *magma*. A magma with the property $G.3^\circ$ is called a *unital magma*, a magma with the property $G.2^\circ$ is called a *semigroup*. A magma in which the equations $a \circ x = b$ and $x \circ a = b$ are solvable for all a and b is called a *quasigroup*. A unital semigroup is called a *monoid*, a unital quasigroup is called a *loop*. All these structures (as also the ones to be introduced yet) are termed infinite, respectively finite, if the underlying set is infinite, respectively finite.

Definition 2.1. *A map $f : G \longrightarrow G'$ between two groups $(G, \circ)$ and $(G', \Box)$ is called* **homomorphism**, *if the following property holds:*

$$(\forall a, b \in G)[f(a \circ b) = f(a) \Box f(b)] \tag{8}$$

Thus a homomorphism "carries" the composition law $\circ$ on G to the composition law $\Box$ on G'. Homomorphisms of groups are well visualized in some important aspects with the help of two concepts, the image Im(f) and the kernel Ker(f) of the homeomorphism.

Definition 2.2. *If $f : G \longrightarrow G'$ is a group homomorphism, then we define:*

$$a)\mathrm{Im(f)} = f(a)/a \in G \tag{9}$$

$$b)\mathrm{Ker(f)} = a \in G/f(a) = e' \in G'. \tag{10}$$

It is well known that Im(f) is a subgroup of G' and Ker(f) is a subgroup of G.

Definition 2.3. *A homomorphism f between two groups G and G' is called* **isomorphism** *if f is bijective. In the case where $G = G'$ an homomorphism f is called* **endomorphism** *and an isomorphism is called* **automorphism**.

Definition 2.4. **Grp** *is the category with groups as objects and homomorphisms as morphisms.*

Definition 2.5. *Let* **Grp** *and* **IsoGrp** *be two categories.*
A Santilli iso-functor $\hat{\mathcal{I}}$ from **Grp** *associates to each object A in* **Grp** *category an object $\hat{I}A$ in* **IsoGrp** *and to each morphism $f \in \mathrm{Hom}_{\mathbf{Grp}}(A, B)$ a morphism $\hat{I}f \in \mathrm{Hom}_{\mathbf{IsoGrp}}(\hat{I}A, \hat{I}B)$, i.e., we reconstruct the elements of the* **category IsoGrp** *as*

$$\mathbf{Grp} \ni A \longrightarrow \hat{A} \equiv A\hat{I} \in \mathbf{IsoGrp}, \tag{11}$$

where the isounit $\hat{I}$ is defined with the help of an invertible element [6]

$$T : \hat{I} = T^{-1}, \tag{12}$$

called isotopic element, and the new composition law is defined by

$$(\forall\, \hat{A}, \hat{B} \in \mathbf{IsoGrp})[\hat{A} \hat{\circ} \hat{B} \equiv \hat{A} T \hat{B}]. \tag{13}$$

It can be proved easily, that **IsoGrp** for the fixed isotopic element T, with the above internal composition, can become a iso-group $\hat{G}$ with unit $\hat{I}$ [6]. The notions of isomorphism, iso-isomorphism, etc, can be defined in a similar way as above.

DEFINITION 2.6. *Let* $\hat{\mathcal{I}} : \mathbf{Grp} \to \mathbf{IsoGrp}$ *and* $\hat{\mathcal{I}}' : \mathbf{Grp} \to \mathbf{IsoGrp}$ *be two functors. A* **natural transformation** $\alpha : \hat{\mathcal{I}} \to \hat{\mathcal{I}}'$ *is given by the following data:*

For every object A *in* **Grp** *there is a morphism* $\alpha_A : \hat{\mathcal{I}}(A) \to \hat{\mathcal{I}}'(A)$ *in* **IsoGrp** *such that for every morphism* $f : A \to B$ *in* **Grp** *the following diagram is commutative*

$$\begin{array}{ccc} \hat{\mathcal{I}}(A) & \xrightarrow{\alpha_A} & \hat{\mathcal{I}}'(A) \\ \hat{\mathcal{I}}(f)\downarrow & & \downarrow \hat{\mathcal{I}}'(f) \\ \hat{\mathcal{I}}(B) & \xrightarrow{\alpha_B} & \hat{\mathcal{I}}'(B). \end{array}$$

Commutativity means (in terms of equations) that the following compositions of morphisms are equal: $\hat{\mathcal{I}}(f) \circ \alpha_A = \alpha_B \circ \hat{\mathcal{I}}'(f)$.

The morphisms α_A, $A \in \mathrm{Ob}(\mathbf{Grp})$, are called the *components of the natural transformation* α.

3 Categories of iso-rings and iso-fields

3.1 IsoRng category

Groups are algebraic systems with one internal composition law. More complicated (and hence, richer) systems are obtained if we introduce a second internal composition law, which is related to the first.

If, in a set of elements of any kind two operations $a+b$ and ab are defined such that

$R.l°$ The set is a commutative group with respect to the operation $a+b$;

$A.2°$ The set is a semigroup with respect to the operation ab;

$R.3^\circ$ The operation ab is distributive with respect to the operation $a+b$:

$$a(b+c) = ab + ac, \qquad (a+b)c = ac + bc, \tag{14}$$

the set is called a *ring*.

The ring $\mathbb{Z}$ of integers is a ring with 1, the rings $2\mathbb{Z}$ of even integers, $3\mathbb{Z}$ of integers of the form $3n$, n from $\mathbb{Z}$, and so on, are rings without 1.

DEFINITION 3.1. *A map $f : R \longrightarrow R'$ between two rings $R(+,\star)$ and $R'(\Box,\circ)$ is called* **ring homomorphism***, if it preserves both composition laws, i.e, if the following property holds:*

$$(\forall a, b \in R)[f(a+b) = f(a)\Box f(b) \wedge f(a \star b) = f(a) \circ f(b)] \tag{15}$$

DEFINITION 3.2. *A homomorphism f between two rings R and R' is called* **isomorphism** *if f is bijective. In the case where $R = R'$, a homomorphism f is called* **endomorphism** *and an isomorphism is called* **automorphism**.

DEFINITION 3.3. *If $f : R \longrightarrow R'$ is a ring homomorphism, then we define*

$$i)\ Im(f) = f(a)\ /\ a \in R, \tag{16}$$

$$ii)\ Ker(f) = a \in R\ /\ f(a) = e' \in R' \tag{17}$$

where e' is the identity element of the ring R'.

DEFINITION 3.4. **Rng** *is the category with rings as objects and ring homomorphisms as morphisms.*

DEFINITION 3.5. *Let* **Rng** *and* **IsoRng** *be two categories. A Santilli isofunctor $\hat{\mathcal{I}}$ from* **Rng** *associates to each object A in* **Rng** *category an object $\hat{I}A$ in* **IsoRng** *and to each morphism $f \in \mathrm{Hom}_{\mathbf{Rng}}(A,B)$ a morphism $\hat{I}f \in \mathrm{Hom}_{\mathbf{IsoRng}}(\hat{I}A, \hat{I}B)$, i.e., we reconstruct the elements of the* **category IsoRng** *as*

$$\mathbf{Rng} \ni A \longrightarrow \hat{A} \equiv A\hat{I} \in \mathbf{IsoRng}, \tag{18}$$

where the isounit $\hat{I}$ is defined with the help of an isotopic element [6]

$$T : \hat{I} = T^{-1}, \tag{19}$$

the first composition law (which is often called addition) is kept unchanged and the second composition law (which is often called multiplication) is defined by

$$(\forall\, \hat{A}, \hat{B} \in \mathbf{IsoRng})[\hat{A}\hat{\star}\hat{B} \equiv \hat{A}T\hat{B}]. \tag{20}$$

It can be proved easily, that **IsoRng** for the fixed isotopic element T, with the above internal composition, can become a iso-ring $\hat{R}$ with isounit $\hat{I}$ [6]. The notions of isomorphism, iso-isomorphism, etc, between two iso-rings can be defined in a similar way as above.

3.2 IsoField category

A ring in which the set of elements without 0 is a commutative group with respect to the operation ab is called a *field*.

DEFINITION 3.6. *A* **field** $\mathbb{F}$ *is a commutative ring with unit where every element (except zero), is invertible. More precisely a field* $\mathbb{F}(+, \cdot)$ *is:*
1) An abelian group with respect to an internal operation, which is usally denoted with + and called addition and
2) is equipped with a second internal operation, denoted with · and called multiplication, so that the following rules hold

$$i)\ (\forall\ \alpha, \beta, \gamma \in \mathbb{F})\ \ [\alpha(\beta\gamma) = (\alpha\beta)\gamma], \tag{21}$$

$$ii)\ (\exists\ 1 \in \mathbb{F})\ (\forall\ \alpha \in \mathbb{F})\ \ [\alpha 1 = 1\alpha = \alpha], \tag{22}$$

$$iii)\ (\forall\ \alpha \in \mathbb{F})\ \ (\exists\ \alpha^{-1} \in \mathbb{F})\ \ [\alpha\alpha^{-1} = \alpha^{-1}\alpha = 1], \tag{23}$$

$$iv)\ (\forall\ \alpha, \beta \in \mathbb{F})\ \ [\alpha\beta = \beta\alpha], \tag{24}$$

$$v)\ (\forall\ \alpha, \beta, \gamma \in \mathbb{F})\ \ [\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma]. \tag{25}$$

A ring in which the set of elements without 0 is a noncommutative group with respect to the operation ab is called a *skew field.*

Definition 3.7. *A map* $f : \mathbb{F} \longrightarrow \mathbb{F}'$ *between two fields* $\mathbb{F}(+,\cdot)$ *and* $\mathbb{F}'(\Box,\circ)$ *is called* **field homomorphism**, *if it preserves both composition laws, i.e, if the following property holds:*

$$(\forall a, b \in \mathbb{F})[f(a+b) = f(a)\Box f(b) \wedge f(a\cdot b) = f(a)\circ f(b)] \tag{26}$$

In the following we shall denote a field for brevity with $\mathbb{F}$ instead of $\mathbb{F}(+,\cdot)$.

The ring $\mathbb{Q}$ of rational numbers, the ring $\mathbb{R}$ of real numbers, the ring $\mathbb{C}$ of complex numbers, the ring $\mathbb{F}_p$ of residues modulo a prime integer p, and the *Galois field* $\mathbb{F}_q$ $(q = p^k)$ obtained from $\mathbb{F}_p$ by adjoining the roots of an irreducible algebraic equation of degree k over $\mathbb{F}_p$, are fields. The ring $\mathbb{H}$ of *quaternions* is a skew field. The fields $\mathbb{F}_p$ and $\mathbb{F}_q$ are finite and contain p and q= p^k elements, respectively. The rings $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$, and $\mathbb{H}$ are infinite, and the three last fields are *continuous* (*topological*). A field $\mathbb{Q}(\alpha, \beta, \ldots)$ of algebraic numbers, obtained from $\mathbb{Q}$ by adjoining the roots $\alpha, \beta, \ldots$ of an irreducible algebraic equation over $\mathbb{Q}$, and the field $\mathbb{Q}_p$, of *p-adic numbers*, which is also an extension of $\mathbb{Q}$, are also infinite, and the last field is continuous.

Definition 3.8. **Field** *is the category with fields as objects and field homomorphisms as morphisms.*

Definition 3.9. *Let* **Field** *and* **IsoField** *be two categories. A Santilli isofunctor* $\hat{\mathcal{I}}$ *from* **Field** *associates to each object* A *in* **Field** *category an object* $\hat{I}A$ *in* **IsoField** *and to each morphism* $f \in \mathrm{Hom}_{\mathbf{Field}}(A,B)$ *a morphism* $\hat{I}f \in \mathrm{Hom}_{\mathbf{IsoField}}(\hat{I}A, \hat{I}B)$, *i.e., we reconstruct the elements of the* **category IsoField** *as*

$$\mathbf{Field} \ni A \longrightarrow \hat{A} \equiv A\hat{I} \in \mathbf{IsoField}, \tag{27}$$

where the isounit $\hat{I}$ *is defined with the help of an isotopic element [6]*

$$T : \hat{I} = T^{-1}, \tag{28}$$

the first composition law (which is often called addition) is kept unchanged and the second composition law (which is often called multiplication) is defined by

$$(\forall\, \hat{A}, \hat{B} \in \mathbf{IsoField})[\hat{A} \hat{\cdot} \hat{B} \equiv \hat{A} T \hat{B}]. \tag{29}$$

4 Subcategories, subisogroups, subisorings

Subsets of categories, isogroups and isoringsthat are also categories, isogroups and isorings respectively, are called *subcategories*, *subisogroups* and *subisorings*, respectively.

Definition 4.1. *A category* **D** *is called a* **subcategory** *of a category* **C** *if* Ob (**D**) ⊆ Ob (**C**)*,* Mor (**D**) ⊆ Mor (**C**)*, and morphism composition in* **D** *coincide with their composition in* **C***.*

A bijection between two sets endowed with algebraic structures which preserves the operations of these structures is called an *isomorphism*, sets between which there is an isomorphism are called *isomorphic*.

A surjection or injection between two sets endowed with algebraic structures which preserves the operations is called a *homomorphism*, sets between which there is a homomorphism are called *homomorphic*.

If two isogroups $\hat{G}$ and $\hat{H}$ are homomorphic, the elements of $\hat{G}$ corresponding to the neutral element of $\hat{H}$ form a subisogroup $\hat{N}$ of $\hat{G}$, such a subisogroup is called an *invariant subisogroup* or *normal subisogroup*. In this case the isogroup isomorphic to $\hat{H}$ is called the *quotient isogroup* and is written as $\hat{G}/\hat{N}$.

If two isorings $\hat{\mathcal{R}}$ and $\hat{\mathcal{S}}$ are homomorphic, the elements of $\hat{\mathcal{R}}$ corresponding to the zero in $\hat{\mathcal{S}}$ form a subisoring $\hat{\mathcal{J}}$ of $\hat{\mathcal{R}}$ called an *ideal*. In this case the isoring isomorphic to $\hat{\mathcal{S}}$ is called the *quotient isoring* and is written as $\hat{\mathcal{R}}/\hat{\mathcal{J}}$.

Isomorphisms of a isogroup or a isoring onto itself are called *isogroup* or *isoring automorphisms*, respectively; homomorphisms of a isogroup or a isoring into itself are called *isogroup* or *isoring endomorphisms*, respectively.

5 Direct sums and products

If $\hat{G}$ and $\hat{H}$ are two isogroups with operations $\hat{g}_1\hat{\circ}\hat{g}_2$ and $\hat{h}_1\hat{\circ}\hat{h}_2$, the pairs $(\hat{g}, \hat{h})$ with operation $(\hat{g}_1, \hat{h}_1)\hat{\circ}(\hat{g}_2, \hat{h}_2) = (\hat{g}_1\hat{\circ}\hat{g}_2, \hat{h}_1\hat{\circ}\hat{h}_2)$ form a isogroup. If $\hat{G}$ and $\hat{H}$ are both additive, respectively multiplicative, isogroups, this isogroup is called a *direct sum*, respectively *direct product*, and is written as $\hat{G}\hat{\oplus}\hat{H}$, respectively $\hat{G}\hat{\otimes}\hat{H}$.

The direct sum $\hat{\mathcal{R}}\hat{\oplus}\hat{\mathcal{S}}$ of rings $\hat{\mathcal{R}}$ and $\hat{\mathcal{S}}$ is similarly defined.

6 Vector isospaces

A set L^n of elements of any kind, called *vectors*, is said to be an *n-dimensional vector space* if in this set the operations of *addition* and of *multiplication by scalars*, that is real numbers, are defined, satisfying:

$VI.1°$ - $5°$ Addition of vectors satisfies axioms $G.1-5°$ for a commutative group;

$VII.1°$ For any vector a and any scalar λ there exists a vector

$$\mathbf{b} = \mathbf{a} \cdot \lambda = \mathbf{a}\lambda \tag{30}$$

called the product of $\mathbf{a}$ by λ;

$VII.2°$ Multiplication by 1 does not change a vector:

$$\mathbf{a} \cdot 1 = \mathbf{a}; \tag{31}$$

$VII.3°$ Multiplication of vectors by scalars is distributive with respect to addition of scalars:

$$\mathbf{a}(\lambda + \mu) = \mathbf{a}\lambda + \mathbf{a}\mu; \tag{32}$$

$VII.4°$ Multiplication of vectors by scalars is distributive with respect to addition of vectors:

$$(\mathbf{a} + \mathbf{b})\lambda = \mathbf{a}\lambda + \mathbf{b}\lambda; \tag{33}$$

$VII.5°$ Multiplication of vectors by scalars is associative:

$$(\mathbf{a}\lambda)\mu = \mathbf{a}(\lambda\mu); \tag{34}$$

and *axioms* $VIII.1°$ - $2°$ *of dimension*, which are based on the notions of linear independence and dependence of vectors. Vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ are said to be *linearly independent* if a linear combination $\mathbf{a}_1\lambda_1 + \mathbf{a}_2\lambda_2 + \ldots + \mathbf{a}_m\lambda_m$ is equal to zero only if all coefficients $\lambda_i = 0$, and *linearly dependent* if there are nonzero coefficients λ_i such that this linear combination is equal to zero.

$VIII.1°$ There exist n linearly independent vectors;

$VIII.2°$ Any $n+1$ vectors are linearly dependent.

If we have chosen n linearly independent vectors $\mathbf{e}_1\,\mathbf{e}_2, \ldots, \mathbf{e}_n$ in L_n, then each vector can be written as

$$\mathbf{x} = \sum_i \mathbf{e}_i x^i = \mathbf{e}_i x^i. \tag{35}$$

The numbers x^i are called the *coordinates* of the vector $\mathbf{x}$, the vectors $\mathbf{e}_i$ are called *basis vectors.* Later we will write the sums (35) only in the last form and when in our formulas the same upper and lower indices appear we will always mean summation with respect to these indices.

DEFINITION 6.1. *A subset U of a vector space V is called* **vector subspace** *if it is a subsystem which obeys the axioms of vector space in itself, that is U is closed under vector addition and scalar multiplication.*

DEFINITION 6.2. *The notions of* Ker(f) *and* Im(f) *are defined by the relations*

$$a)\ \mathrm{Ker(f)} = [x \in V\ /\ f(x) = 0 \in U], \tag{36}$$

$$b)\ \mathrm{Im(f)} = [f(x) \in U\ /\ x \in V]. \tag{37}$$

It easy proved that Ker(f) and Im(f) are subspaces of V and U respectively.

DEFINITION 6.3. *Let V and U two vector spaces over the same field $\mathbb{F}$ (not necessarily of the same dimension). A map $f : V \longrightarrow U$ is called* **linear map** *or* **linear transformation** *if the following property is holds:*

$$(\forall\alpha, \beta \in \mathbb{F})\,(\forall x, y \in V)\,[f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)]. \tag{38}$$

In case $V = U$, the map f is called **linear operator**.

Two vector spaces are called *isomorphic* if there is a bijection between them that preserves addition of vectors and multiplication of vectors by scalars, and *homomorphic* if there is a surjection or an injection between them that preserves these operations. Isomorphisms of a vector space onto itself and homomorphisms of a vector space into itself are called *automorphisms* and *endomorphisms* of this vector space respectively. Automorphisms and endomorphisms of a vector space are called *linear transformations* in it.

DEFINITION 6.4. $\mathbf{Vect}_k$ **category** *consisting of vector spaces over a field k as objects and k-linear maps as morphisms;*

From the definition of vector space one can see that we cannot construct an isotopy of a vector space without first introducing an isotopy of the field, because the multiplicative unit I of the space is that of the underlying field. Note that we are lifting of the field, but the elements of the linear space remain unchanged.

DEFINITION 6.5. *Let* **Vect** *and* **IsoVect** *be two categories, V be a vector space over the field $\mathbb{F}$ and $\hat{\mathbb{F}}$ an isofield of $\mathbb{F}$. A Santilli iso-functor $\hat{\mathcal{I}}$ from* **Vect** *associates to each object A in* **Vect** *category an object $\hat{I}A$ in* **IsoVect category** *by "isovector space" as the vector space $\hat{V}$, (which is the same set as V), over the isofield $\hat{\mathbb{F}}$ equipped with a new external operation $\diamond$ which is such to verify all the axioms for a vector space, i.e,*

$$(\forall\ \hat{\alpha}, \hat{\beta} \in \hat{\mathbb{F}})\ \ (\forall\ x \in \hat{V})[\hat{\alpha} \diamond (\hat{\beta} \diamond x) = (\hat{\alpha} \star \hat{\beta}) \diamond x], \tag{39}$$

$$(\forall\ \hat{\alpha}, \in \hat{\mathbb{F}})\ \ (\forall\ x, y \in \hat{V})[\hat{\alpha} \diamond (x + y) = (\hat{\alpha} \diamond x + \hat{\beta}) \diamond y], \tag{40}$$

$$(\forall\ \hat{\alpha}, \hat{\beta} \in \hat{\mathbb{F}})\ \ (\forall\ x \in \hat{V})[(\hat{\alpha} + \hat{\beta}) \diamond x) = (\hat{\alpha} \diamond x + \hat{\beta}) \diamond x], \tag{41}$$

$$(\forall\ x \in V)[\hat{I} \diamond x = x \diamond \hat{I} = x]. \tag{42}$$

and to each morphism $f \in \mathrm{Hom}_{\mathbf{Vect}}(A, B)$ an isolinear transformation as a morphism $\hat{I}f \in \mathrm{Hom}_{\mathbf{IsoVect}}(\hat{I}A, \hat{I}B)$, which is an isomap:

$$\hat{f} : \hat{V} \longrightarrow \hat{V'}, \tag{43}$$

between two isolinear vector spaces $\hat{V}$ and $\hat{V}'$ over the same isofield $\hat{\mathbb{F}}$ which preserves the sum and isomultiplication, i.e., which is such that

$$(\forall\ \hat{\alpha}, , \hat{\beta} \in \hat{\mathbb{F}})\ \ (\forall\ x, y \in V)[\hat{f}(\hat{\alpha} \star x + \hat{\beta} \star y) = \hat{\alpha} \star \hat{f}(x) + \hat{\beta} \star \hat{f}(y)]. \quad (44)$$

7 Algebras

If the set A is simultaneously a vector space L^n and a ring $\mathcal{R}$ or a field $\mathbb{F}$ and if the following axiom holds:

$A.1°$ For any two elements a and b and any two scalars λ and μ we have

$$(a\lambda)(b\mu) = (ab)(\lambda\mu), \quad (45)$$

then the set A is called an *n-algebra.*

The product of two basis elements of an algebra can be written as

$$e_i e_j = C_{ij}^k e_k. \quad (46)$$

The numbers C_{ij}^k are called the *structure constants* of the algebra. In this book we will consider both associative and nonassociative algebras. The field $\mathbb{C}$ of complex numbers is a commutative associative algebra with basis $1, i \quad (i^2 = -1)$, the skew field $\mathbb{H}$ of quaternions is a noncommutative associative algebra with basis $1, i, j, k$ $(i^2 = j^2 = k^2 = -l,\ ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j)$.

An algebra which is a field or a skew field is called a *division algebra.*

Two algebras are called *isomorphic* if there is a bijection between them that preserves addition and multiplication of vectors, and multiplication of vectors by scalars, and *homomorphic* if there is a surjection or an injection between them that preserves these operations. Isomorphisms of an algebra onto itself and homomorphisms of an algebra into itself are *algebra automorphisms* and *algebra endomorphisms* of this algebra, respectively.

DEFINITION 7.1. *A subset V of an algebra A is called a subalgebra if it satisfies the algebra axioms, i.e., if it is an algebra in its own right.*

It is easy to proove that a subset $\mathsf{V} \subset \mathsf{A}$ of an algebra A is a subalgebra iff:

$$(\forall x, y \in \mathsf{A})(\forall \alpha \in \mathbb{F})[x + y \in \mathsf{A}, \alpha x \in \mathsf{A}, xy \in \mathsf{A}] \tag{47}$$

that is, the subset V is closed under all the composition laws.

Definition 7.2. *A map $f : \mathsf{A} \longrightarrow \mathsf{A}'$ between algebras over the same field $\mathbb{F}$ is called* **morphism** *if:*

$$(\forall x, y \in \mathsf{A})(\forall \alpha, \beta \in \mathbb{F})[f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) \,\wedge\, f(xy) = f(x)f(y)]. \tag{48}$$

Definition 7.3. *The usual statements about the image and kernel of a morphisms (48) hold.*

Definition 7.4. **Alg** *is the category with algebras as objects and maps (48) as morphisms.*

Definition 7.5. *Let* **Alg** *and* **IsoAlg** *be two categories.*
A Santilli iso-functor $\hat{\mathcal{I}}$ from **Alg** *associates to each object A in* **Alg** *category an object $\hat{\mathsf{A}}$ in* **IsoAlg** *and to each morphism $f \in \mathrm{Hom}_{\mathbf{Alg}}(\mathsf{A}, \mathsf{B})$ a morphism $\hat{f} \in \mathrm{Hom}_{\mathbf{IsoAlg}}(\hat{\mathsf{A}}, \hat{\mathsf{B}})$, i.e., we reconstruct the elements of the* **category IsoAlg** *as an "isotope" $\hat{\mathsf{A}}$ of an algebra A with elements $x, y, z, \cdots$ and product xy over a field $\mathbb{F}$, is the same vector space L^n but defined over the isofield $\hat{\mathbb{F}}$, equipped with a new product $x \star y$, called "isotopic product", which is such to verify the original axioms of A.*

A possible way to overcome basic issues in present-day Category Formulation of Santilli's Isotopies is to solve another three problems:

Problem 1, it has to be formulated the basic results of categories of topological, incidence and metric isostructures: Category of topological isospaces; Category of Affine isospaces; Category of projective isospaces; Category of Euclidean isospaces; Category of pseudo-Euclidean isospaces; Category of conformal isospaces; Category of pseudoconformal isospaces.

Problem 2, it has to be shown how Lie-Santilli Isogroups and Isoalgebras. can be represented in category theory.

Problem 3, it has to be described the categorical concept of infinite-dimensional isospaces: Category of infinite-dimensional linear isospaces; Category of Hilbert isospaces; Category of Banach isospaces and Categories of Hilbert and Banach isoalgebras.

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