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Measurable Iso-Functions

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Abstract: In this article are given definitions definition for measurable is-functions of the first, second, third, fourth and fifth kind. They are given examples when the original function is not measurable and the corresponding iso-function is measurable and the inverse. They are given conditions for the isotopic element under which the corresponding is-functions are measurable. It is introduced a definition for equivalent iso-functions. They are given examples when the iso-functions are equivalent and the corresponding real functions are not equivalent. They are deducted some criterions for measurability of the iso-functions of the first, second, third, fourth and fifth kind. They are investigated for measurability the addition, multiplication of two iso-functions, multiplication of iso-function with an iso-number and the powers of measurable iso-functions. They are given definitions for step iso-functions, iso-step iso-functions, characteristic iso-functions, iso-characteristic iso-functions. It is investigate for measurability the limit function of sequence of measurable iso-functions. As application they are formulated the iso-Lebesgue's theorems for iso-functions of the first, second, third, fourth and fifth kind. These iso-Lebesgue's theorems give some information for the structure of the iso-functions of the first, second, third, fourth and fifth kind.

Keywords: Measurable Iso-Sets, Measurable Is-Functions, Is-Lebesgue Theorems

1. Introduction

Genious idea is the Santilli's generalization of the basic unit of quantum mechanics into an integro-differential operator $\hat{I}$ which is as positive-definite as +1 and it depends of local variables and it is assumed to be the inverse of the isotopic element $\hat{T}$

$$+1 > 0 \to \hat{I}(t, r, p, a, E, \cdots) = \frac{1}{\hat{T}} > 0$$

and it is called Santilli isounit. Santilli introduced a generalization called lifting of the conventional associative product ab into the form

$$ab \to a\hat{\times}b = a\hat{T}b$$

Called isoproduct for which

$$\hat{I}\hat{\times}a = \frac{1}{\hat{T}}\hat{T}a = a\hat{\times}\hat{I} = a\hat{T}\frac{1}{\hat{T}} = a$$

For every element a of the field of real numbers, complex numbers and quaternions. The Santilli isonumbers are defi ned as follows: for given real number or complex number or quaternion a,

$$\hat{a} = a\hat{I}$$

With isoproduct

$$\hat{a}\hat{\times}\hat{b} = \hat{a}\hat{T}\hat{b} = a\frac{1}{\hat{T}}\hat{T}b\frac{1}{\hat{T}} = ab\frac{1}{\hat{T}} = \widehat{ab}.$$

If $a \neq 0$, the corresponding isoelement of $\frac{1}{a}$ will be denoted with $\hat{a}^{-1}$or $\hat{I} \searrow \hat{a}$.

With $\hat{F}_{\mathbb{R}}$ we will denote the field of the is-numbers $\hat{a}$ for which $a \in \mathbb{R}$ and basic isounit $\hat{I}_1$.

In [1], [3]-[12] are defined isocontinuous isofunctions and isoderivative of isofunction and in [1] are proved some of their properties. If $\widehat{D}_1$ is an isoset in $\hat{F}_{\mathbb{R}}$, the class of isofunctions is denoted by $\widehat{FC}_{D_1}$ and the class of isodifferentiable isofunctions is denoted by $\widehat{FC}^1_{D_1}$, with the same basic isounit $\hat{I} = \frac{1}{\hat{T}}$, it is supposed

$$\hat{T} \in C^1(D_1), \hat{T} > 0 \; in \; D_1.$$

Here D_1 is the corresponding real set of $\widehat{D}_1$. If x is an independent variable, then the corresponding lift is $\frac{x}{\hat{T}(x)}$, if f is real-valued function on D_1, then the corresponding lift of first kind is defined as follows

$$\hat{f}^{\wedge}(\hat{x}) = \frac{f\left(\hat{t}(x)\frac{x}{\widehat{T(x)}}\right)}{\hat{T}(x)} = \frac{f(x)}{\hat{T}(x)}$$

and we will denote it by $\ddot{f}^{\wedge\wedge}$.

In accordance with [1], the isodifferential is defined as follows

$$\hat{d}(\cdot) = \hat{T}(x)d(\cdot).$$

Then

$$\hat{d}(\hat{x}) = \hat{T}(x)d(\hat{x}) = \hat{T}(x)d\left(\frac{x}{\widehat{T(x)}}\right) = \hat{T}(x)\left(\frac{1}{\widehat{T(X)}} -\right.$$

$$XT'(x)T2(x)dx{=}1-xT'(x)T(x)dx,,$$

$$\hat{d}\left(\hat{f}^{\wedge}(\hat{x})\right) = \hat{T}(x)d\left(\hat{f}^{\wedge}(\hat{x})\right) = \widehat{T(x)d}\left(\frac{f(x)}{\hat{T}(x)}\right) = \left(f'(x) - f(x)\frac{\hat{T}'(x)}{\hat{T}(x)}\right)dx.$$

In accordance with [1], the first is-derivative of the is-function $\hat{f}^{\wedge\wedge}$ is defined as follows

$$\left(\hat{f}^{\wedge\wedge}\ \ (x)\right)^{\otimes}_{x} = \hat{d}\left(\hat{f}^{\wedge}(\hat{x})\right) \nearrow \hat{d}(\hat{x}) = \frac{1}{\hat{T}(x)}\frac{\hat{d}\left(\hat{f}^{\wedge}(\hat{x})\right)}{\hat{d}(\hat{x})} = \frac{f'(x)\hat{T}(x) - f(x)\widehat{T'(x)}}{\hat{T}^2(x) - x\hat{T}(x)\hat{T}'(x)}.$$

When $\hat{T}(x) \equiv 1$, then

$$\left(\hat{f}^{\wedge}(\hat{x})\right)^{\otimes}_{x} = f'(x).$$

Our aim in this article is to be investigated some aspects of theory of measurable iso-functions. The paper is organized as follows. In Section 2 are defined measurable iso-functions and they are deducted some of their properties. In Section 3 is investigated the structure of the measurable iso-functions.

2. The Definition and the Simplest Properties of Measurable Is-Functions

We suppose that A is a given point set, $\hat{T} : A \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$ for every $x \in A$, $\hat{T}_1 > 0$ be a given constant, $f: A \longrightarrow \mathbb{R}$ bea given real-valued function. With $\hat{f}$ we will denote the corresponding is-function of the first, second, third, fourth and fifth kind. More precisely,

1. $\hat{f}(x) \equiv \hat{f}^{\wedge}(\hat{x}) = \frac{f(x)}{\hat{T}(x)}$, when $\hat{f}$ is an is-function of the first kind.
2. $\hat{f}(x) = \hat{f}^{\wedge}(x) = \frac{f(x\hat{T}(x))}{\hat{T}(x)}$, when $\frac{x}{\hat{T}(x)} \in A$ for $x \in A$, when $\hat{f}$ is an is-function of the second kind.
3. $\hat{f}(x) = \hat{f}(\hat{x}) = \frac{f\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)}$, $\frac{x}{\hat{T}(x)} \in A$ for $x \in A$, when $\hat{f}$ is an is-function of the third kind.
4. $\hat{f}(x) \equiv f^{\wedge}(x) = f\left(x\hat{T}(x)\right)$, when $x\hat{T}(x) \in A$ for $x \in A$, when $\hat{f}$ is an is-function of the fourth kind.
5. $\hat{f}(x) = f^{\vee}(x) = f\left(\frac{x}{\hat{T}(x)}\right)$, $\frac{x}{\hat{T}(x)} \in A$ for $x \in A$, when $\hat{f}$ is an is-function of the fifth kind.

For $a \in A$ with $A(\hat{f} > a)$ we will denote the set

$$A(\hat{f} > a) := \{x \in A : \hat{f}(x) > a\}.$$

We define the symbols $A(\hat{f} \geq a)$, $A(\hat{f} = a)$, $A(\hat{f} < a)$, $A(a < \hat{f} < b)$ and etc., in the same way.

If the set on which the is-function $\hat{f}$ is defined is designated by a letter C or D, we shall write $C(\hat{f}$>a) or $D(\hat{f}$>a).

Definition 2.1. The is-function $\hat{f}$ is said to be measurable if

1. The set A is measurable.
2. The set $A(\hat{f} > a)$ is measurable for all $a \in A$.

Theorem 2.3. Let $\hat{f}$ be a measurable is-function defined on the set A. If B is a measurable subset of A, then the is-function $\hat{f}(x)$, considered only for$x \in B$, is measurable.

Proof. Let $a \in \mathbb{R}$ be arbitrarily chosen and fixed. We will prove that

$$B(\hat{f} > a) = B \cap A(\hat{f} > a). \tag{1}$$

Really, let $x \in B(\hat{f} > a)$ be arbitrarily chosen. Then $x \in B$ and $\hat{f}$(x)>a. Since $B \subset A$, we have that $x \in A$. From $x \in A$ and $\hat{f}(x) > a$ it follows that $x \in A(\hat{f} > a)$. Because $x \in B(\hat{f} > a)$ was arbitrarily chosen and for it we get that it is an element of the set $B \cap A(\hat{f} > a)$, we conclude that

$$B \subset (\hat{f} > a) \subset B \cap A(\hat{f} > a). \tag{2}$$

Let now $x \in B \cap A(\hat{f} > a)$ be arbitrarily chosen. Then $x \in B$ and $x \in A(\hat{f} > a)$. Hence $x \in B$ and $\hat{f}(x) > a$. Therefore $x \in B(\hat{f} > a)$. Because $x \in B \cap A(\hat{f} > a)$ was arbitrarily chosen and we get that it is an element of $B(\hat{f} > a,$ we conclude that

$$B \cap A(\hat{f} > a) \subset B(\hat{f} > a).$$

From the last relation and from (2) we prove the relation (1).

Since the iso-function $\hat{f}$ is a measurable function on the set A, we have that $A(\hat{f} > a)$ is a measurable set. As the intersection of two measurable sets is a measurable set, we have that $B \cap A(\hat{f} > a)$ is a measurable set. Consequently, using (1), the set $B(\hat{f}$>a) is measurable set. In this way we have

1. B is a measurable set,
2. $B(\hat{f} > a)$ is a measurable set for all $a \in \mathbb{R}$.

Therefore the iso-function $\hat{f}$, considered only for $x \in B$, is a measurable is-function.

Theorem 2.4. Let $\hat{f}$ be defined on the set A, which is the union of a finite or denumerable number of measurable sets A_K, $A = \bigcup_K A_k$. If $\hat{f}$ is measurable on each of the sets A_k, then it is also measurable on A.

Proof. Let $a \in \mathbb{R}$ be arbitrarily chosen. We will prove that

$$A(\hat{f} > a) = \cup_k A_k(\hat{f} > a). \qquad (3)$$

Let $x \in A(\hat{f} > a)$ be arbitrarily chosen. Then $x \in A$ and $\hat{f}(x) > a$. Since $x \in A$ and $A = \cup_k A_k$, there exists k such that $x \in A_k$. Therefore $x \in A_k$ and $\hat{f}(x) > a$. Hence, $x \in A_k(\hat{f} > a)$ and $x \in \cup_k A_k(\hat{f} > a)$. Because $x \in A(\hat{f} > a)$ was arbitrarily chosen and for it we get that it is an element of $\cup_k A_k(\hat{f} > a)$, we conclude that

$$A(\hat{f} > a) \subset \cup_k A_k(\hat{f} > a). \qquad (4)$$

Let now $y \in \cup_k A_k(\hat{f} > a)$ be arbitrarily chosen. Then there exists l such that $y \in A_l(\hat{f}$>a). From here $x \in A_l$ and $\hat{f}(y) > a$. Hence, $y \in A = \cup_k A_k$ and $\hat{f}(y) > a$. Consequently $\in A(\hat{f} > a)$. Because $y \in \cup_k A_k(\hat{f} > a)$ was arbitrarily chosen and for it we get that it is an element of $A(\hat{f} > a)$ we conclude that

$$A(\hat{f} > a) \subset \bigcup_k A_k(\hat{f} > a).$$

From the last relation and from (4) we prove the relation (3).

Since the union of finite or denumerable number of measurable sets is a measurable set, using that the sets $A_k(\hat{f} > a)$ are measurable, we obtain that A and $A(\hat{f} > a)$ are measurable sets. Therefore $\hat{f}$ is a measurable is-function.

Definition 2.5. Two is-functions $\hat{f}$ and $\hat{g}$, defined on the same setr A, are said to be equivalent if

$$\mu\left(A(\hat{f} \neq \hat{g})\right) = 0.$$

We will write

$$\hat{f} \sim \hat{g}.$$

Remark 2.6. There is a possibility $f \nsim g$ and in the same time$\hat{f} \sim \hat{g}$.

Let

$$A = [1,2], f(x) = x, g(x) = x + 1,$$

$$\hat{T}(x) = \frac{-1 + \sqrt{1 + 4x^2}}{2x}, x \in A.$$

Then

$$f \nsim g.$$

On the oth-1+er hand,

$$\hat{f}^\wedge(\hat{x}) = \frac{f(x)}{\hat{T}(x)} = \frac{x}{\frac{-1 + \sqrt{1 + 4x^2}}{x}}$$

$$= \frac{2x^2}{-1 + \sqrt{1 + 4x^2}} = \frac{2x^2\left(1 + \sqrt{1 + 4x^2}\right)}{\left(\sqrt{1 + 4x^2} - 1\right)\left(\sqrt{1 + 4x^2} - 1\right)}$$
$$= \frac{2x^2\left(1 + \sqrt{1 + 4x^2}\right)}{4x^2} = \frac{1 + \sqrt{1 + 4x^2}}{2},$$

$$g^\wedge(x) = g\left(x\hat{T}(x)\right) = x\hat{T}(x) + 1$$
$$= x\frac{-1 + \sqrt{1 + 4x^2} + 1}{2x} = \frac{-1 + \sqrt{1 + 4x^2}}{2} + 1$$
$$= \frac{1 + \sqrt{1 + 4x^2}}{2}.$$

We have that

$$\mu\left(A(\hat{f}^{\wedge\wedge} \neq g^\wedge)\right) = 0$$

Or

$$\hat{f}^{\wedge\wedge} \sim g^\wedge.$$

Remark 2.7. There is a possibility $f \sim g$ and in the same time $\hat{f} \nsim \hat{g}$. Let

$$A = [1,2], f(x) = g(x) = x^2, \hat{T}(x) = x + 1, x \in A.$$

Then

$$f \sim g.$$

On the other hand,

$$f^\wedge(x) = f\left(x\hat{T}(x)\right) = x^2\hat{T}^2(x) = x^2(x+1)^2, g^\vee(x) = g\left(\frac{x}{\hat{T}(x)}\right) = \frac{x^2}{\hat{T}^2(x)} = \frac{x^2}{(x+1)^2}.$$

Then

$$f^\wedge(x) = g^\vee(x) \iff x^2(x+1)^2 = \frac{x^2}{(x+1)^2} \iff (x+1)^4 = 1 \iff x = 0 \notin A.$$

Therefore

$$\mu\left(A(f^\wedge = g^\vee)\right) = 0,$$

Hence,

$$\mu\left(A(f^\wedge \neq g^\vee)\right) = 1.$$

Consequently

$$f^\wedge \sim g^\vee.$$

Proposition 2.8. The fuAnctions f and g are equivalent if and only if the functions $\hat{f}^{\wedge\wedge}$ and $\hat{g}^{\wedge\wedge}$ are equivalent

Proof. We have

$$\mu\left(A(f \neq g)\right) = 0 \iff \mu\left(A\left(\frac{f}{\hat{T}} \neq \frac{g}{\hat{T}}\right)\right) = 0$$
$$\iff \mu\left(A(\hat{f}^{\wedge\wedge} \neq \hat{g}^{\wedge\wedge})\right) = 0.$$

Definition 2.9. Let some property P holds for all the points of the set A, except for the points of a subset B of the set A. If $= 0$, we say that the property P holds almost everywhere on the set A, or for almost all points of A.

Definition 2.10. We say that two is-functions defined on the set A are equivalent if they are equal almost everywhere on the set A.

Theorem 2.11. If $\hat{f}(x)$ is a measurable is-function defined on the set A, and if $\hat{f} \sim \hat{g}$, then the is-function $\hat{g}(x)$ is also measurable.

Proof. Let

$$B := A(\hat{f} \neq \hat{g}), D := A \setminus B.$$

Because $\hat{f} \sim \hat{g}$ we have that

$$\mu\left(A(\hat{f} \neq \hat{g})\right) = 0$$

or $\mu B = 0$.

Since every function, definite on a set with measure zero is measurable on it, we have that the is-function $\hat{g}$ is measurable on the set B.

We note that the is-functions $\hat{f}(x)$ and $\hat{g}(x)$ are identical on D and since the is0-function $\hat{f}$ is measurable on D, we get that the is-function $\hat{g}$ is measurable on D.

Consequently the is-function $\hat{g}$ is measurable on

$$B \cup D = A.$$

Theorem 2.12. If the is-function $\hat{f}(x)$, defined on the set A, is measurable, then the sets

$$A(\hat{f} \geq a), A(\hat{f} = a), A(\hat{f} \leq a), A(\hat{f} < a)$$

Are measurable for all $a \in \mathbb{R}$.

Proof. We will prove that

$$A(\hat{f} \geq a) = \prod_{n=1}^{\infty} A\left(\hat{f} > a - \frac{1}{n}\right). \qquad (5)$$

Really, let $x \in A(\hat{f} \geq a)$ be arbitrarily chosen. Then $x \in A$ and $\hat{f}(x) \geq a$. Hence, for every $n \in \mathbb{N}$ we have $\hat{f}(x) > a - \frac{1}{n}$. Therefore

$$x \in \prod_{n=1}^{\infty} A\left(\hat{f} > a - \frac{1}{n}\right).$$

Because $x \in A(\hat{f} \geq a)$ was arbitrarily chosen and for it we obtain $x \in \prod_{n=1}^{\infty} A\left(\hat{f} > a - \frac{1}{n}\right)$,

We conclude that

$$A(\hat{f} \geq a) \subset \prod_{n=1}^{\infty} A\left(\hat{f} > a - \frac{1}{n}\right). \qquad (6)$$

Let now $x \in \prod_{n=1}^{\infty} A\left(\hat{f} > a - \frac{1}{n}\right)$ be arbitrarily chosen. Then $x \in A\left(\hat{f} > a - \frac{1}{n}\right)$ for every natural number n. From here $x \in A$ and

$$\hat{f}(x) > a - \frac{1}{n}$$

For all natural number n. Consequently

$$\lim_{n\to\infty} \hat{f}(x) \geq \lim_{n\to\infty} \left(a - \frac{1}{n}\right)$$

or

$$\hat{f}(x) \geq a$$

and $x \in A(\hat{f} \geq a)$. Since $x \in \prod_{n=1}^{\infty} A\left(\hat{f} > a - \frac{1}{n}\right)$ was arbitrarily chosen and we get that $x \in A(\hat{f} \geq a)$, we conclude

$$\prod_{n=1}^{\infty} A\left(\hat{f} > a - \frac{1}{n}\right) \subset A(\hat{f} \geq a).$$

From the last relation and from (6) we obtain the relation (5).

Because the intersection of denumerable measurable sets is a measurable set, using the relation (5) and the fact that all sets $A\left(\hat{f} > a - \frac{1}{n}\right)$ are measurable for all natural numbers n, we conclude that the set $A(\hat{f} \geq a)$ is a measurable set.

The set $A(\hat{f} = a)$ is a measurable set because

$$A(\hat{f} = a) = A(\hat{f} \geq a) \setminus A(\hat{f} > a).$$

The set $A(\hat{f} \leq a)$ is measurable set since

$$A(\hat{f} \leq a) = A \setminus A(\hat{f} > a).$$

The set $A(\hat{f} < a)$ is measurable since

$$A(\hat{f} < a) = A \setminus A(\hat{f} \geq a).$$

Remark 2.13. We note that if at least one of the sets

$$A(\hat{f} \geq a), A(\hat{f} = a), A(\hat{f} \leq a), A(\hat{f} < a)$$

Is measurable for all $a \in \mathbb{R}$, then the iso-function $\hat{f}$ is measurable on the set A.

Really, let $A(\hat{f} \geq a)$ is measurable for all $a \in \mathbb{R}$. Then, using the relation

$$A(\hat{f} > a) = \prod_{n=1}^{\infty} A\left(\hat{f} \geq a - \frac{1}{n}\right), \qquad (7)$$

we obtain that the set $A(\hat{f} > a)$ is measurable for all $a \in \mathbb{R}$.

If $A(\hat{f} \leq a))$ is measurable for all $a \in \mathbb{R}$, then using the relation

$$A(\hat{f} > a) = A \setminus A(\hat{f} \leq a),$$

we get that the set $A(\hat{f} > a)$ is measurable for all $a \in \mathbb{R}$.

If $A(\hat{f} < a)$ is measurable for all $a \in \mathbb{R}$, then using the relation

$$A(\hat{f} > a) = A \setminus A(\hat{f} \leq a),$$

We conclude that the set $A(\hat{f} > a)$ is measurable for all $a \in \mathbb{R}$.

Theorem 2.14. If $\hat{f}(x) = c = const$ for all points of a measurable set A, then the is-function $\hat{f}(x)$ is measurable.

Proof. For all $a \in \mathbb{R}$ we have that

$$A(\hat{f} > a) = A \; if \; c > a \; and \; A(\hat{f} > a) = \emptyset \; if \; c \leq a.$$

Since the sets A and $\emptyset$ are measurable sets, then $A(\hat{f} > a)$

is measurable for all $a \in \mathbb{R}$. Therefore the is-function $\hat{f}(x)$ is measurable.

Definition 2.15. An is-function $\hat{f}(x)$ defined on the closed interval [a, b] is said to be a step is-function if there is a finite number of points

$$a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$$

Such that $\hat{f}(x)$ is a constant on (a_i, a_{i+1}) , $i = 0, 1, 2, \cdots, n-1$.

Proposition 2.16. A step is-function is measurable.

Proof. Let $\hat{f}$(x) is a step is-function on the closed interval [a, b]. Let also,

$$a = a_0 < a_1 < a_2 < \cdots < a_{n-1} < a_n = b$$

be such that $\hat{f}(x)$ is a constant on (a_i, a_{i+1}) , $i = 0, 1, 2, \cdots, n-1$. From the previous theorem we have that $\hat{f}(x)$ is measurable on (a_i, a_{i+1}), $i = 0, 1, 2, \cdots, n$. We note that

the sets $\{a_i\}$, $i = 0, 1, 2, \cdots, n-1$, are sets with measure zero. Therefore the is-function

$\hat{f}(x)$ is measurable on $\{a_i\}$, $i = 0, 1, 2, \cdots, n$. From here, using that

$$[a, b] = \bigcup_{i=0}^{n} (a_i, a_{i+1}) \bigcup \bigcup_{i=0}^{n} \{a_i\},$$

We conclude that the is-function $\hat{f}(x)$ is measurable on [a, b].

Theorem 2.17. If the is-function $\hat{f}(x)$, defined on the set A is measurable and $c \in \mathbb{R}, c \neq 0$, then the is-functions

1. $\hat{f}(x) + c$,
2. $c\hat{f}(x)$,
3. $|\hat{f}(x),|$
4. $\hat{f}^2(x)$,
5. $\frac{1}{\hat{f}(x)}$,

are also measurable.

Proof. Let $a \in \mathbb{R}$ be arbitrarily chosen. The assertion follows from the following relations.

1. $A(\hat{f} + c > a) = A(\hat{f} > c - a)$.
2. $A(c\hat{f} > a) = A\left(\hat{f} > \frac{a}{c}\right)$ if c>0, $A(c\hat{f} > a) = A\left(\hat{f} < \frac{a}{c}\right)$ if c<0.
3. $A(|\hat{f}| > a) = A$ if a<0, $A(|\hat{f}| > a) = A(\hat{f} > a) \cup A(\hat{f} < -a)$ if $a \geq 0$.
4. $A(\hat{f}^2 > a) = A$ if a<0, [$A(\hat{f}^2 > a) = A(|\hat{f}| > \sqrt{a})$ if $a \geq 0$.
5. $A\left(\frac{1}{\hat{f}} > a\right) = A(\hat{f} > 0) \cap A\left(\hat{f} < \frac{1}{a}\right)$ if a>0, $A\left(\frac{1}{\hat{f}} >\right.$ *a=A(f>0)∪A(f<0)∩Af<*$1a$ if a<0, *A*1*f>a=A(f>0)* if a=0.

Definition 2.18. An is-function $\hat{f}$, defined on the closed interval [pa, b], is said to be is-step is-function, if there is a finite number of points

$$a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b,$$

such that

$$\hat{f}(x) = \frac{c_i}{\hat{T}(x)}, x \in [a_i, a_{i+1}), c_i = const, i = 0, 1, \cdots, n-1.$$

Theorem 2.19. Let $\hat{T}$(x)>0 for every x ∈ [a, b] and $\hat{T}$(x) is measurable on [a, b]. Let also, $\hat{T}$(x) is an iso-step is-function on [a, b]. Then $\hat{f}$(x) is measurable on [a, b].

Proof. Let

$$a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b,$$

be such that

$$\hat{f}(x) = \frac{c_i}{\hat{T}(x)}, x \in [a_i, a_{i+1}), c_i = const, i = 0, 1, \cdots, n-1.$$

From the last theorem it follows that $\frac{c_i}{\hat{T}(x)}$ is a measurable is-function on $[a_{i,} a_{i+1})$, $i = 0, 1, 2, \cdots, n-1$. Fromn-1 here and from

$$[a, b] = \bigcup_{i=0}^{n-1} [a_i, a_{i+1}) \cup \{b\}.$$

Since {b} is a set with measure zero, we conclude that the is-step is-function $\hat{f}$ is measurable on [a, b].

Definition 2.20. Let M be a subset of the closed interval [a, b]. The function $\varphi_M(x) = 0$ for $x \in [a, b] \setminus M$ and $\varphi_M = 1$ for $x \in M$, is called the characteristic function of the set M.

Theorem 2.21. If the set M is a measurable subset of the closed interval A=[a, b], then the characteristic function $\varphi_M(x)$ is measurable on [a, b].

Proof. The assertion follows from the following relations. $A(\varphi_M > a) = \emptyset$ if $a \geq 1$, $A(\varphi_M > a) = M$ if $1 > a \geq 0, A(\varphi_M > a) = A$ if a<0.

Definition 2.22. Let M be a subset of the set A=[a, b]. The iso- function $\hat{\varphi}_M(x) = 0$ if $x \in A \setminus M$ and $\hat{\varphi}_M = \frac{1}{\hat{T}(x)}$ if $x \in M$, will be called characteristic is-function of the set M.

Theorem 2.23. Let $\hat{T}(x)$ be a measurable function on A=[a, b], M be a measurable subset of A. Then the characteristic is-function $\hat{\varphi}_M(x)$ of the set M is measurable.

Proof. Let $a \in \mathbb{R}$ be arbitrarily chosen. Then

$$A(\hat{\varphi}_M > a) = (A \setminus M)(0 > a) \cup M\left(\frac{1}{\hat{T}(x)} > a\right),$$

From here, using that the sets $(A \setminus M)(0 > a)$ and $M\left(\frac{1}{\hat{T}(x)} > a\right)$ are measurable sets, we conclude that $A(\hat{\varphi}_M > a)$ is a measurable set. Because the constant a was arbitrarily chosen, we have that the characteristic function $\hat{\varphi}_M$ is a measurable is-function.

Theorem 2.24. Let f and $\hat{T}$ are continuous functions on the closed set A. Then the is-function $\hat{f}^\wedge(\hat{x})$ is measurable.

Proof. Let $a \in \mathbb{R}$ be arbitrarily chosen. Since every closed set is a measurable set, we conclude that the set A is a measurable set.

We will prove that the set $A(\hat{f}^{\wedge\wedge} \leq a)$ is a closed set.

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of elements of the set $A(\hat{f}^{\wedge\wedge} \leq a$ such that

$$\lim_{n\to\infty} x_n = x_{0.}$$

Since $A(\hat{f}^{\wedge\wedge} \leq a)$ is a subset of the set A we have that $\{x_n\}_{n=1}^{\infty} \subset A$. Because the set A is a closed set, we obtain that $x_0 \in A$. From the definition of the set $A(\hat{f}^{\wedge\wedge} \leq a)$ we have that

$$\hat{f}^{\wedge}(\hat{x}_n) = \frac{f(x_n)}{\hat{T}(x_n)} \leq a,$$

Hence, when $n \to \infty$, using that f and $\hat{T}$ are continuous functions on the set A, we get

$$\lim_{n\to\infty} \hat{f}^{\wedge}(\hat{x}_n) = \lim_{n\to\infty} \frac{f(x_n)}{\hat{T}(x_n)} = \frac{f(x_0)}{\hat{T}(x_0)} = \hat{f}^{\wedge}(\hat{x}_0) \leq a,$$

i.e., $x_0 \in A(\hat{f}^{\wedge\wedge} \leq a)$. Therefore the set $A(\hat{f}^{\wedge\wedge} \leq a)$ is a closed set. From here, the set $A(\hat{f}^{\wedge\wedge} \leq a)$ is a measurable set. Because the difference of two measurable sets is a measurable set, we have that the set

$$A(\hat{f}^{\wedge\wedge} > a) = A \setminus A(\hat{f}^{\wedge\wedge} \leq a)$$

Is a measurable set.

Since $a \in \mathbb{R}$ was arbitrarily chosen, we obtain that the is-function of the first kind $\hat{f}^{\wedge\wedge}$ is measurable.

Theorem 2.25. Let f and $\hat{T}$ are continuous functions on the closed set A. The the is-functions

$$\hat{f}^{\wedge}(x), \hat{f}(\hat{x}), f^{\wedge}(x), f^{\vee}(x)$$

are measurable on A.

Theorem 2.26. If two measurable is-functions $\hat{f}$ and $\hat{g}$ are defined on the set A, then the set $A(\hat{f} > \hat{g})$ is measurable.

Proof. We enumerate all rational numbers

$$r_1, r_2, r_3, \cdots.$$

We will prove that

$$A(\hat{f} > \hat{g}) = \cup_{k=1}^{\infty}\left(A(\hat{f} > r_k) \cap A(\hat{g} < r_k)\right). \qquad (8)$$

Let

$$x \in A(\hat{f} > \hat{g})$$

Be arbitrarily chosen. Then

$$x \in A, \hat{f}(x) > \hat{g}(x).$$

There exists a rational number r_k such that

$$\hat{f}(x) > r_k > \hat{g}(x).$$

Therefore

$$x \in A \; and \; \hat{f}(x) > r_k; x \in A \; and \; r_k > \hat{g}(x),$$

i.e.,

$$x \in A(\hat{f} > r_k), x \in A(\hat{g} < r_k).$$

Consequently

$$x \in A(\hat{f} > r_k) \cap A(\hat{g} < r_k)$$

And

$$x \in \bigcup_{k=1}^{\infty}\left(A(\hat{f} > r_k) \cap A(\hat{g} < r_k)\right).$$

Because $x \in A(\hat{f} > \hat{g})$ was arbitrarily chosen and for it we get $x \in \cup_{k=1}^{\infty}\left(A(\hat{f} > r_k) \cap A(\hat{g} < r_k)\right)$, we conclude that

$$A(\hat{f} > \hat{g}) \subset \cup_{k=1}^{\infty}\left(A(\hat{f} > r_k) \cap A(\hat{g} < r_k)\right). \qquad (9)$$

Let no

$$x \in \bigcup_{k=1}^{\infty}\left(A(\hat{f} > r_k) \cap A(\hat{g} < r_k)\right)$$

be arbitrarily chosen. Then there exists a natural k so that

$$x \in A(\hat{f} > r_k) \cap A(\hat{g} < r_k).$$

Hence,

$$x \in A(\hat{f} > r_k), x \in A(\hat{g} < r_k).$$

Then

$$x \in A, \hat{f}(x) > r_{k,} r_k < \hat{g}(x)$$

or

$$x \in A, \hat{f}(x) > r_k > \hat{g}(x).$$

Therefore

$$x \in A(\hat{f} > \hat{g}).$$

Because

$$x \in \bigcup_{k=1}^{\infty}\left(A(\hat{f} > r_k) \cap A(\hat{g} < r_k)\right)$$

Was arbitrarily chosen and for it we get that $x \in A(\hat{f} > \hat{g})$, we conclude that

$$\bigcup_{k=1}^{\infty}\left(A(\hat{f} > r_k) \cap A(\hat{g} < r_k)\right) \subset A(\hat{f} > \hat{g}).$$

From the last relation and from the relation (9) we get the relation (8).

Since $\hat{f}$ and $\hat{g}$ are measurable iso-functions on A, we have that the sets

$$A(\hat{f} > r_k), A(\hat{g} < r_k)$$

are measurable sets for every natural k, whereupon the sets

$$A(\hat{f} > r_k) \cap A(\hat{g} < r_k)$$

Are measurable sets for every natural k.

Therefore, using the relation (8), we obtain that the set $A(\hat{f} > \hat{g})$ is a a measurable set.

Theorem 2.27. Let $\hat{f}(x)$ and $\hat{g}(x)$ be finite measurable is-functions on the set A. Then each of the is-functions

1. $\hat{f}(x) - \hat{g}(x)$,
2. $\hat{f}(x) + \hat{g}(x)$,
3. $\hat{f}(x)\hat{g}(x)$,
4. $\frac{\hat{f}(x)}{\hat{g}(x)}$ if $\hat{g}(x) \neq 0$ on A,

Is measurable.

Proof.

1. Let $a \in \mathbb{R}$ be arbitrarily chosen. Since $\hat{g}(x)$ is measurable, then $a + \hat{g}(x)$ is measurable. From here and from the last theorem it follows that the set

$$A(\hat{f}(x) - \hat{g}(x) > a) = A(\hat{f}(x) > a + \hat{g}(x))$$

Is measurable. Because $a \in \mathbb{R}$ was arbitrarily chosen, we conclude that the function $\hat{f}(x) - \hat{g}(x)$ is measurable.

2. Since $\hat{g}$ is a measurable is-function, we have that the function $-\hat{g}$ is a measurable is-function. From here and from 1) we conclude that the is-function

$$\hat{f} + \hat{g} = \hat{f} - (-\hat{g})$$

Is measurable.

3. We note that

$$\hat{f}(x)\hat{g}(x) = \tfrac{1}{2}(\hat{f}(x) + \hat{g}(x))^2 - \tfrac{1}{2}(\hat{f}(x) - \hat{g}(x))^2. \quad (10)$$

Since $\hat{f}(x)$ and $\hat{g}(x)$ are measurable iso-functions, using 1) and 2) we have that

$$\hat{f}(x) + \hat{g}(x) and\ \hat{f}(x) - \hat{g}(x)$$

Are measurable is-functions. Hence the is-functions

$$(\hat{f}(x) + \hat{g}(x))^2, (\hat{f}(x) - \hat{g}(x))^2$$

Are measurable, whereupon

$$\frac{1}{2}(\hat{f}(x) + \hat{g}(x))^2 \ and\ \frac{1}{2}(\hat{f}(x) - \hat{g}(x))^2$$

Are measurable. From here, using 1) and (10), we conclude that $\hat{f}(x)\hat{g}(x)$ is measurable.

4. Since $\hat{g}(x)$ is measurable and $\hat{g}(x) \neq 0$ on A, we have that the is-function $\frac{1}{\hat{g}(x)}$ is measurable. From here and from 3) the is-function

$$\frac{\hat{f}(x)}{\hat{g}(x)} = \hat{f}(x)\frac{1}{\hat{g}(x)}$$

Is measurable.

Theorem 2.28. Let $\{\hat{f}_n(x)\}_{n=1}^{\infty}$ be a sequence of measurable is-functions defined on the set A. If

$$\lim_{n\to\infty} \hat{f}_n(x) = \hat{f}(x) \quad (11)$$

Exists for every $x \in A$, then the is-function $\hat{f}(x)$ is measurable.

Proof. Let $a \in \mathbb{R}$ be arbitrarily chosen. For $n, k, m \in \mathbb{N}$ we define the sets

$$A_{m,k} := A\left(\hat{f}_k > a + \frac{1}{m}\right), B_{m,n} := \prod_{k=n}^{\infty} A_{m,k}.$$

We will prove that

$$A(\hat{f} > a) = \cup_{n,m} B_{m,n}. \quad (12)$$

Let

$$x \in A(\hat{f} > a)$$

Be arbitrarily chosen. Then

$$x \in A\ and\ \hat{f}(x) > a.$$

Hence, there is enough large natural number m_1 such that

$$\hat{f}(x) > a + \frac{1}{m_1}.$$

Using (11), there are enough large natural numbers k and m such that

$$\hat{f}_k(x) > a + \frac{1}{m},$$

i.e., $x \in A_{m,k}$.

From here, it follows that there is enough large n so that $x \in A_{m,k}$ for every $k \geq n$, i.e., $x \in B_{m,n}$ and then $x \in \cup_{m,n} B_{m,n}$.

Since $x \in A(\hat{f} > a)$ was arbitrarily chosen and we get that it is an element of the set $\cup_{m,n} B_{m,n}$, we conclude that

$$A(\hat{f} > a) \subset \cup_{m,n} B_{m,n}. \quad (13)$$

Let now $x \in \cup_{m,n} B_{m,n}$ be arbitrarily chosen.

Then, there are $m_2, n \in \mathbb{N}$ so that

$$x \in B_{m_2,n_1} = \prod_{k=n_1}^{\infty} A_{m_2,k_1}$$

or

$$\hat{f}_{k_1}(x) > a + \tfrac{1}{m_2}\ for\ \forall k \geq n_1.$$

Hence,

$$\lim_{k_1\to\infty} \hat{f}_{k_1}(x) \geq \lim_{k_1\to\infty}\left(a + \frac{1}{m_2}\right)$$

or

$$\hat{f}(x) \geq a + \frac{1}{m_2} > a.$$

Therefore

$$x \in A(\hat{f} > a).$$

Since $x \in \bigcup_{m,n} B_{m,n}$ was arbitrarily chosen and for it we obtain $x \in A(\hat{f} > a)$, we conclude that

$$\bigcup_{m,n} B_{m,n} \subset A(\hat{f} > a).$$

From the last relation and from (13) it follows the relation (12).

Since $\hat{f}_k(x)$ are measurable, we have that the sets $A_{m,k}$ are measurable for every $m, k \in \mathbb{N}$, hence $B_{m,n}$ are measurable for every $m, n \in \mathbb{N}$ and then, using (12), the set $A(\hat{f} > a)$ is measurable. Consequently the is-function $\hat{f}$ is measurable.

Theorem 2.29. be a sequence of measurable is-functions defined on the set A. If

$$\lim_{n\to\infty} \hat{f}_n(x) = \hat{f}(x) \quad (14)$$

Exists for almost everywhere $x \in A$, then the is-function $\hat{f}(x)$ is measurable.

Proof. Let B be the subset of A so that the relation (14) holds for every $x \in B$. From the previous theorem it follows that the is-function $\hat{f}(x)$ is measurable on the set B.

We note that

$$\mu(A \setminus B) = 0.$$

Therefore the is-function $\hat{f}(x)$ is measurable on $A \setminus B$. Hence, the is-function $\hat{f}(x)$ is measurable on A.

Let

$$\hat{T}_n, \hat{T}: A \to (0, \infty), f_n, f: A \to \mathbb{R},$$

$$0 < q_1 \leq \hat{T}_n(x), \hat{T}(x) \leq q_2 for\ x \in A, n \in \mathbb{N}.$$

Then

1. $\hat{f}_n^{\wedge}(\hat{x}) = \frac{f_n(x)}{\hat{T}_n(x)}, \hat{f}^{\wedge}(x) = \frac{f(x)}{\hat{T}(x)},$
2. $\hat{f}_n^{\wedge}(x) = \frac{f_n(x\hat{T}_n(x))}{\hat{T}_n(x)}, \hat{f}^{\wedge}(x) = \frac{f(x\hat{T}(x))}{\hat{T}(x)}$
   If

$$x\hat{T}_n(x), x\hat{T}(x), x \in A,$$

3. $\hat{f}_n(\hat{x}) = \frac{f_n\left(\frac{x}{\hat{T}_n(x)}\right)}{\hat{T}_n(x)}, \hat{f}(\hat{x}) = \frac{f\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)}$
   If

$$\frac{x}{\hat{T}_n(x)}, \frac{x}{\hat{T}(x)}, x \in A,$$

4. $f_n^{\wedge}(x) = f_n(x\hat{T}_n(x)), f^{\wedge}(x) = f(x\hat{T}(x)),$
   If

$$x\hat{T}_n(x), x\hat{T}(x), x \in A,$$

5. $f_n^{\vee}(x) = f_n\left(\frac{x}{\hat{T}_n(x)}\right), f^{\vee}(x) = f\left(\frac{x}{\hat{T}(x)}\right)$
   If

$$\frac{x}{\hat{T}_n(x)}, \frac{x}{\hat{T}(x)}, x \in A.$$

3. The Structure of the Measurable Is-Functions

Theorem 3.1. (is-Lebesgue theorem for is-functions of the first kind) Let there be given a sequence $\{f_n(x)\}_{n=1}^{\infty}$ of measurable functions on a set A, all of which are finite almost everywhere. Let also, $\{\hat{T}_n(x)\}_{n=1}^{\infty}$ be a sequence of measurable functions on the set A,

$$0 < q_1 \leq \hat{T}_n(x) \leq q_2$$

For all natural numbers n and for all $x \in A$, where q_1 and q_2 are positive constants. Suppose that

$$\lim_{n\to\infty} f_n(x) = f(x),$$

$$\lim_{n\to\infty} \hat{T}_n(x) = \hat{T}(x)$$

Almost everywhere on the set A, and $f(x)$ is finite almost everywhere on A,

$$q_1 \leq \hat{T}(x) \leq q_2$$

For all $x \in A$. Then

$$\lim_{n\to\infty} \mu A\left(\left|\hat{f}_n^{\wedge}(\hat{x}) - \hat{f}^{\wedge}(\hat{x})\right| \geq \sigma\right) = 0$$

For all $\sigma \geq 0$.

Proof. We will note that the limit functions f(x) and $\hat{T}(x)$ are measurable and the sets under considerations are measurable.

Let

$$A := A(|f| = \infty),$$

$$B_n := A(|f_n| = \infty),$$

$$C := A(f_n \nrightarrow f),$$

$$D := B \cup \left(\bigcup_{n=1}^{\infty} B_n\right) \cup C.$$

Since

$$\mu B = 0, \mu C = 0, \mu B_n = 0,$$

using the properties of the measurable sets, we have that

$$\mu\mathbb{Q} = 0.$$

Let

$$A_k(\sigma) = A\left(\left|\frac{f_k}{\hat{T}_k} - \frac{f}{\hat{T}}\right| \geq \sigma\right),$$

$$R_n(\sigma) = \bigcup_{k=n}^{\infty} A_k(\sigma),$$

$$M = \prod_{n=1}^{\infty} R_n(\sigma).$$

We have that

$$R_1(\sigma) \supset R_2(\sigma) \supset \cdots.$$

Hence,

$$\lim_{n\to\infty} \mu\, R_n(\sigma) = \mu M.$$

Let us assume that $x_0 \notin \mathbb{Q}$. Then, using the definition of the set $\mathbb{Q}$, we have

$$\lim_{n\to\infty} \frac{f_k(x_0)}{\hat{T}_k(x_0)} = \frac{f(x_{0).}}{\hat{T}(x_0)}$$

Since

$$0 < q_1 \le \hat{T}_n(x), \hat{T}(x) \le q_{2\,,k=1,2,\cdots,n,}$$

we have that

$$\frac{f_1(x_0)}{\hat{T}_1(x_0)}, \frac{f_2(x_0)}{\hat{T}_2(x_0)}, \cdots, \frac{f_k(x_0)}{\hat{T}_k(x_0)}, \cdots$$

and their limit

$$\frac{f(x_{0).}}{\hat{T}(x_0)}$$

are finite.

Therefore there is an enough large natural n such that

$$\left|\frac{f_k(x_0)}{\hat{T}_k(x_0)} - \frac{f(x_{0).}}{\hat{T}(x_0)}\right| < \sigma$$

for every $k \ge n$. Then $x_0 \notin A_k(\sigma), k \ge n,$ where $x_0 \notin R_n(\sigma)$ and from here $x_0 \notin M$.

Consequently $M \subset \mathbb{Q}$.

Because $\mu\mathbb{Q} = 0$, from the last relation, we have that $\mu M = 0$, i.e.,

$$\lim_{n\to\infty} R_n(\sigma) = 0,$$

and since

$$A_n(\sigma) \subset R_n(\sigma),$$

$$\lim_{n\to\infty} R_n(\sigma) = 0$$

or

$$\lim_{n\to\infty} \mu A\left(\left|\hat{f}_n^{\wedge}(\hat{x}) - \hat{f}^{\wedge}(\hat{x})\right| \ge \sigma\right) = 0.$$

As in above one can prove the following results for the other kinds of is-functions.

Theorem 3.2. (is-Lebesgue theorem for is-functions of the second kind) Let there be given a sequence $\{f_n(x)\}_{n=1}^{\infty}$ of measurable functions on a set A, all of which are finite almost everywhere. Let also, $\{\hat{T}_n(x)\}_{n=1}^{\infty}$ be a sequence of measurable functions on the set A,

$$0 < q_1 \le \hat{T}_n(x) \le q_2$$

For all natural numbers n and for all $x \in A$, where q_1 and q_2 are positive constants. Suppose that

$$\lim_{n\to\infty} f_n(x) = f(x),$$

$$\lim_{n\to\infty} \hat{T}_n(x) = \hat{T}(x)$$

Almost everywhere on the set A, and $f(x)$ is finite almost everywhere on A,

$$q_1 \le \hat{T}(x) \le q_2$$

For all $x \in A$. Then

$$\lim_{n\to\infty} \mu A\left(\left|\hat{f}_n^{\wedge}(x) - \hat{f}^{\wedge}(x)\right| \ge \sigma\right) = 0$$

for all $\sigma \ge 0$.

Theorem 3.3. (is-Lebesgue theorem for is-functions of the third kind) Let there be given a sequence $\{f_n(x)\}_{n=1}^{\infty}$ of measurable functions on a set A, all of which are finite almost everywhere. Let also, $\{\hat{T}_n(x)\}_{n=1}^{\infty}$ be a sequence of measurable functions on the set A,

$$0 < q_1 \le \hat{T}_n(x) \le q_2$$

For all natural numbers n and for all $x \in A$, where q_1 and q_2 are positive constants. Suppose that

$$\lim_{n\to\infty} f_n(x) = f(x),$$

$$\lim_{n\to\infty} \hat{T}_n(x) = \hat{T}(x)$$

Almost everywhere on the set A, and $f(x)$ is finite almost everywhere on A,

$$q_1 \le \hat{T}(x) \le q_2$$

For all $x \in A$. Then

$$\lim_{n\to\infty} \mu A\left(\left|\hat{f}_n\ (\hat{x}) - \hat{f}\ \ (\hat{x})\right| \ge \sigma\right) = 0$$

for all $\sigma \ge 0$.

Theorem 3.4. (is-Lebesgue theorem for is-functions of the fourth kind) Let there be given a sequence $\{f_n(x)\}_{n=1}^{\infty}$ of measurable functions on a set A, all of which are finite almost everywhere. Let also, $\{\hat{T}_n(x)\}_{n=1}^{\infty}$ be a sequence of measurable functions on the set A,

$$0 < q_1 \le \hat{T}_n(x) \le q_2$$

For all natural numbers n and for all $x \in A$, where q_1 and q_2 are positive constants. Suppose that

$$\lim_{n\to\infty} f_n(x) = f(x),$$

$$\lim_{n\to\infty}\hat{T}_n(x)=\hat{T}(x)$$

Almost everywhere on the set A, and $f(x)$ is finite almost everywhere on A,

$$q_1 \le \hat{T}(x) \le q_2$$

For all $x \in A$. Then

$$\lim_{n\to\infty}\mu A(|f_n^{\wedge}(x)-f^{\wedge}(x)|\ge\sigma)=0$$

For all $\sigma \ge 0$.

Theorem 3.5. (is-Lebesgue theorem for is-functions of the fifth kind) Let there be given a sequence $\{f_n(x)\}_{n=1}^{\infty}$ of measurable functions on a set A, all of which are finite almost everywhere. Let also, $\{\hat{T}_n(x)\}_{n=1}^{\infty}$ be a sequence of measurable functions on the set A,

$$0 < q_1 \le \hat{T}_n(x) \le q_2$$

For all natural numbers n and for all $x \in A$, where q_1 and q_2 are positive constants. Suppose that

$$\lim_{n\to\infty} f_n(x)=f(x),$$

$$\lim_{n\to\infty}\hat{T}_n(x)=\hat{T}(x)$$

Almost everywhere on the set A, and $f(x)$ is finite almost everywhere on A,

$$q_1 \le \hat{T}(x) \le q_2$$

For all $x \in A$. Then

$$\lim_{n\to\infty}\mu A(|f_n^{\vee}(x)-f^{\vee}(x)|\ge\sigma)=0$$

for all $\sigma \ge 0$.

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