Category of Lie-Santilli Isogroups[1]

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Abstract. In this work we demonstrate the possibility of category theory to provide a compact method of encoding mathematical isostructures in a uniform way. In this way we constructed a category of isogroups $\widehat{\mathbf{Grp}}$, a category of vector isospaces $\widehat{\mathbf{Vect}}$, a category of topological vector isospaces $\widehat{\mathbf{TopVec}}$ and a category of topological isogroups $\widehat{\mathbf{TopGrp}}$. These categories helped us to build a category of Lie-Santilli isogroups $\widehat{\mathbf{LSGrp}}$ too.

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INTRODUCTION

Isomathematics was proposed by Santilli [1] in 1978, and subsequently studied by numerious pure and applied mathematicians as: S. Okubo, H. Myung, M. Tomber, Gr. Tsagas, D. Sourlas, C. Corda, J. Kadeisvili, A. Aringazin, A. Kirhukin, R. Ohemke, G. Wene, G. M. Benkart, J. Osborn, D. Britten, J. Lohmus, E. Paal, L. Sorgsepp, D. Lin, J. Voujouklis, P. Broadbridge, P. Chernooff, J. Sniatycku, S. Guiasu, E. Prugovecki, A. Sagle, C. Jiang, R. Falcon Ganfornina, J. Nunez Valdes, A. Davvaz, S. G. Georgiev and others.

As a result of these efforts, the new mathematics can be constructed via the systematic application of axiom - preservibg liftings, called isotopies, of the totality of all structures in the mathematics: inluding all operators and their operations, icluding the isotopic lifting of numbers, functinal analysis, differential calculas, geometries, topologies, Lie theory and others [2, 3].

Category theory groups together in categories the mathematical objects with some common structure (e.g., sets, partially ordered sets, groups, rings, and so forth) and the appropriate morphisms between such objects [4, 5]. These morphisms are required to satisfy certain properties which make the set of all such relations 'coherent'. Given a category, it is not the case that every two objects have a relation between them, some do and others don't. For the ones that do, the number of relations can vary depending on which category we are considering.

DEFINITION 1 *A* **category** *is a quadruple* $(\mathrm{Ob}, \mathrm{Hom}, \mathrm{id}, \circ)$ *consisting of:*
(Cl) a class Ob *of objects;*
(C2) for each ordered pair (A,B) *of objects a set* $\mathrm{Hom}(A,B)$ *of morphisms;*
(C3) for each object A a morphism $\mathrm{id}_A \in \mathrm{Hom}(A,A)$*, the identity of A;*
(C4) a composition law associating to each pair of morphisms $f \in \mathrm{Hom}(A,B)$ *and* $g \in \mathrm{Hom}(B,C)$ *a morphism* $g \circ f \in \mathrm{Hom}(A,C)$*;*
which is such that:
(M1) $h \circ (g \circ f) = (h \circ g) \circ f$ *for all* $f \in \mathrm{Hom}(A,B)$, $g \in \mathrm{Hom}(B,C)$ *and* $h \in \mathrm{Hom}(C,D)$*;*
(M2) $\mathrm{id}_B \circ f = f \circ \mathrm{id}_A = f$ *for all* $f \in Hom(A,B)$*;*
(M3) the sets $\mathrm{Hom}(A,B)$ *are pairwise disjoint.*

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CATEGORIES OF ISOGROUPS AND VECTOR ISOSPACES

In [6, 7] we built a *category of isogroups* $\widehat{\mathbf{Grp}}$ which consists of isogroups $\hat{G}$ as objects i.e., we constructed the elements for each object $\hat{G}$ of the category $\widehat{\mathbf{Grp}}$ as $\hat{a} \equiv a\hat{I} \in \widehat{\mathbf{Grp}}$, where the isounit $\hat{I}$ is defined with the help of an invertible element $T : \hat{I} = T^{-1}$, called *isotopic element*, and the new composition law is defined by

$$(\forall \hat{a}, \hat{b} \in \widehat{\mathbf{Grp}}) \quad [\hat{a} \hat{\circ} \hat{b} \equiv \hat{a}\, T\, \hat{b}]. \tag{1}$$

satisfying the following axioms:

$\widehat{G}.1^\circ$ For any two elements $\hat{a}$ and $\hat{b}$ there exists an element

$$\hat{c} = \hat{a} \hat{\circ} \hat{b} \tag{2}$$

$\widehat{G}.2^\circ$ This operation is associative, that is, for any three elements $\hat{a}, \hat{b}, \hat{c}$

$$(\hat{a} \hat{\circ} \hat{b}) \hat{\circ} \hat{c} = \hat{a} \hat{\circ} (\hat{b} \hat{\circ} \hat{c}). \tag{3}$$

$\widehat{G}.3^\circ$ There exists a *isounit* $\hat{I}$, i. e. an element such that for every element $\hat{a}$

$$\hat{a} \hat{\circ} \hat{I} \equiv \hat{a}\, T\, \hat{I} = \hat{a}\, T\, T^{-1} = \hat{a}. \tag{4}$$

$\widehat{G}.4^\circ$ For each element $\hat{a}$ there exists a *symmetric element* $\hat{a}^{-1}$ such that

$$\hat{a} \hat{\circ} \hat{a}^{-1} = \hat{a}^{-1} \hat{\circ} \hat{a} = \hat{I}. \tag{5}$$

If the isogroup $\hat{G}$ satisfies in addition the axiom

$\widehat{G}.5^\circ$. For any two elements $\hat{a}$ and $\hat{b}$

$$\hat{a} \hat{\circ} \hat{b} = \hat{b} \hat{\circ} \hat{a}, \tag{6}$$

then the isogroup $\hat{G}$ is called *commutative* or *Abelian*.

And the *category of isogroups* $\widehat{\mathbf{Grp}}$ also consists of isohomomorphisms $\hat{f} : \hat{G} \longrightarrow \hat{G}'$ between isogroups $(\hat{G}, \hat{\circ})$ and $(\hat{G}', \widehat{\Box})$ as morphisms $\hat{I} f \in \mathrm{Hom}_{\widehat{\mathbf{Grp}}}(\hat{G}, \hat{G}')$, where the following property holds:

$$(\forall \hat{a}, \hat{b} \in \hat{G}) \;\; [\hat{f}(\hat{a} \hat{\circ} \hat{b}) = \hat{f}(\hat{a}) \widehat{\Box} \hat{f}(\hat{b})]. \tag{7}$$

Thus an isohomomorphism “carries” the composition law $\hat{\circ}$ on $\hat{G}$ to the composition law $\widehat{\Box}$ on $\hat{G}'$. It can be proved easily, that if category $\widehat{\mathbf{Grp}}$ is a monoid and also a groupoid for the fixed isotopic element T, with the above internal composition, it can become an isogroup $\hat{G}$ with unit $\hat{I}$.

Let $\mathscr{I} : \mathbf{Grp} \to \widehat{\mathbf{Grp}}$ and $\mathscr{I}' : \mathbf{Grp} \to \widehat{\mathbf{Grp}}$ be two Santilli functors [6]. A *natural transformation* $\hat{\alpha} : \mathscr{I} \to \mathscr{I}'$ is given by the following data: For every object A in $\mathbf{Grp}$ there is a morphism $\hat{\alpha}_A : \mathscr{I}(A) \to \mathscr{I}'(A)$ in $\widehat{\mathbf{Grp}}$ such that for every morphism $f : A \to B$ in $\mathbf{Grp}$ the following diagram is commutative

$$\begin{array}{ccc}
\mathscr{I}(A) & \xrightarrow{\hat{\alpha}_A} & \mathscr{I}'(A) \\
\mathscr{I}(f)\downarrow & & \downarrow \mathscr{I}'(f) \\
\mathscr{I}(B) & \xrightarrow{\hat{\alpha}_B} & \mathscr{I}'(B).
\end{array}$$

Commutativity means that the isotopic elements T have to satisfy the following equations i.e., the following compositions of morphisms are equal: $\mathscr{I}(f) \star \hat{\alpha}_A = \hat{\alpha}_B \star \mathscr{I}'(f)$. The morphisms $\hat{\alpha}_A$, $A \in \mathrm{Ob}(\mathbf{Grp})$, are called the *components of the natural transformation* $\hat{\alpha}$.

Let $\widehat{V}$ be a vector isospace over the isofield $\widehat{\mathbb{F}}$ [6, 7]. A *category of vector isospaces* $\widehat{\mathbf{Vect}}$ consists of vector isospaces $\widehat{V}$ as objects equipped with a new external operation $\diamond$ which is such to verify all the axioms for a vector isospace, i.e,

$\widehat{V}I.1^\circ$ - 5° Addition of vectors satisfies axioms $\widehat{G}.1 - 5^\circ$ for a commutative isogroup;

$\widehat{V}II.1^\circ$ For any vector $\mathbf{x}$ and any isoscalar $\hat{\alpha}$ there exists a vector

$$(\forall\; \hat{\alpha}, \in \widehat{\mathbb{F}}) \;\; (\forall\, \mathbf{x}, \mathbf{y} \in \mathbf{V}) \;\; [\mathbf{y} = \hat{\alpha} \diamond \mathbf{x}]; \tag{8}$$

called the product of $\mathbf{x}$ by $\hat{\alpha}$;

$\widehat{V}II.2^\circ$ Multiplication by $\hat{I}$ does not change a vector:

$$(\forall\, \mathbf{x} \in V)\ [\hat{I} \diamond \mathbf{x} = \mathbf{x} \diamond \hat{I} = \mathbf{x}]; \tag{9}$$

$\widehat{V}II.3^\circ$ Multiplication of vectors by isoscalars is distributive with respect to addition of isoscalars:

$$(\forall\ \hat{\alpha}, \hat{\beta} \in \hat{\mathbb{F}})\ (\forall\, \mathbf{x} \in \mathbf{V})\ [(\hat{\alpha} + \hat{\beta}) \diamond \mathbf{x}) = \hat{\alpha} \diamond \mathbf{x} + \hat{\beta} \diamond \mathbf{x}]; \tag{10}$$

$\widehat{V}II.4^\circ$ Multiplication of vectors by isoscalars is distributive with respect to addition of isovectors:

$$(\forall\ \hat{\alpha}, \in \hat{\mathbb{F}})\ (\forall\, \mathbf{x}, \mathbf{y} \in \mathbf{V})\ [\hat{\alpha} \diamond (\mathbf{x} + \mathbf{y}) = \hat{\alpha} \diamond \mathbf{x} + \hat{\beta} \diamond \mathbf{y}]; \tag{11}$$

$\widehat{V}II.5^\circ$ Multiplication of vectors by isoscalars is associative:

$$(\forall\ \hat{\alpha}, \hat{\beta} \in \hat{\mathbb{F}})\ (\forall\, \mathbf{x} \in \mathbf{V})\ [\hat{\alpha} \diamond (\hat{\beta} \diamond \mathbf{x}) = (\hat{\alpha} \star \hat{\beta}) \diamond \mathbf{x}]; \tag{12}$$

$\widehat{V}III.1^\circ$ There exist n linearly independent vectors;

$\widehat{V}III.2^\circ$ Any $n+1$ vectors are linearly dependent.

And the *category of vector isospaces* $\widehat{\mathbf{Vect}}$ also consists of isocontinuous isolinear transformations as morphisms $\hat{I}f \in \mathrm{Hom}_{\widehat{\mathbf{Vect}}}(\hat{I}A, \hat{I}B)$:

$$\hat{f} : \widehat{V} \longrightarrow \widehat{V}', \tag{13}$$

between vector isospaces $\widehat{V}$ and $\widehat{V'}$ over the same isofield $\hat{\mathbb{F}}$ which preserve the sum and isomultiplication, i.e., which are such that

$$(\forall\ \hat{\alpha}, , \hat{\beta} \in \hat{\mathbb{F}})\ (\forall\, \mathbf{x}, \mathbf{y} \in V)\ [\hat{f}(\hat{\alpha} \diamond \mathbf{x} + \hat{\beta} \diamond \mathbf{y}) = \hat{\alpha} \diamond \hat{f}(\mathbf{x}) + \hat{\beta} \diamond \hat{f}(\mathbf{y})]. \tag{14}$$

CATEGORIES OF TOPOLOGICAL ISOGROUPS AND VECTOR ISOSPACES

Let $\widehat{T}$ be a topological vector isospace over the isofield $\hat{\mathbb{F}}$ [8]. A mapping from a topological vector isospace $\widehat{T}$ onto a topological vector isospace $\widehat{T}'$ is called *isocontinuous* if for each neighborhood $V(\hat{x}')$ of a point $\hat{x}'$ in $\widehat{T}'$ there is a neighborhood $U(\hat{x})$ of the corresponding point $\hat{x}$ in $\widehat{T}$ such that the images of all points in $U(\hat{x})$ belong to $V(\hat{x}')$.

A *category of topological vector isospaces* $\widehat{\mathbf{TopVec}}$ consists of topological vector isospaces $\widehat{T}$ as objects, which satisfy the following axioms:

$\widehat{T}.1^\circ$ The union of a finite number of closed subsets is closed;

$\widehat{T}.2^\circ$ The intersection of arbitrary many closed subsets is closed;

$\widehat{T}.3^\circ$ The whole vector isospace $\widehat{T}$ is a closed set;

$\widehat{T}.4^\circ$ The empty set $\emptyset$ is a closed set.

And the *category of topological vector isospaces* $\widehat{\mathbf{TopVec}}$ also consists of isocontinuous isolinear transformations as morphisms $\hat{I}f \in \mathrm{Hom}_{\widehat{\mathbf{TopVec}}}(\hat{I}A, \hat{I}B)$:

$$\hat{f} : \widehat{T} \longrightarrow \widehat{T}', \tag{15}$$

between topological vector isospaces $\widehat{T}$ and $\widehat{T}'$ over the same isofield $\hat{\mathbb{F}}$ which preserves the sum and isomultiplication, i.e., which is such that

$$(\forall\ \hat{\alpha}, , \hat{\beta} \in \hat{\mathbb{F}})\ (\forall\, \mathbf{x}, \mathbf{y} \in T)\ [\hat{f}(\hat{\alpha} \diamond \mathbf{x} + \hat{\beta} \diamond \mathbf{y}) = \hat{\alpha} \diamond \hat{f}(\mathbf{x}) + \hat{\beta} \diamond \hat{f}(\mathbf{y})]. \tag{16}$$

The isonatural topology in the isofield $\hat{\mathbb{R}}$, with closed and open sets defined as in usual real iso-Calculus, can be specified by the countable system of neighborhoods consisting of the intervals with rational ends.

A bijection from $\widehat{T}$ onto $\widehat{T}'$ which is isocontinuous together with its inverse bijection is called a *isohomeomorphism*; in this case the topological vector isospaces $\widehat{T}$ and $\widehat{T}'$ are called *isohomeomorphic*.

Topological isospaces each point of which has a neighborhood isohomeomorphic to the Euclidean isospace [9] endowed with its natural topology are called *n-dimensional topological isomanifolds* or *n-isomanifolds*.

In [8] we built a *category of topological isogroups* $\widehat{\mathbf{TopGrp}}$ which consists of topological isogroups $\widehat{TG}$ as objects i.e., we reconstruct a set of elements for each object $\widehat{TG}$ of the category $\widehat{\mathbf{TopGrp}}$ as a set of elements is said to be a *topological isogroup* if

$\widehat{TG}.1^\circ$ This set have to be an isogroup (see (2)–(5));

$\widehat{TG}.2^\circ$ This set have to be a topological vector isospace (see $\widehat{T}.1^\circ$–$\widehat{T}.4^\circ$);

$\widehat{TG}.3^\circ$ The isogroup operations $\hat{x} \to \hat{\alpha} \hat{\circ} \hat{x}$, $\hat{x} \to \hat{x} \;\hat{\circ}\; \hat{\alpha}$ and $\hat{x} \to \hat{x}^{-1}$, where α is a isoscalar, are isocontinuous mappings from this topological vector isospace onto itself.

And the *category of topological isogroups* $\widehat{\mathbf{TopGrp}}$ also consists of isocontinuous isohomomorphisms as morphisms.

CATEGORY OF LIE-SANTILLI ISOGROUPS

The most important for isogeometry are topological isogroups that are also *topological n-isomanifolds*. These topological groups are called n-dimensional *Lie-Santilli isogroups*.

DEFINITION 2 *A category of Lie-Santilli isogroups* $\widehat{\mathbf{LSGrp}}$ *consists of Lie-Santilli isogroups as objects, and isohomomorphisms as morphisms.*

DEFINITION 3 *A Lie-Santilli algebra can be defined as an object A in* $\widehat{\mathbf{Vect}}_{\hat{\mathbb{F}}}$, *the category of vector isospaces over a isofield* $\hat{\mathbb{F}}$ *of characteristic not 2, together with a morphism* $[\cdot,\cdot] : A \otimes A \to A$, *where* $\otimes$ *refers to the monoidal product of* $\widehat{\mathbf{Vect}}_{\hat{\mathbb{F}}}$, *such that*

$$[\cdot,\cdot] \circ (\mathrm{id} + \tau_{A,A}) = 0, \quad [\cdot,\cdot] \circ ([\cdot,\cdot] \otimes \mathrm{id}) \circ (\mathrm{id} + \sigma + \sigma^2) = 0,$$

where $\tau(a \otimes b) := b \otimes a$ *and* σ *is the cyclic permutation braiding* $(\mathrm{id} \otimes \tau_{A,A}) \star (\tau_{A,A} \otimes \mathrm{id})$.

The mapping from a category of Lie-Santilli groups $\widehat{\mathbf{LSGrp}}$ to a category of Lie-Santilli algebras $\widehat{\mathbf{LSAlg}}$ is functorial, which implies that isohomomorphisms of Lie-Santilli groups lift to isohomomorphisms of Lie-Santilli algebras, and various properties are satisfied by this lifting: it commutes with composition, it maps Lie-Santilli subgroups, kernels, quotients and cokernels of Lie-Santilli groups to subalgebras, kernels, quotients and cokernels of Lie-Santilli algebras, respectively. The functor $\mathscr{L}$ which takes each Lie-Santilli group to its Lie-Santilli algebra and each isohomomorphism to its isodifferential one is faithful and exact.

When we use Lie-Santilli groups in physics to describe symmetry, we think of each element $\hat{g}$ of the group $\hat{G}$ as a "process". The element $\hat{I}$ corresponds to the "process of doing nothing at all". We can compose processes $\hat{g}$ and $\hat{h}$ — do $\hat{h}$ and then $\hat{g}$ — and get the product $\hat{g}\hat{\circ}\hat{h}$. Crucially, every process $\hat{g}$ can be "undone" using its inverse $\hat{g}^{-1}$. So: in contrast to a Lie-Santilli group, which consists of a static collection of "things", a category of Lie-Santilli groups consists not only of objects or "things" but also morphisms which can viewed as "processes" transforming one thing into another. Similarly, in a 2-category, the 2-morphisms can be regarded as "processes between processes", and so on.

REFERENCES

1. R. M. Santilli, *Hadronic J.* **1**, 223–423 (1978).
2. R. M. Santilli, *Elements of Hadronic Mechanics*, Naukova Dumka, Kyiv, 1995, 1 edn.
3. D. S. Sourlas, and G. T. Tsagas, *Mathematical Foundations of the Lie-Santilli theory*, Naukova Dumka, Kyiv, 1993, 1 edn.
4. S. S. Moskaliuk, *From Cayley-Klein Groups to Categories*, Methods of Mathematical Modelling, Vol. 11, TIMPANI, Kyiv, 2006, 1 edn.
5. S. S. Moskaliuk, *Algebras, Groups and Geometries* **31**, No. 2, p. 117–156 (2014).
6. S. S. Moskaliuk, *Algebras, Groups and Geometries* **31**, No. 1, p. 1–22 (2014).
7. S. S. Moskaliuk, *Algebras, Groups and Geometries* **31**, No. 1, p. 101–116 (2014).
8. S. S. Moskaliuk, *Algebras, Groups and Geometries* **31**, No. 3, p. 237–266 (2014).
9. S. S. Moskaliuk, *Algebras, Groups and Geometries* **31**, No. 4, p. 361–390 (2014).