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# Hypermathematics, $H_v$-Structures, Hypernumbers, Hypermatrices and Lie-Santilli Addmissibility

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**Abstract:** We present the largest class of hyperstructures called $H_v$-structures. In $H_v$-groups and $H_v$-rings, the fundamental relations are defined and they connect the algebraic hyperstructure theory with the classical one. Using the fundamental relations, the $H_v$-fields are defined and their elements are called hypernumbers or $H_v$-numbers. $H_v$-matrices are defined to be matrices with entries from an $H_v$-field. We present the related theory and results on hypermatrices and on the Lie-Santilli admissibility.

**Keywords:** Representations, Hope, Hyperstructures, $H_v$-Structures

## 1. Introduction to Hypermathematics, the $H_v$-Structures

Hyperstructure is called an algebraic structure containing at least one hyperoperation. More precisely, a set H equipped with at least one multivalued map $\cdot: H\times H \rightarrow P(H)$, is called hyperstructure and the map hyperoperation, we abbreviate hyperoperation by hope. The first hyperstructure was the hypergroup, introduced by F. Marty in 1934 [25], [26], where the strong generalized axioms of a group wrere used. We deal with the largest class of hyperstructures called $H_v$-structures introduced in 1990 [40],[44],[45] which satisfy the weak axioms where the non-empty intersection replaces the equality.

Some basic definitions:

Definitions 1.1 In a set H with a hope $\cdot: H\times H\rightarrow P(H)$, we abbreviate by WASS the weak associativity: $(xy)z\cap x(yz)\neq\varnothing$, $\forall x,y,z\in H$ and by COW the weak commutativity: $xy\cap yx\neq\varnothing$, $\forall x,y\in H$.

The hyperstructure $(H,\cdot)$ is called $H_v$-semigroup if it is WASS and is called $H_v$-group if it is reproductive $H_v$-semigroup:

$$xH=Hx=H,\ \forall x\in H.$$

The hyperstructure $(R,+,\cdot)$ is called $H_v$-ring if $(+)$ and $(\cdot)$ are WASS, the reproduction axiom is valid for $(+)$ and $(\cdot)$ is weak distributive with respect to $(+)$:

$$x(y+z)\cap(xy+xz)\neq\varnothing,\ (x+y)z\cap(xz+yz)\neq\varnothing,\ \forall x,y,z\in R.$$

For definitions, results and applications on $H_v$-structures, see books [44],[4],[10],[12] and papers [6],[7],[8],[9],[11],[17],[18],[19],[22],[24],[46]. An extreme class is defined as follows [41],[44]: An $H_v$-structure is very thin iff all hopes are operations except one, with all hyperproducts singletons except only one, which is a subset of cardinality more than one. Thus, a very thin $H_v$-structure is an H with a hope $(\cdot)$ and a pair $(a,b)\in H^2$ for which ab=A, with cardA>1, and all the other products, are singletons.

The main tools to study hyperstructures are the so called, fundamental relations. These are the relations $\beta^*$ and $\gamma^*$ which are defined, in $H_v$-groups and $H_v$-rings, respectively, as the smallest equivalences so that the quotient would be group and ring, respectively [38],[40],[44],[48],[49]. The way to find the fundamental classes is given as follows [44]:

Theorem 1.2 Let $(H,\cdot)$ be an $H_v$-group and let us denote by U the set of all finite products of elements of H. We define the relation $\beta$ in H as follows: $x\beta y$ iff $\{x,y\}\subset u$ where $u\in U$. Then the fundamental relation $\beta^*$ is the transitive closure of the relation $\beta$.

The main point of the proof is that $\beta$ guaranties that the following is valid: Take elements x,y such that $\{x,y\}\subset u\in U$ and any hyperproduct where one of these elements is used. Then, if this element is replaced by the other, the new hyperproduct is inside the same fundamental class where the first hyperproduct is. Thus, if the 'hyperproducts'of the above

$\beta$-classes are 'products', then, they are fundamental classes. Analogously for the $\gamma$ in $H_v$-rings.

An element is called single if its fundamental class is a singleton.

Motivation for $H_v$-structures:

1. The quotient of a group with respect to an invariant subgroup is a group.

2. Marty states that, the quotient of a group with respect to any subgroup is a hypergroup.

3. The quotient of a group with respect to any partition is an $H_v$-group.

In $H_v$-structures a partial order can be defined [44].

Definition 1.3 Let $(H,\cdot)$, $(H,\otimes)$ be $H_v$-semigroups defined on the same H. $(\cdot)$ is smaller than $(\otimes)$, and $(\otimes)$ greater than $(\cdot)$, iff there exists automorphism $f\in Aut(H,\otimes)$ such that $xy\subset f(x\otimes y)$, $\forall x\in H$.

Then $(H,\otimes)$ contains $(H,\cdot)$ and write $\cdot\leq\otimes$. If $(H,\cdot)$ is structure, then it is called basic and $(H,\otimes)$ is an $H_b$-structure.

The Little Theorem [26]. Greater hopes of the ones which are WASS or COW, are also WASS and COW, respectively.

The fundamental relations are used for general definitions of hyperstructures. Thus, to define the general $H_v$-field one uses the fundamental relation $\gamma^*$:

Definition 1.4 [40],[43],[44]. The $H_v$-ring $(R,+,\cdot)$ is an $H_v$-field if the quotient $R/\gamma^*$ is a field.

The elements of an $H_v$-field are called hypernumbers. Let $\omega^*$ be the kernel of the canonical map and from $H_v$-ring R to $R/\gamma^*$; then we call it reproductive $H_v$-field if:

$$x(R-\omega^*) = (R-\omega^*)x = R-\omega^*,\ \forall x\in R-\omega^*.$$

From this definition a new class is defined [51],[56]:

Definition 1.5 The $H_v$-semigroup $(H,\cdot)$ is called h/v-group if the $H/\beta^*$ is a group.

An $H_v$-group is called cyclic [33],[44], if there is an element, called generator, which the powers have union the underline set, the minimal power with this property is the period of the generator. If there exists an element and a special power, the minimum one, is the underline set, then the $H_v$-group is called single-power cyclic.

To compare classes we can see the small sets. To enumerate and classify $H_v$-structures, is complicate because we have great numbers. The partial order [44],[47], restrict the problem in finding the minimal, up to isomorphisms, $H_v$-structures. We have results by Bayon & Lygeros as the following [2],[3]: In sets with three elements: Up to isomorphism, there are 6.494 minimal $H_v$-groups. The 137 are abelians; 6.152 are cyclic. The number of $H_v$-groups with three elements is 1.026.462. 7.926 are abelians; 1.013.598 are cyclic, 16 are very thin. Abelian $H_v$-groups with 4 elements are, 8.028.299.905 from which the 7.995.884.377 are cyclic.

Some more complicated hyperstructures can be defined, as well. In this paper we focus on $H_v$-vector spaces and there exist an analogous theory on $H_v$-modules.

Definition 1.6 [44],[50]. Let $(F,+,\cdot)$ be an $H_v$-field, $(M,+)$ be COW $H_v$-group and there exists an external hope

$$F\times M\rightarrow P(M): (a,x)\rightarrow ax,$$

such that, $\forall a,b\in F$ and $\forall x,y\in M$ we have

$$a(x+y)\cap(ax+ay)\neq\emptyset,\ (a+b)x\cap(ax+bx)\neq\emptyset,\ (ab)x\cap a(bx)\neq\emptyset,$$

then M is called an $H_v$-vector space over F.

The fundamental relation $\varepsilon^*$ is defined to be the smallest equivalence such that the quotient $M/\varepsilon^*$ is a vector space over the fundamental field $F/\gamma^*$. For this fundamental relation there is an analogous to the Theorem 1.2.

Definitions 1.7 [51],[53],[55]. Let $(H,\cdot)$ be hypergroupoid. We remove $h\in H$, if we consider the restriction of $(\cdot)$ in the set $H-\{h\}$. We say that $h\in H$ absorbs $h\in H$ if we replace h by h and h does not appear in the structure. We say that $h\in H$ merges with $h\in H$, if we take as product of any $x\in H$ by h, the union of the results of x with both h, h, and consider h and h as one class, with representative h, therefore the element h does not appeared in the hyperstructure.

Let $(H,\cdot)$ be an $H_v$-group, then, if an element h absorbs all elements of its own fundamental class then this element becomes a single in the new $H_v$-group.

Theorem 1.8 In an $H_v$-group $(H,\cdot)$, if an element h absorbs all elements of its fundamental class then this element becomes a single in the new $H_v$-group.

Proof. Let $h\in\beta^*(h)$, then, by the definition of the 'absorb', h is replaced by h that means that $\beta^*(h)=\{h\}$. Moreover, for all $x\in H$, the fundamental property of the product of classes

$$\beta^*(x)\cdot\beta^*(h) = \beta^*(xh) \text{ becomes } \beta^*(x)\cdot h = \beta^*(xh),$$

and from the reproductivity ([44] p.19) we obtain $x\cdot h=\beta^*(xh)$, $\forall x\in\beta^*(x)$. This is the basic property that enjoys any single element [44].

Remark that in case we have a single element then we can compute all fundamental classes.

A well known and large class of hopes is given as follows [33],[37],[39],[44],[20]:

Definitions 1.9 Let $(G,\cdot)$ be a groupoid, then for every subset $P\subset G$, $P\neq\emptyset$, we define the following hopes, called P-hopes: $\forall x,y\in G$

$$P:\ xPy= (xP)y\cup x(Py),$$

$$P_r:\ xP_ry= (xy)P\cup x(yP),\quad P_l:\ xP_ly= (Px)y\cup P(xy).$$

The $(G,P)$, $(G,P_r)$ and $(G,P_l)$ are called P-hyperstructures. In the case of semigroup $(G,\cdot)$: $xPy=(xP)y\cup x(Py)=xPy$ and $(G,P)$ is a semihypergroup but we do not know about $(G,P_r)$ and $(G,P_l)$. In some cases, depending on the choice of P, the $(G,P_r)$ and $(G,P_l)$ can be associative or WASS.

A generalization of P-hopes is the following [13],[14]: Let $(G,\cdot)$ be abelian group and P a subset of G with more than one elements. We define the hope $\times_P$ as follows:

$$x\times_P y = x\cdot P\cdot y = \{x\cdot h\cdot y \mid h\in P\} \text{ if } x\neq e \text{ and } y\neq e$$

$$x\cdot y \text{ if } x=e \text{ or } y=e$$

we call this hope, $P_e$-hope. The hyperstructure $(G,\times_P)$ is an abelian $H_v$-group.

A general definition of hopes, is the following [57],[58]:

Definitions 1.10 Let H be a set with n operations (or hopes) $\otimes_1,\otimes_2,\ldots,\otimes_n$ and one map (or multivalued map) $f:H\to H$, then n hopes $\partial_1,\partial_2,\ldots,\partial_n$ on H are defined, called $\partial$-hopes by putting

$$x\partial_i y = \{f(x)\otimes_i y,\ x\otimes_i f(y)\},\ \forall x,y\in H,\ i\in\{1,2,\ldots,n\}$$

or in case where $\otimes_i$ is hope or f is multivalued map we have

$$x\partial_i y = (f(x)\otimes_i y)\cup(x\otimes_i f(y)),\ \forall x,y\in H,\ i\in\{1,2,\ldots,n\}$$

Let $(G,\cdot)$ groupoid and $f_i:G\to G$, $i\in I$, set of maps on G. Take the map $f_\cup:G\to P(G)$ such that $f_\cup(x)=\{f_i(x)\,|\,i\in I\}$, call it the union of the $f_i(x)$. We call the union $\partial$-hope ($\partial$), on G if we consider the map $f_\cup(x)$. An important case for a map f, is to take the union of this with the identity id. Thus, we consider the map $f\equiv f\cup(id)$, so $f(x)=\{x,f(x)\}$, $\forall x\in G$, which is called b-$\partial$-hope, we denote it by ($\partial$), so we have

$$x\partial y = \{xy,\ f(x)\cdot y,\ x\cdot f(y)\},\ \forall x,y\in G.$$

Remark If $\otimes_i$ is associative then $\partial_i$ is WASS. If $\partial$ contains the operation $(\cdot)$, then it is b-operation. Moreover, if $f:G\to P(G)$ is multivalued then the b-$\partial$-hopes is defined by using the $f(x)=\{x\}\cup f(x)$, $\forall x\in G$.

Motivation for the definition of $\partial$-hope is the derivative where only multiplication of functions is used. Therefore, for functions s(x), t(x), we have $s\partial t=\{s't,st'\}$, ($'$) is the derivative.

Example. For all first degree polynomials $g_i(x)=a_ix+b_i$, we have

$$g_1\partial g_2 = \{a_1a_2x+a_1b_2,\ a_1a_2x+b_1a_2\},$$

so it is a hope in the set of first degree polynomials. Moreover all polynomials x+c, where c be a constant, are units.

There exists the uniting elements method introduced by Corsini–Vougiouklis [5] in 1989. With this method one puts in the same class, two or more elements. This leads, through hyperstructures, to structures satisfying additional properties.

Definition 1.11 The uniting elements method is the following: Let G be an algebraic structure and let d be a property, which is not valid. Suppose that d is described by a set of equations; then, consider the partition in G for which it is put together, in the same partition class, every pair of elements that causes the non-validity of the property d. The quotient by this partition G/d is an $H_v$-structure. Then, quotient out the $H_v$-structure G/d by the fundamental relation $\beta^*$, a stricter structure $(G/d)\beta^*$ for which the property d is valid, is obtained.

An interesting application of the uniting elements is when more than one property is desired, because some of the properties lead straight to the classes. The commutativity and the reproductivity property are easily applicable. The following is valid:

Theorem 1.12 [44] Let $(G,\cdot)$ be a groupoid, and

$$F = \{f_1,\ldots,f_m,f_{m+1},\ldots,f_{m+n}\}$$

be a system of equations on G consisting of two subsystems

$$F_m = \{f_1,\ldots,f_m\} \text{ and } F_n = \{f_{m+1},\ldots,f_{m+n}\}.$$

Let $\sigma$, $\sigma_m$ be the equivalence relations defined by the uniting elements procedure using the systems F and $F_m$ respectively, and let $\sigma_n$ be the equivalence relation defined using the induced equations of $F_n$ on the grupoid $G_m = (G/\sigma_m)/\beta^*$. Then

$$(G/\sigma)/\beta^* \cong (G_m/\sigma_n)/\beta^*.$$

i.e. the following diagram is commutative

$$\begin{tikzcd} G \arrow[r, "\rho_m"] \arrow[d, "\rho"'] & G/\sigma_m \arrow[r, "\varphi_m"] & G_m \arrow[d, "\rho_n"] \\ G/\sigma \arrow[d, "\varphi"'] & & G_m/\sigma_n \arrow[d, "\varphi_n"] \\ (G/\sigma)/\beta^* \arrow[rr, "\cong"] & & (G_m/\sigma_n)/\beta^*. \end{tikzcd}$$

From the above it is clear that the fundamental structure is very important, and even more so if this is known from the beginning. This is the problem to construct hyperstructures with desired fundamental structures [44].

Theorem 1.13 Let $(S,\cdot)$ be a commutative semigroup with one element $w\in S$ uch that the set wS is finite. Consider the transitive closure $L^*$ of the relation L defined as follows: xLy iff there exists $z\in S$ such that zx=zy .

Then $\langle S/L^*,\circ\rangle/\beta^*$ is finite commutative group, where ($\circ$) is the induced operation on classes of $S/L^*$.

For the proof see [5],[44].

An application combining hyperstructures and fuzzy theory, is to replace the 'scale' of Likert in questionnaires by the bar of Vougiouklis & Vougiouklis [69],[70],[21],[27]:

Definition 1.14 In every question substitute the Likert scale with the 'bar' whose poles are defined with '0' on the left end, and '1' on the right end:

0 ―――――――――――――――――――― 1

The subjects/participants are asked instead of deciding and checking a specific grade on the scale, to cut the bar at any point they feel expresses their answer to the question.

The use of the bar of Vougiouklis & Vougiouklis instead of a scale of Likert has several advantages during both the filling-in and the research processing. The final suggested length of the bar, according to the Golden Ratio, is 6.2cm. The hyperstructure theory, offer innovating new suggestions to connect finite groups of objects. These suggestions are obtained from properties and special elements inside the hyperstructure.

## 2. Hyper-Representations

Representations (abbreviate by rep) of $H_v$-groups can be faced either by generalized permutations or by $H_v$-matrices [34],[36],[39],[43],[44],[52],[54],[66]. Reps by generalized permutations can be achieved by using translations [42]. We present an outline of the hypermatrix rep in $H_v$-structures and there exist the analogous theory for the h/v-structures.

Definitions 2.1 [44],[66] $H_v$-matrix is a matrix with entries elements of an $H_v$-field. The hyperproduct of two $H_v$-matrices $A=(a_{ij})$ and $B=(b_{ij})$, of type m×n and n×r respectively, is defined, in the usual manner,

$$A\cdot B = (a_{ij})\cdot(b_{ij}) = \{\, C=(c_{ij}) \mid c_{ij}\in \oplus\Sigma a_{ik}\cdot b_{kj} \,\},$$

and it is a set of m×r $H_v$-matrices. The sum of products of elements of the $H_v$-field is the union of the sets obtained with all possible parentheses put on them, called n-ary circle hope on the hyperaddition.

The hyperproduct of $H_v$-matrices does not satisfy WASS.

The problem of the $H_v$-matrix reps is the following:

Definitions 2.2 For a given $H_v$-group $(H,\cdot)$, find an $H_v$-field $(F,+,\cdot)$, a set $M_R=\{(a_{ij}) \mid a_{ij}\in F\}$ and a map $T: H\to M_R: h\to T(h)$ such that

$$T(h_1h_2)\cap T(h_1)T(h_2) \neq \varnothing,\ \forall h_1,h_2\in H.$$

The map T is called $H_v$ -matrix rep. If $T(h_1h_2)\subset T(h_1)T(h_2)$, $\forall h_1,h_2\in H$, then T is called inclusion rep. T is a good rep if $T(h_1h_2)=T(h_1)T(h_2)=\{T(h) \mid h\in h_1h_2\}, \forall h_1,h_2\in H$. If T is one to one and good then it is a faithful rep.

The problem of reps is complicated since the hyperproduct is big. It can be simplified in cases such as: The $H_v$-matrices are over $H_v$-fields with scalars 0 and 1. The $H_v$-matrices are over very thin $H_v$-fields. On 2×2 $H_v$-matrices, since the circle hope coincides with the hyperaddition. On $H_v$-fields which contain singles, which act as absorbings.

The main theorem of reps is the following [44],[52]:

Theorem 2.3 A necessary condition in order to have an inclusion rep T of an $H_v$-group $(H,\cdot)$ by n×n $H_v$-matrices over the $H_v$-field $(F,+,\cdot)$ is the following:

For all classes $\beta^*(x)$, $x\in H$ there must exist elements $a_{ij}\in H$, $i,j\in\{1,...,n\}$ such that

$$T(\beta^*(a)) \subset \{A=(a'_{ij}) \mid a'_{ij}\in \gamma^*(a_{ij}),\ i,j\in\{1,...,n\}\}$$

Thus, every inclusion rep $T:H\to M_R:a\to T(a)=(a_{ij})$ induces a homomorphic rep T* of the group $H/\beta^*$ over the field $F/\gamma^*$ by setting

$$T^*(\beta^*(a)) = [\gamma^*(a_{ij})],\ \forall\beta^*(a)\in H/\beta^*,$$

where $\gamma^*(a_{ij})\in R/\gamma^*$ is the ij entry of the matrix $T^*(\beta^*(a))$. T* is called fundamental induced rep of T.

Denote $tr_\varphi(T(x)) = \gamma^*(T(x_{ii}))$ the fundamental trace, then the mapping

$$X_T: H \to F/\gamma^*: x\to X_T(x) = tr_\varphi(T(x)) = trT^*(x)$$

is called fundamental character.

Using special classes of $H_v$-structures one can have several reps [52],[66]:

Definition 2.4 Let $M=M_{m\times n}$ be vector space of m×n matrices over a field F and take sets

$$S=\{s_k:k\in K\}\subseteq F,\ Q=\{Q_j:j\in J\}\subseteq M,\ P=\{P_i:i\in I\}\subseteq M.$$

Define three hopes as follows

$$S: F\times M\to P(M):(r,A)\to rSA=\{(rs_k)A: k\in K\}\subseteq M$$

$$Q_+: M\times M\to P(M):(A,B)\to AQ_+B=\{A+Q_j+B: j\in J\}\subseteq M$$

$$P: M\times M\to P(M):(A,B)\to APB=\{AP^t_iB: i\in I\}\subseteq M$$

Then $(M,S,Q_+,P)$ is a hyperalgebra over F called general matrix P-hyperalgebra.

The bilinear hope P, is strong associative and the inclusion distributivity with respect to addition of matrices

$$AP(B+C)\subseteq APB+APC,\ \forall A,B,C\in M$$

is valid. So (M,+,P) defines a multiplicative hyperring on non-square matrices.

In a similar way a generalization of this hyperalgebra can be defined considering an $H_v$-field instead of a field and using $H_v$-matrices instead of matrices.

In the representation theory several constructions are used, one can find some of them as follows [43],[44],[ 52], [54]:

Construction 2.5 Let $(H,\cdot)$ be $H_v$-group, then for all $(\oplus)$ such that $x\oplus y\supset\{x,y\}$, $\forall x,y\in H$, the $(H,\oplus,\cdot)$ is an $H_v$-ring. These $H_v$-rings are called associated to $(H,\cdot)$ $H_v$-rings.

In rep theory of hypergroups, in sense of Marty where the equality is valid, there are three associated hyperrings $(H,\oplus,\cdot)$ to $(H,\cdot)$. The $(\oplus)$ is defined respectively, $\forall x,y\in H$, by:

type a: $x\oplus y=\{x,y\}$, type b: $x\oplus y=\beta^*(x)\cup\beta^*(y)$, type c: $x\oplus y=H$

In the above types the strong associativity and strong or inclusion distributivity, is valid.

Construction 2.6 Let $(H,\cdot)$ be an $H_V$-semigroup and $\{v_1,\dots,v_n\}\cap H=\varnothing$, an ordered set, where if $v_i<v_j$, when $i<j$. Extend $(\cdot)$ in $H_n=H\cup\{v_1,v_2,\dots,v_n\}$ as follows:

$$x\cdot v_i = v_i\cdot x = v_i,\ v_i\cdot v_j = v_j\cdot v_i = v_j,\ \forall i<j \text{ and}$$

$$v_i\cdot v_i = H\cup\{v_1,\dots,v_{i-1}\},\ \forall x\in H,\ i\in\{1,2,\dots,n\}.$$

Then $(H_n,\cdot)$ is an $H_V$-group, called Attach Elements Construction, and $(H_n,\cdot)/\beta^*\cong Z_2$, where $v_n$ is single [51],[55].

Some problems arising on the topic, are:

Open Problems.

a. Find standard $H_v$-fields to represent all $H_v$-groups.

b. Find reps by $H_v$-matrices over standard finite $H_v$-fields analogous to $Z_n$.

c. Using matrices find a generalization of the ordinary multiplication of matrices which it could be used in $H_v$-rep theory (see the helix-hope [68]).

d. Find the 'minimal' hypermatrices corresponding to the minimal hopes.

e. Find reps of special classes of hypergroups and reduce these to minimal dimensions.

Recall some definitions from [68],[16],[32]:

Definitions 2.7 Let $A=(a_{ij})\in M_{m\times n}$ be m×n matrix and $s,t\in N$ be natural numbers such that $1\le s\le m$, $1\le t\le n$. Then we define a characteristic-like map cst: $M_{m\times n}\to M_{s\times t}$ by corresponding to the matrix A, the matrix $Acst=(a_{ij})$ where $1\le i\le s$, $1\le j\le t$. We call it cut-projection of type st. We define the mod-like map st: $M_{m\times n}\to M_{s\times t}$ by corresponding to A the matrix Ast=($\underline{a}_{ij}$) which has as entries the sets

$$\underline{a}_{ij}=\{a_{i+\kappa s,j+\lambda t}\mid 1\le i\le s, 1\le j\le t \text{ and } \kappa,\lambda\in N,\ i+\kappa s\le m,\ j+\lambda t\le n\}.$$

Thus we have the map

$$st: M_{m\times n}\to M_{s\times t}: A\to Ast=(\underline{a}_{ij}).$$

We call this multivalued map helix-projection of type st. So Ast is a set of s×t-matrices $X=(x_{ij})$ such that $x_{ij}\in \underline{a}_{ij}, \forall i,j$.

Let $A=(a_{ij})\in M_{m\times n}$, $B=(b_{ij})\in M_{u\times v}$ matrices and s=min(m,u), t=min(n,u). We define a hope, called helix-addition or helix-sum, as follows:

$$\oplus: M_{m\times n}\times M_{u\times v}\to P(M_{s\times t}):$$

$$(A,B)\to A\oplus B=Ast+Bst=(\underline{a}_{ij})+(\underline{b}_{ij})\subset M_{s\times t},$$

where

$$(\underline{a}_{ij})+(\underline{b}_{ij})=\{(c_{ij})=(a_{ij}+b_{ij})\mid a_{ij}\in\underline{a}_{ij} \text{ and } b_{ij}\in\underline{b}_{ij}\}.$$

And define a hope, called helix-multiplication or helix-product, as follows:

$$\otimes: M_{m\times n}\times M_{u\times v}\to P(M_{m\times v}): (A,B)\to A\otimes B=Ams\cdot Bsv=(\underline{a}_{ij})\cdot(\underline{b}_{ij})\subset M_{m\times v},$$

where

$$(\underline{a}_{ij})\cdot(\underline{b}_{ij})=\{(c_{ij})=(\textstyle\sum a_{it}b_{tj})\mid a_{ij}\in\underline{a}_{ij} \text{ and } b_{ij}\in\underline{b}_{ij}\}.$$

Remark. In $M_{m\times n}$ the addition of matrices is an ordinary operation, therefore we are interested only in the 'product'. From the fact that the helix-product on non square matrices is defined, the definition of the Lie-bracket is immediate, therefore the helix-Lie Algebra is defined [62], as well. This algebra is an $H_v$-Lie Algebra where the fundamental relation ε* gives, by a quotient, a Lie algebra, from which a classification is obtained.

For more results on the topic see [16],[32],[61],[62].

In the following we denote $E_{ij}$ any type of matrices which have the ij-entry 1 and in all the other entries we have 0.

Example 2.8 Consider the 2×3 matrices of the following form,

$$A_\kappa=E_{11}+\kappa E_{21}+E_{22}+E_{23},\ B_\kappa=\kappa E_{21}+E_{22}+E_{23},\ \forall\kappa\in\mathbb{N}.$$

Then we obtain $A_\kappa\otimes A_\lambda=\{A_{\kappa+\lambda},A_{\lambda+1},B_{\kappa+\lambda},B_{\lambda+1}\}$

Similarly, $B_\kappa\otimes A_\lambda=\{B_{\kappa+\lambda},B_{\lambda+1}\}$, $A_\kappa\otimes B_\lambda=B_\lambda=B_\kappa\otimes B_\lambda$.

Thus the set $\{A_\kappa,B_\lambda\mid\kappa,\lambda\in\mathbb{N}\}$ becomes an $H_v$-semigroup which is not COW because for κ≠λ we have

$$B_\kappa\otimes B_\lambda=B_\lambda\neq B_\kappa=B_\lambda\otimes B_\kappa,$$

however

$$(A_\kappa\otimes A_\lambda)\cap(A_\lambda\otimes A_\kappa)=\{A_{\kappa+\lambda},B_{\kappa+\lambda}\}\neq\varnothing,\ \forall\kappa,\lambda\in\mathbb{N}.$$

All elements $B_\lambda$ are right absorbing and $B_1$ is a left scalar, because $B_1\otimes A_\lambda=B_{\lambda+1}$ and $B_1\otimes B_\lambda=B_\lambda$, $A_0$ is a unit.

# 3. Hyper-Lie-Algebras

Lie-Santilli admisibility

The general definition of an $H_v$-Lie algebra over an $H_v$-field is given as follows [61],[62]:

Definition 3.1 (L,+) be $H_v$-vector space on $H_v$-field $(F,+,\cdot)$, $\varphi:F\to F/\gamma^*$ the canonical map and $\omega_F=\{x\in F:\varphi(x)=0\}$, where 0 is the zero of the fundamental field $F/\gamma^*$. Moreover, let $\omega_L$ be the core of the canonical map $\varphi': L\to L/\varepsilon^*$ and denote by the same symbol 0 the zero of $L/\varepsilon^*$. Consider the bracket (commutator) hope:

$$[\ ,\ ]: L\times L\to P(L): (x,y)\to[x,y]$$

then L is called an $H_v$-Lie algebra over F if the following axioms are satisfied:

(L1) The bracket hope is bilinear, i.e.

$$[\lambda_1x_1+\lambda_2x_2,y]\cap(\lambda_1[x_1,y]+\lambda_2[x_2,y])\neq\varnothing$$

$$[x,\lambda_1y_1+\lambda_2y]\cap(\lambda_1[x,y_1]+\lambda_2[x,y_2])\neq\varnothing,$$

$$\forall x,x_1,x_2,y,y_1,y_2\in L \text{ and } \lambda_1,\lambda_2\in F$$

$$\text{(L2) } [x,x]\cap\omega_L\neq\varnothing,\ \forall x\in L$$

$$\text{(L3) } ([x,[y,z]]+[y,[z,x]]+[z,[x,y]])\cap\omega_L\neq\varnothing,\ \forall x,y\in L$$

Example 3.2 Consider all traceless matrices $A=(a_{ij})\in M_{2\times 3}$, in the sense that $a_{11}+a_{22}=0$. In this case, the cardinality of the helix-product of any two matrices is 1, or $2^3$, or $2^6$. These correspond to the cases: $a_{11}=a_{13}$ and $a_{21}=a_{23}$, or only $a_{11}=a_{13}$ either only $a_{21}=a_{23}$, or if there is no restriction, respectively. For the Lie-bracket of two traceless matrices the corresponding cardinalities are up to 1, or $2^6$, or $2^{12}$, resp. We remark that, from the definition of the helix-projection, the initial 2×2, block guaranties that in the result there exists at least one traceless matrix.

From this example it is obvious the following:

Theorem 3.3 Using the helix-product the Lie-bracket of any two traceless matrices $A=(a_{ij})$, $B=(b_{ij})\in M_{m\times n}$, m<n, contain at least one traceless matrix.

Last years, hyperstructures have a variety of applications in mathematics and other sciences. The hyperstructures theory can now be widely applicable in industry and production, too. In several books [4],[10],[12] and papers [1],[11],[17],[23],[31],[35],[50],[67],[70] one can find numerous applications.

The Lie-Santilli theory on isotopies was born in 1970's to solve Hadronic Mechanics problems. Santilli proposed [28] a 'lifting' of the trivial unit matrix of a normal theory into a nowhere singular, symmetric, real-valued, new matrix. The original theory is reconstructed such as to admit the new matrix as left and right unit. The isofields needed in this theory correspond into the hyperstructures were introduced by Santilli and Vougiouklis in 1996 and they are called e-hyperfields [29],[30],[59],[60],[64],[13],[14],[15] which are used in physics or biology. The $H_v$-fields can give e-hyperfields which can be used in the isotopy theory for applications.

The IsoMathematics Theory is very important subject in applied mathematics. It is a generalization by using a kind of the Rees analogous product on matrix semigroup with a sandwich matrix, like the P-hopes. It contains the classical theory but also can find easy solutions in different branches of mathematics. To compare this novelty we give two analogous examples: (1) The unsolved, from ancient times, problems in Geometry was solved in a different branch of mathematics, the Algebra with the genius Galois Theory. (2) With the Representation Theory one can solve problems in Lie Algebras and to transfer these in Lie Groups using the exponential map, and the opposite. One very important thing of the IsoMathematics Theory is that admits generalizations, as well. Two very important of them are the following: First, is the so called Admissible Lie-Santilli Algebras [28],[30], [62],[65] by using again a kind of Rees sandwich product. Second, is that one can extend this theory into the multivalued case, i.e. into $H_v$-structures.

Definitions 3.4 A hyperstructure $(H,\cdot)$ containing a unique scalar unit e, is called e-hyperstructure. We assume that $\forall x$, there is an inverse $x^{-1}$, i.e. $e\in x\cdot x^{-1}\cap x^{-1}\cdot x$. A hyperstructure $(F,+,\cdot)$, where $(+)$ is an operation and $(\cdot)$ is a hope, is called e-hyperfield if the following are valid:

$(F,+)$ is abelian group with the additive unit 0, $(\cdot)$ is WASS,

$(\cdot)$ is weak distributive with respect to $(+)$, 0 is absorbing: $0\cdot x=x\cdot 0=0$, $\forall x\in F$, there exist a multiplicative scalar unit 1, i.e. $1\cdot x=x\cdot 1=x$, $\forall x\in F$, and $\forall x\in F$ there exists a unique inverse $x^{-1}$, such that $1\in x\cdot x^{-1}\cap x^{-1}\cdot x$.

The elements of an e-hyperfield are called e-hypernumbers. In the case that the relation: $1=x\cdot x^{-1}=x^{-1}\cdot x$, is valid, then we say that we have a strong e-hyperfield.

A general construction based on the partial ordering of the $H_v$-structures:

Construction 3.5 [13],[14],[15],[30] Main e-Construction. Given a group $(G,\cdot)$, where e is the unit, then we define in G, a large number of hopes $(\otimes)$ by extended $(\cdot)$, as follows:

$$x\otimes y=\{xy,g_1,g_2,\ldots\},\ \forall x,y\in G\text{-}\{e\},\text{ and } g_1,\ g_2,\ldots\in G\text{-}\{e\}$$

Then $(G,\otimes)$ becomes an $H_v$-group, in fact is $H_b$-group which contains the $(G,\cdot)$. The $H_v$-group $(G,\otimes)$ is an e-hypergroup. Moreover, if $\forall x,y$ such that $xy=e$, so we have $x\otimes y=xy$, then $(G,\otimes)$ becomes a strong e-hypergroup.

Definition 3.6 Let $(H_o,+,\cdot)$ be the attached, by one element, $H_v$-field of the $H_v$-semigroup $(H,\cdot)$. Thus, for $(H,\cdot)$, take an element v outside of H, and extend $(\cdot)$ in $H_n=H\cup\{v\}$ by:

$$x\cdot v=v\cdot x=v,\ v\cdot v=H,\ \forall x\in H.$$

$(H_n,\cdot)$ is an $H_V$-group, called Attach Elements Construction, and $(H_n,\cdot)/\beta^*\cong Z_2$, where v, is single. If $(H,\cdot)$ has a left and right scalar unit e then $(H_o,+,\cdot)$ is an e-hyperfield, the attached $H_v$-field of $(H,\cdot)$.

Remark. The above main e-construction gives an extremely large class of e-hopes. These e-hopes can be used in the several more complicate hyperstructures to obtain appropriate e-hyperstructures. However, the most useful are the ones where only few products are enlarged.

Example 3.7 Take the finite-non-commutative quaternion group $Q=\{1,-1,\ i,-i,\ j,-j,\ k,-k\}$. Using this operation one can obtain several hopes which define very interesting e-groups. For example, denoting $i=\{i,-i\}$, $j=\{j,-j\}$, $k=\{k,-k\}$ we may define the $(*)$ hope by the Cayley table:

***Table 1.***

| * | 1 | -1 | i | -i | j | -j | k | -k |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | -1 | i | -i | j | -j | k | -k |
| -1 | -1 | 1 | -i | i | -j | j | k | k |
| i | i | -i | -1 | 1 | k | -k | -j | j |
| -i | -i | i | 1 | -1 | -k | k | j | -j |
| j | j | -j | -k | k | -1 | 1 | i | -i |
| -j | -j | j | k | -k | 1 | -1 | -i | i |
| k | k | k | j | -j | -i | i | -1 | 1 |
| -k | -k | k | -j | j | i | -i | 1 | -1 |

The hyperstructure $(Q,*)$ is strong e-hypergroup because 1 is scalar unit and the elements -1,i,-i,j,-j,k and -k have unique inverses the elements -1,-i,i,-j,j,-k and k, resp., which are the inverses in the basic group. Thus, from this example one can have more strict hopes.

In [30],[62],[65] a kind of P-hopes was introduced which is appropriate to extent the Lie-Santilli admissible algebras in hyperstructures:

The general definition is the following:

Construction 3.8 Let $(L=M_{m\times n},+)$ be an $H_v$-vector space of $m\times n$ hyper-matrices over the $H_v$-field $(F,+,\cdot)$, $\varphi:F\to F/\gamma^*$, the canonical map and $\omega_F=\{x\in F:\varphi(x)=0\}$, where 0 is the zero of the fundamental field $F/\gamma^*$, $\omega_L$ be the core of the canonical map $\varphi':L\to L/\varepsilon^*$ and denote again by 0 the zero of $L/\varepsilon^*$. Take any two subsets $R,S\subseteq L$ then a Santilli's Lie-admissible hyperalgebra is obtained by taking the Lie bracket, which is a hope:

$$[,]_{RS}: L\times L\to P(L):\ [x,y]_{RS}=xR^ty-yS^tx.$$

Notice that $[x,y]_{RS}=xR^ty-yS^tx=\{\ r^ty-ys^tx\mid r\in R\text{ and }s\in S\}$.

Special cases, but not degenerate, are the 'small' and 'strict':

$$\text{(a) } R=\{e\} \text{ then } [x,y]_{RS} = xy-yS^tx = \{xy-ys^tx\mid s\in S\}$$

$$\text{(b) } S=\{e\} \text{ then } [x,y]_{RS} = xR^ty-yx = \{xr^ty-yx\mid r\in R\}$$

$$\text{(c) } R=\{r_1,r_2\} \text{ and } S=\{s_1,s_2\} \text{ then}$$

$$[x,y]_{RS} = xR^ty-yS^tx =$$

$$\{xr_1{}^ty-ys_1{}^tx,\ xr_1{}^ty-ys_2{}^tx,\ xr_2{}^ty-ys_1{}^tx,\ xr_2{}^ty-ys_2{}^tx\}$$

# 4. Galois $H_v$-Fields and Low Dimensional $H_v$-Matrices

Recall some results from [63], which are referred to finite $H_v$-fields which we will call, according to the classical theory, Galois $H_v$-fields. Combining the uniting elements procedure with the enlarging theory we can obtain stricter structures or hyperstructures. So enlarging operations or hopes we can obtain more complicated structures.

Theorem 4.1 In the ring $(Z_n,+,\cdot)$, with n=ms we enlarge the multiplication only in the product of elements $0\cdot m$ by setting $0\otimes m=\{0,m\}$ and the rest results remain the same. Then

$$(Z_n,+,\otimes)/\gamma^* \cong (Z_m,+,\cdot).$$

Proof. First we remark that the only expressions of sums and products which contain more, than one, elements are the expressions which have at least one time the hyperproduct $0\otimes m$. Adding to this special hyperproduct the element 1, several times we have the equivalence classes modm. On the other side, since m is a zero divisor, adding or multiplying elements of the same class the results are remaining in one class, the class obtained by using only the representatives. Therefore, $\gamma^*$-classes form a ring isomorphic to $(Z_m,+,\cdot)$.

Remark. In the above theorem we can enlarge other products as well, for example $2\cdot m$ by setting $2\otimes m=\{2,m+2\}$, then the result remains the same. In this case the elements 0 and 1 remain scalars, so they are refered in e-hyperstructures.

From the above theorem it is immediate the following:

Corollary 4.2 In the ring $(Z_n,+,\cdot)$, with n=ps where p is a prime number, we enlarge the multiplication only in the product of the elements $0\cdot p$ by setting $0\otimes p=\{0,p\}$ and the rest results remain the same. Then the hyperstructure $(Z_n,+,\otimes)$ is a very thin $H_v$-field.

The above theorem provides the researchers with $H_v$-fields appropriate to the rep theory since they may be smaller or minimal hyperstructures.

Remarks 4.3 The above theorem in connection with Uniting Elements method leads to the fact that in $H_v$-structure theory it is able to equip algebraic structures or hyperstructures with properties as associativity, commutativity, reproductivity. This equipment can be applied independently of the order of the desired properties. This is crucial point since some properties are easy to be applied, so we can apply them first, and then the difficult ones. For example from an $H_v$-ring we first go to an $H_v$-integral domain, by uniting the zero divisors, and then to the $H_v$-field by reaching the reproductivity.

Construction 4.5 (Galois $H_v$-fields) In the ring $(Z_n,+,\cdot)$, with n=ps where p is prime, enlarge only the product of the elements 2 by p+2, i.e. $2\cdot(p+)$, by setting $2\otimes(p+2)=\{2,p+2\}$ and the rest remain the same. Then $(Z_n,+,\otimes)$ is a COW very thin $H_v$-field where 0 and 1 are scalars and we have:

$$(\mathbf{Z}_n,+,\otimes)/\gamma^* \cong (\mathbf{Z}_p,+,\cdot).$$

Proof. Straightforward.

Remark 4.6 Galois Hv-fields of the above type are the most appropriate in the representation theory since the cardinality of the products is low. Moreover, one can use more enlargements using elements of the same fundamental class, therefore, one can have several cardinalities. The low dimensional reps can be based on the above Galois Hv-fields, since they use infinite Hv-fields although the fundamental fields are finite.

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