Category of Lie-Santilli Isogroups

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Abstract. In this work we demonstrate the possibility of category theory to provide a compact method of encoding mathematical isostructures in a uniform way. In this way we constructed a category of isogroups $\hat{\text{Grp}}$, a category of vector isospaces $\hat{\text{Vect}}$, a category of topological vector isospaces $\hat{\text{TopVec}}$ and a category of topological isogroups $\hat{\text{TopGrp}}$. These categories helped us to build a category of Lie-Santilli isogroups $\hat{\text{LSGrp}}$ too.

Keywords: Category theory; Lie groups; Santilli functor

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INTRODUCTION


As a result of these efforts, the new mathematics can be constructed via the systematic application of axiom-preserving liftings, called isotopies, of the totality of all structures in the mathematics: including all operators and their operations, including the isotopic lifting of numbers, functional analysis, differential calculus, geometries, topologies, Lie theory and others [2, 3].

Category theory groups together in categories the mathematical objects with some common structure (e.g., sets, partially ordered sets, groups, rings, and so forth) and the appropriate morphisms between such objects [4, 5]. These morphisms are required to satisfy certain properties which make the set of all such relations ‘coherent’. Given a category, it is not the case that every two objects have a relation between them, some do and others don’t. For the ones that do, the number of relations can vary depending on which category we are considering.

Definition 1 A category is a quadruple $(\text{Ob, Hom, id, } \circ)$ consisting of:

(C1) a class Ob of objects;

(C2) for each ordered pair $(A, B)$ of objects a set Hom$(A, B)$ of morphisms;

(C3) for each object $A$ a morphism $\text{id}_A \in \text{Hom}(A, A)$, the identity of $A$;

(C4) a composition law associating to each pair of morphisms $f \in \text{Hom}(A, B)$ and $g \in \text{Hom}(B, C)$ a morphism $g \circ f \in \text{Hom}(A, C)$;

which is such that:

$(M1) \ h \circ (g \circ f) = (h \circ g) \circ f \text{ for all } f \in \text{Hom}(A, B), \ g \in \text{Hom}(B, C) \text{ and } h \in \text{Hom}(C, D)$;

$(M2) \ \text{id}_A \circ f = f \circ \text{id}_A = f \text{ for all } f \in \text{Hom}(A, B)$;

$(M3) \ \text{the sets } \text{Hom}(A, B) \text{ are pairwise disjoint}$.

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CATEGORIES OF ISOGROUPS AND VECTOR ISOSPACES

In [6, 7] we built a category of isogroups $\mathbf{Grp}$ which consists of isogroups $G$ as objects i.e., we constructed the elements for each object $\hat{G}$ of the category $\mathbf{Grp}$ as $\hat{a} \equiv aI \in \mathbf{Grp}$, where the isounit $I$ is defined with the help of an invertible element $T : \hat{I} = T^{-1}$, called isotopic element, and the new composition law is defined by

\[
(\forall a, b \in \mathbf{Grp}) \quad [\hat{a} \circ \hat{b}] \equiv \hat{a}T \hat{b},
\]

satisfying the following axioms:

$\hat{G}.1^o$ For any two elements $\hat{a}$ and $\hat{b}$ there exists an element

\[
\hat{c} = \hat{a} \circ \hat{b}
\]

$\hat{G}.2^o$ This operation is associative, that is, for any three elements $\hat{a}, \hat{b}, \hat{c}$

\[
(\hat{a} \circ \hat{b}) \circ \hat{c} = \hat{a} \circ (\hat{b} \circ \hat{c})
\]

$\hat{G}.3^o$ There exists a isounit $\hat{I}$, i.e., an element such that for every element $\hat{a}$

\[
\hat{a} \circ \hat{I} \equiv \hat{a}T \hat{I} = \hat{a}TT^{-1} = \hat{a}
\]

$\hat{G}.4^o$ For each element $\hat{a}$ there exists a symmetric element $\hat{a}^{-1}$ such that

\[
\hat{a} \circ \hat{a}^{-1} = \hat{a}^{-1} \circ \hat{a} = \hat{I}
\]

If the isogroup $\hat{G}$ satisfies in addition the axiom

$\hat{G}.5^o$. For any two elements $\hat{a}$ and $\hat{b}$

\[
\hat{a} \circ \hat{b} = \hat{b} \circ \hat{a}
\]

then the isogroup $\hat{G}$ is called commutative or Abelian.

And the category of isogroups $\mathbf{Grp}$ also consists of isohomomorphisms $\hat{f} : \hat{G} \rightarrow \hat{G}'$ between isogroups $(\hat{G}, \circ)$ and $(\hat{G}', \circ')$ as morphisms $\hat{f} \in \text{Hom}_{\mathbf{Grp}}(\hat{G}, \hat{G}')$, where the following property holds:

\[
(\forall \hat{a}, \hat{b} \in \hat{G}) \quad [\hat{f}(\hat{a} \circ \hat{b})] = \hat{f}(\hat{a}) \circ \hat{f}(\hat{b})
\]

Thus an isohomomorphism “carries” the composition law $\circ$ on $\hat{G}$ to the composition law $\circ'$ on $\hat{G}'$. It can be proved easily, that if category $\mathbf{Grp}$ is a monoid and also a groupoid for the fixed isotopic element $T$, with the above internal composition, it can become an isogroup $\hat{G}$ with unit $\hat{I}$.

Let $\mathcal{F} : \mathbf{Grp} \rightarrow \mathbf{Grp}$ and $\mathcal{F}' : \mathbf{Grp} \rightarrow \mathbf{Grp}$ be two Santilli functors [6]. A natural transformation $\alpha : \mathcal{F} \rightarrow \mathcal{F}'$ is given by the following data: For every object $A$ in $\mathbf{Grp}$ there is a morphism $\alpha_A : \mathcal{F}(A) \rightarrow \mathcal{F}'(A)$ in $\mathbf{Grp}$ such that for every morphism $f : A \rightarrow B$ in $\mathbf{Grp}$ the following diagram is commutative

\[
\begin{array}{ccc}
\mathcal{F}(A) & \xrightarrow{\alpha_A} & \mathcal{F}'(A) \\
\mathcal{F}(f) \downarrow & & \downarrow \mathcal{F}'(f) \\
\mathcal{F}(B) & \xrightarrow{\alpha_B} & \mathcal{F}'(B)
\end{array}
\]

Commutativity means that the isotopic elements $T$ have to satisfy the following equations i.e., the following compositions of morphisms are equal: $\mathcal{F}(f) \circ \alpha_A = \alpha_B \circ \mathcal{F}'(f)$. The morphisms $\alpha_A, A \in \text{Ob}(\mathbf{Grp})$, are called the components of the natural transformation $\alpha$.

Let $\mathcal{V}$ be a vector isospace over the isofield $\mathcal{F}$ [6, 7]. A category of vector isospaces $\mathbf{Vect}$ consists of vector isospaces $\mathcal{V}$ as objects equipped with a new external operation $\circ$ which is such to verify all the axioms for a vector isospace, i.e., $\mathcal{V}.1^o - 5^o$ Addition of vectors satisfies axioms $\hat{G}.1 - 5^o$ for a commutative isogroup;

$\mathcal{V}.6^o$ For any vector $x$ and any isoscalar $\alpha$ there exists a vector

\[
(\forall \alpha, x \in \mathcal{V}) \quad (\forall x, y \in \mathcal{V}) \quad y = \alpha \circ x;
\]
a topological vector isospace 

\[ \hat{U} \] between topological vector isospaces satisfy the following axioms:

\( (\forall x \in V) \; (I \circ x = x \circ \hat{I} = x) \); (9)

\( (\forall \alpha, \beta \in \hat{F}) \; (\forall x \in V) \; [(\alpha + \beta) \circ x = \alpha \circ x + \beta \circ x]; \) (10)

\( (\forall \alpha, x, y \in V) \; [\alpha \circ (x + y) = \alpha \circ x + \beta \circ y]; \) (11)

\( (\forall \alpha, \beta \in \hat{F}) \; (\forall x \in V) \; [\alpha \circ (\beta \circ x) = (\alpha \circ \beta) \circ x]; \) (12)

\( \hat{V} \) is called the product of \( x \) by \( \hat{\alpha} \); \n
\( \hat{V} \) Multiplication by \( I \) does not change a vector:

\( (\forall \alpha, \beta \in \hat{F}) \; (\forall x \in V) \; [(\alpha + \beta) \circ x = \alpha \circ x + \beta \circ x]; \)

\( (\forall \alpha, x, y \in V) \; [\alpha \circ (x + y) = \alpha \circ x + \beta \circ y]; \)

\( (\forall \alpha, \beta \in \hat{F}) \; (\forall x \in V) \; [\alpha \circ (\beta \circ x) = (\alpha \circ \beta) \circ x]; \)

\( \hat{V} \) There exist \( n \) linearly independent vectors;

\( \hat{V} \) Any \( n + 1 \) vectors are linearly dependent.

And the category of vector isospaces \( \text{Vect} \) also consists of isocontinuous isolinear transformations as morphisms

\[ f : \hat{V} \rightarrow \hat{V}, \]

between vector isospaces \( \hat{V} \) and \( \hat{V}' \) over the same isofield \( \hat{F} \) which preserve the sum and isomultiplication, i.e., which are such that

\( (\forall \alpha, \beta \in \hat{F}) \; (\forall x, y \in V) \; [f(\alpha \circ (\beta \circ x) = (\alpha \circ \beta) \circ f(x)]; \)

CATEGORIES OF TOPOLOGICAL ISOGROUPS AND VECTOR ISOSPACES

Let \( \hat{T} \) be a topological vector isospace over the isofield \( \hat{F} \) [8]. A mapping from a topological vector isospace \( \hat{T} \) onto a topological vector isospace \( \hat{T}' \) is called iscontinuous if for each neighborhood \( V(\hat{x}') \) of a point \( \hat{x}' \) in \( \hat{T}' \) there is a neighborhood \( U(\hat{x}) \) of the corresponding point \( \hat{x} \) in \( \hat{T} \) such that the images of all points in \( U(\hat{x}) \) belong to \( V(\hat{x}'); \)

A category of topological vector isospaces \( \text{TopVec} \) consists of topological vector isospaces \( \hat{T} \) as objects, which satisfy the following axioms:

\( \hat{T} \). The union of a finite number of closed subsets is closed;

\( \hat{T} \). The intersection of arbitrary many closed subsets is closed;

\( \hat{T} \). The whole vector isospace \( \hat{T} \) is a closed set;

\( \hat{T} \). The empty set \( \emptyset \) is a closed set.

And the category of topological vector isospaces \( \text{TopVec} \) also consists of isocontinuous isolinear transformations as morphisms \( f \in \text{Hom}_{\text{TopVec}}(I\hat{A}, I\hat{B}); \)

\[ f : \hat{T} \rightarrow \hat{T}', \]

between topological vector isospaces \( \hat{T} \) and \( \hat{T}' \) over the same isofield \( \hat{F} \) which preserves the sum and isomultiplication, i.e., which is such that

\( (\forall \alpha, \beta \in \hat{F}) \; (\forall x, y \in T) \; [f(\alpha \circ (\beta \circ x) = (\alpha \circ \beta) \circ f(x)]; \)

The isonatural topology in the isofield \( \hat{F} \), with closed and open sets defined as in usual real iso-Calculus, can be specified by the countable system of neighborhoods consisting of the intervals with rational ends.

A bijection from \( \hat{T} \) onto \( \hat{T}' \), which is iscontinuous together with its inverse bijection is called a isohomeomorphism; in this case the topological vector isospaces \( \hat{T} \) and \( \hat{T}' \) are called isohomeomorphic.

Topological isospaces each point of which has a neighborhood isohomeomorphic to the Euclidean isospace [9] endowed with its natural topology are called \( n \)-dimensional topological isomanifolds or \( n \)-isomanifolds.
In [8] we built a category of topological isogroups \( \text{TopGrp} \) which consists of topological isogroups \( \hat{T}G \) as objects i.e., we reconstruct a set of elements for each object \( \hat{T}G \) of the category \( \text{TopGrp} \) as a set of elements is said to be a topological isogroup if

\( \hat{T}G.1^\circ \) This set have to be an isogroup (see (2)–(5));

\( \hat{T}G.2^\circ \) This set have to be a topological vector isospace (see \( \hat{T}.1^\circ \sim \hat{T}.4^\circ \));

\( \hat{T}G.3^\circ \) The isogroup operations \( \hat{x} \rightarrow \hat{\alpha} \circ \hat{x}, \hat{x} \rightarrow \hat{x} \circ \hat{\alpha} \) and \( \hat{x} \rightarrow \hat{x}^{-1} \), where \( \alpha \) is a isoscalar, are isocontinuous mappings from this topological vector isospace onto itself.

And the category of topological isogroups \( \text{TopGrp} \) also consists of isocontinuous isohomomorphisms as morphisms.

**CATEGORY OF LIE-SANTILLI ISOGROUPS**

The most important for isogometry are topological isogroups that are also topological \( n \)-isomanifolds. These topological groups are called \( n \)-dimensional Lie-Santilli isogroups.

**DEFINITION 2** A category of Lie-Santilli isogroups \( \text{LSGrp} \) consists of Lie-Santilli isogroups as objects, and isohomomorphisms as morphisms.

**DEFINITION 3** A Lie-Santilli algebra can be defined as an object \( A \) in \( \text{Vect} \), the category of vector isospaces over a isofield \( \hat{F} \) of characteristic not 2, together with a morphism \([\cdot, \cdot]: A \otimes A \rightarrow A\), where \( \otimes \) refers to the monoidal product of \( \text{Vect} \), such that

\[
[A, B] \circ (\text{id} + \tau_{A,A}) = 0, \quad [A, B] \circ ([A, C] \circ \text{id}) \circ (\text{id} + \sigma + \sigma^2) = 0,
\]

where \( \tau(a \otimes b) := b \otimes a \) and \( \sigma \) is the cyclic permutation braiding \( (\text{id} \otimes \tau_{A,A}) \ast (\tau_{A,A} \otimes \text{id}) \).

The mapping from a category of Lie-Santilli groups \( \text{LSGrp} \) to a category of Lie-Santilli algebras \( \text{LSAlg} \) is functorial, which implies that isohomomorphisms of Lie-Santilli groups lift to isohomomorphisms of Lie-Santilli algebras, and various properties are satisfied by this lifting: it commutes with composition, it maps Lie-Santilli subgroups, kernels, quotients and cokernels of Lie-Santilli groups to subalgebras, kernels, quotients and cokernels of Lie-Santilli algebras, respectively. The functor \( \mathcal{L} \) which takes each Lie-Santilli group to its Lie-Santilli algebra and each isohomomorphism to its isodifferential one is faithful and exact.

When we use Lie-Santilli groups in physics to describe symmetry, we think of each element \( \hat{g} \) of the group \( \hat{G} \) as a “process”. The element \( \hat{I} \) corresponds to the “process of doing nothing at all”. We can compose processes \( \hat{g} \) and \( \hat{h} \) — do \( \hat{h} \) and then \( \hat{g} \) — and get the product \( \hat{g} \circ \hat{h} \). Crucially, every process \( \hat{g} \) can be “undone” using its inverse \( \hat{g}^{-1} \). So: in contrast to a Lie-Santilli group, which consists of a static collection of “things”, a category of Lie-Santilli groups consists not only of objects or “things” but also morphisms which can viewed as “processes” transforming one thing into another. Similarly, in a 2-category, the 2-morphisms can be regarded as “processes between processes”, and so on.

**REFERENCES**