# LIE-ADMISSIBLE HYPERALGEBRAS

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**Abstract.** We review Albert's axioms of Lie-admissible algebras, Santilli's geno- and hyper-mathematics with a Lie-admissible structure, and Vougiouklis $H_v$-hyperstructures; we then introduce Lie-admissible hyperstructures; and point out their expected relevance for biological structures and other fields.

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## 1. Introduction

In 1948, A.A. Albert [1] defined a generally non associative algebra $U$ with elements $a, b, c, ...$, and product $ab$ over a field of characteristic zero as being *Lie-admissible* (*Jordan-admissible*) when the attached antisymmetric algebra $U^-$ (attached symmetric algebra $U^+$) which is the same vector space as $U$ equipped with the product $[a, b] = ab - ba$ (the product $\{a, b\} = ab + ba$) verifies all axioms of a Lie algebras (Jordan algebra). In the same paper [1], Albert identified the *flexible algebras* with product

$$(A, B) = \lambda AB - (1 - \lambda)BA, \tag{1.1}$$

(where $A, B, ...$ are Hermitean $n$-dimensional matrices, $\lambda$ is a scalar and $AB$ is the conventional associative product) as a realization of Lie-admissible and Jordan-admissible algebras.

In 1967, R.M. Santilli [2] noted that the algebras with product (1.1) do not admit Lie algebras in their classification and, consequently, cannot be used for the construction of covering theories. Consequently, Santilli introduced the product

$$(A, B) = \lambda AB - \mu BA = \alpha(AB - BA) + \beta(AB + BA), \quad \lambda = \alpha + \beta, \mu = \beta - \alpha, \tag{1.2}$$

(where $\lambda, \mu, \lambda \pm \mu$ are non-null scalars) which is jointly Lie-admissible and Jordan-admissible while admitting Lie algebras in their classification.

By recalling that the theories based on Lie algebras are reversible over time because of the anti-Hermiticity of the Lie product $[A, B] = -[A, B]^\dagger$, Santilli introduced product Lie-admissible product (1.2) to break the anti-Hermiticity of the Lie product for the intent of initiating quantitative studies on processes that are irreversible over time via a covering Lie-admissible theory admitting reversible Lie processes as particular cases. For that scope, Santilli [3], [4] introduced in 1968 the following infinitesimal and finite generalizations of Heisenberg equations

$$i\frac{dA}{dt} = (A, H) = \lambda AH - \mu HA, \tag{1.3$a$}$$

$$A(t) = U(t)A(0)V(t)^\dagger = e^{Ht\mu i}A(0)e^{-i\lambda tH}, \quad U = e^{Ht\mu i}, \ V = e^{i\lambda tH}UV^\dagger \neq I, \tag{1.3$b$}$$

where $H$ is the Hamiltonian.

By noting that equation (1.3b) is non-unitary, thus not invariant under the time evolution, Santilli introduced in 1978 [5], [6] the following most general known realization of products that are jointly lie-admissible and Jordan-admissible

$$\begin{aligned}(A, B) &= ARB - BSA = (ATB - BTA) + \{AWB + BWA\} = \\ &= [A, B]^* + \{A, B\}^* = (ATH - HTA) + \{AWH + HWA\}, \\ R &= T + W, S = W - T,\end{aligned} \tag{1.4}$$

where $R, S, R \pm S$ are now non singular operators. In this broader realization, the attaches product $[A, B]* = ATB - BTA$ and $\{A, B\}* = AWB + BWA$ still verify the Lie and Jordan axioms, respectively, but characterize broader algebras called *Lie-Santilli and Jordan-Santilli isoalgebras,* where the prefix "iso" is interpreted in the Greek sense of preserving Lie axioms.

In the same papers [5], [6], Santilli introduced the most general possible broadening of Heisenberg's equations in their infinitesimal and finite form with a Lie-isotopic and Jordan-isotopic structure defined over a field of characteristic zero

$$i\frac{dA}{dt} = (A, H) = ARH - BSA, \tag{1.5$a$}$$

$$(1.5b)\qquad \begin{aligned} &A(t) = U(t)A(0)V(t)^\dagger = e^{HSti}A(0)e^{-itRH}, \\ &U = e^{HSti},\ V = e^{HRti},\ UV^\dagger \neq I, \end{aligned}$$

and assumed them as the foundation of a Lie-admissible covering of quantum mechanics for irreversible processes proposed under the name of *hadronic mechanics* [6].

In the 1990s, Santilli noted that equation (1.5) are indeed the most general possible equations with a Lie-isotopic-admissible and Jordan-isotopic-admissible structure that persist under arbitrary (non-singular) non-unitary transforms. However, the operators $R$ and $S$ are not invariant under time evolution (1.5b). But said operators represent the irreversible component of the systems considered. Therefore, as proposed in 1978, generalized dynamical equations (1.5) do not admit consistent physical applications.

In order to achieve the invariance over time of the $R$ and $S$ operators, Santilli had to build a new mathematics, today known as *Santilli genomathematics* (where the prefix "geno" is now intended in the Greek sense of inducing new axioms), which can be defined as a mathematics admitting the generalized units $1^> = 1/S$ ($^<1 = 1/R$) and corresponding ordered multiplications $A > B = ASB$ ($A < B = ARB$) at all levels, including numeric field, functional analysis, differential calculus, etc, for the representation of motion forward (backward) in time and basic rules

$$1^> = 1/S, A > B = ASB, \quad 1^> > A = A > 1^> = A, \tag{1.6a}$$

$$^<1 = 1/R, A < B = ARB, \quad ^<1 < A = A <^< 1 = A, \tag{1.6b}$$

Real, complex and quaternionic numbers, when equipped with the above forward and, separately, backward generalized units and products, were shown in Ref. [7] of 1993 to verify the abstract axioms of a numeric field and are now called *Santilli forward and backward genoreal, genocomplex and genoquaternionic numbers.* Corresponding forward and backward generalization of functional analysis, differential calculus, and other aspects of applied mathematics were introduced by Santilli in memoir [8] of 1996, The invariance of the $R$ and $S$ operator under a Lie-admissible and Jordan-admissible time evolution (1.5) was first proved in Ref. [9] of 1997.

Santilli additionally noted that genomathematics is indeed effective for the initiation of quantitative studies on irreversible processes while admitting, for the first time, a direct connection to thermodynamics [10-13], but said forward and backward genomathematics are insufficient for quantitative studies of biological structure because they are *single-valued* (in the sense that each ordered products $A > B$ and $A < B$ yield one single result). To initiate quantitative studies on complex biological structures such as the DNA, Santilli introduced in 1996 [8] the following ordered *multi-valued* realization of genomathematics today known as *Santilli multi-valued, forward and backward hypermathematics*

$$\hat{1}^> = \{1^>_1, 1^>_2, 1^>_3, ..., 1^>_n\} = \{1/S_1, 1/S_2, ..., 1/S_n\}$$

(1.7a) $1^{>}_{k} = 1/S_k < 0, k = 1, 2, ..., n$

(1.7b) $A\hat{>}B = AS_1B + AS_2B + AS_3B + ... + AS_nB,$

(1.7c) $^{<}\hat{1} = \{^{<}\hat{1}_1, ^{<}\hat{1}_2, ^{<}\hat{1}_3, ..., ^{<}\hat{1}_n\} = \{R_1, R_2, R_3, ..., R_n\},$

(1.7d) $A\hat{<}B = AR_1B + AR_2B + AR_3B + .... + AR_n,$

whose multi-valued character is evident.

It should be indicate that Santilli forward and backward hypermathematics *are different* than hyperstructures (see, e.g., Refs. [14]–[19]) for numerous reasons, such as: the former are based on classical operations while the latter are characterized by hyper operations; the former are formulated over rings verifying the axioms of a numeric field, while the latter are not; etc.

Jointly, T. Vougiouklis [20]–[30] introduced the most general known formulation of hyperstructures, today known as *Vougiouklis $H_v$ hyperstructures,* which are relevant for the study of irreversible processes, such as biological structure, since Santilli forward and backward genomathematics are expected to show limitations due to their formulations via classical operations. Therefore, in this paper, we shall reformulated Santilli hypermathematics in terms of Vougiouklis.

## 2. The hyperstructures

In this section we present an introduction on the theory of hyperstructures.

The largest class of hyperstructures were introduced in 1990 [24] and are called $H_v$-structures. They satisfy the weak axioms where the non-empty intersection replaces the equality. Some basic definitions are the following:

In a set H equipped with a hyperoperation (abbreviation as *hope*)

$$\cdot : H \times H \to P(H) - \{\varnothing\},$$

we have the following properties in abbrevated notation

*WASS,* the *weak associativity*: $(xy)z \cap x(yz) \neq \varnothing, \forall x, y, z \in H$ and

*COW,* the *weak commutativity*: $xy \cap yx \neq \varnothing, \forall x, y \in H$.

The hyperstructure $(H, \cdot)$ is called $H_v$*-semigroup* if it is *WASS*, and it is called $H_v$*-group* if it is a reproductive $H_v$-semigroup, i.e.,

$$xH = Hx = H, \forall x \in H.$$

As it is well known, in the classical theory, F. Marty stated in 1934 that the quotient of a group with respect to any subgroup is a hypergroup. Also, the quotient of a group with respect to any partition (or any equivalence relation) is an $H_v$-group illustrating the motivation to introduce the $H_v$-structures [24], [26].

In an $H_v$-semigroup the powers of an element $h \in H$ are defined as follows: $h^1 = \{h\}, h^2 = h \cdot h, ..., h^n = h \circ h \circ ... \circ h$, where $(\circ)$ denotes the *n-ary circle hope*, i.e., take the union of hyperproducts, n times, with all possible patterns of parentheses put on them. An $H_v$-semigroup $(H, \cdot)$ is called *cyclic of period s*, if there exists an element $h$, called *generator*, and a natural number $s$, the minimum one, such that $H = h^1 \cup h^2 ... \cup h^s$. We define analogously the cyclicity for the infinite period [13]. If there is an element $h$ and a natural number $s$, the minimum one, such that $H = h^s$, then $(H, \cdot)$ is called *single-power cyclic of period s.*

More complicated hyperstructures can be defined: in a similar way, such as $(R, +, \cdot)$ is called $H_v$*-ring* if $(+)$ and $(\cdot)$ are WASS, the reproduction axiom is valid for $(+)$, and $(\cdot)$ is *weak distributive* with respect to $(+)$, i.e.,

$$x(y+z) \cap (xy+xz) \neq \varnothing,\ (x+y)z \cap (xz+yz) \neq \varnothing,\ \forall x, y, z \in R.$$

Let $(R, +, \cdot)$ be an $H_v$-ring, $(M, +)$ be a COW $H_v$-group and there exists an external hope

$$\cdot : R \times M \to P(M) : (a, x) \to ax$$

such that $\forall a, b \in R$ and $\forall x, y \in M$ we have

$$a(x+y) \cap (ax+ay) \neq \varnothing,\ (a+b)x \cap (ax+bx) \neq \varnothing,\ (ab)x \cap a(bx) \neq \varnothing,$$

then $M$ is called an $H_v$*-module* over $F$. In the case of an $H_v$-field $F$, which is defined later, instead of an $H_v$-ring $R$, then the $H_v$*-vector space* is defined.

The main tool to study hyperstructures is the fundamental relation. In 1970 M. Koscas defined in hypergroups the relation $\beta$ and its transitive closure $\beta^*$. This relation connects the hyperstructures with the corresponding classical structures and is defined in $H_v$-groups as well. T. Vougiouklis [26, 28] introduced the $\gamma^*$ and $\epsilon^*$ relations, which are defined in $H_v$-rings and $H_v$-vector spaces, respectively. Vougiouklis also named all these relations $\beta^*$, $\gamma^*$ and $\epsilon^*$ *Fundamental Relations* because they play a very important role to the study of hyperstructures especially in their representation theory.

**Definition 2.1.** The fundamental relations $\beta^*$, $\gamma^*$ and $\epsilon^*$, are defined in $H_v$-groups, $H_v$-rings and $H_v$-vector space, respectively, as the smallest equivalences so that the quotient would be group, ring and vector space, respectively. ([23], [24], [26], [28])

To specifying the above motivation, we note the following: Let $(G, \cdot)$ be a group and $R$ be an equivalence relation (or a partition) in $G$, then $(G/R, \cdot)$ is an $H_v$-group, therefore we have the quotient $(G/R, \cdot)/\beta^*$ which is a group, the fundamental one. Remark that the classes of the fundamental group $(G/R, \cdot)/\beta^*$ are a union of some of the $R$-classes. Otherwise, the $(G/R, \cdot)/\beta^*$ has elements classes of $G$ where they form a partition which classes are larger than the classes of the original partition $R$.

The way to find the fundamental classes is given by the following:

**Theorem 2.2.** *Let $(H, \cdot)$ be an $H_v$-group and denote by $U$ the set of all finite products of elements of $H$. We define the relation $\beta$ in $H$ by setting $x\beta y$ iff $\{x, y\} \subset u$ where $u \in U$. Then $\beta^*$ is the transitive closure of $\beta$.*

Analogous to the above theorem, in the case of an $H_v$-ring [26], is the following:

**Theorem 2.3.** *Let $(R,+,\cdot)$ be an $H_v$-ring. Denote by $U$ the set of all finite polynomials of elements of* $\boldsymbol{R}$*. We define the relation $\gamma$ in* $\boldsymbol{R}$ *as follows:*

$$x\gamma y \textit{ iff } \{x,y\} \subset u \textit{ where } u \in U.$$

*Then the relation $\gamma^*$ is the transitive closure of the relation $\gamma$.*

An element is called *single* if its fundamental class is singleton [26].

Fundamental relations are used for general definitions. Thus, an $H_v$-ring $(R,+,\cdot)$ is called $\mathbf{H_v}$***-field*** if $R/\gamma^*$ is a field. The elements of an $H_v$-field are called $\mathbf{H_v}$***-numbers*** or hyper-numbers

Let $(H,\cdot),(H,*)$ be $H_v$-semigroups defined on the same set $H$. The hope $(\cdot)$ is called *smaller* than the hope $(*)$, and $(*)$ *greater* than $(\cdot)$, iff there exists an

$$f \in Aut(H,*) \text{ such that } xy \subset f(x*y),\ \forall x,y \in H.$$

Then we write $\cdot \leq *$ and we say that $(H,*)$ contains $(H,\cdot)$. If $(H,\cdot)$ is a structure then it is called *basic structure* and $(H,*)$ is called $H_b - structure$.

**Theorem 2.4.** (The Little Theorem). *Greater hopes than the ones which are WASS or COW, are also WASS or COW, respectively.*

This Theorem leads to a partial order on $H_v$-structures and mainly to a correspondence between hyperstructures and posets.Therefore we can obtain an extreme large number of $H_v$-structures just putting more elements on any result. Using the partial ordering with the fundamental relations, one can give several definitions to obtain constructions used in several applications [28]:

Let $(H,\cdot)$ be hypergroupoid. We *remove* $h \in H$, if we consider the restriction of $(\cdot)$ in the set $H-\{h\}$. $\underline{h} \in H$ *absorbs* $h \in H$ if we replace $h$ by $\underline{h}$ and h does not appear in the structure. $\underline{h} \in H$ merges with $h \in H$, if we take as product of any $x \in H$ by $\underline{h}$, the union of the results of $x$ with both $h$, $\underline{h}$, and consider h and $\underline{h}$ as one class with representative $\underline{h}$.

A class of $H_v$-structures is the following:

An $H_v$-structure is called *very thin* iff all hopes are operations except one, which has all hyperproducts singletons except only one, which is a subset of cardinality more than one. Therefore, in a very thin $H_v$-structure in a set H there exists a hope $(\cdot)$ and a pair $(a,b) \in H^2$ for which $ab = A$, with $cardA > 1$, and all the other products, with respect to any other hopes (so they are operations), are singletons.

A large class of $H_v$-structures is the following [29]:

Let $(G,\cdot)$ be groupoid (resp., hypergroupoid) and $f : G \to G$ be a map. We define a hope $(\partial)$, called *theta-hope*, we write *$\partial$-hope*, on G as follows

$$x\partial y = \{f(x)\cdot y, x\cdot f(y)\},\ \ \forall x,y \in G. \text{ (resp. } x\partial y = (f(x)\cdot y)\cup(x\cdot f(y)),\ \ \forall x,y \in G)$$

If $(\cdot)$ is commutative then $\partial$ is commutative. If $(\cdot)$ is COW, then $\partial$ is COW.

Let $(G, \cdot)$ be a groupoid (or hypergroupoid) and $f : G \to P(G) - \{\varnothing\}$ be any multivalued map. We define the $(\partial)$, on G as follows

$$x\partial y = (f(x) \cdot y) \cup (x \cdot f(y)), \quad \forall x, y \in G$$

Let $(G, \cdot)$ be a groupoid, $f_i : G \to G, i \in I$, be a set of maps on G. The

$$f_\cup : G \to \mathbf{P}(G) : f_\cup(x) = \{f_i(x) | i \in I, \}$$

is *the union* of $f_i(x)$. We have the *union $\partial$-hope* $(\partial)$, on G if we take $f_\cup(x)$. If $\underline{f} \equiv f \cup (id)$, then we have the $b - \partial - hope$.

Motivation for the definition of the theta-hope is the map *derivative* where only the multiplication of functions can be used. The basic property is that if $(G, \cdot)$ is a semigroup then for every f, the $(\partial)$ is *WASS*.

Another well known and large class of hopes is given as follows [22], [26]:

Let $(G, \cdot)$ be a groupoid then for every $P \subset G$, $P \neq \varnothing$, we define the following hopes called P-*hopes*: for all $x, y \in G$

$$\underline{P} : x\underline{P}y = (xP)y \cup x(Py), \ \ \underline{P}_r : x\underline{P}_r y = (xy)P \cup x(yP), \ \ \underline{P}_l : x\underline{P}_l y = (Px)y \cup P(xy).$$

The $(G, \underline{P})$,$(G, \underline{P}_r)$ and $(G, \underline{P}_l)$ are called P-*hyperstructures*. The most usual case is if $(G, \cdot)$ is semigroup, then $x\underline{P}y = (xP)y \cup x(Py) = xPy$ and $(G, \underline{P})$ is a semihypergroup but we do not know about $(G, \underline{P}_r)$ and $(G, \underline{P}_l)$. In some cases, depending on the choice of P, the $(G, \underline{P}_r)$ and $(G, \underline{P}_l)$ can be associative or WASS.

A generalization of $P$-hopes, introduced by Davvaz, Santilli, Vougiouklis in [17], [18] is the following:

**Construction 2.5.** *Let $(G, \cdot)$ be an abelian group and $P$ any subset of $G$ with more than one elements. We define the hope $\times P$ as follows:*

$$x \times_p y = \begin{cases} x \times_P y = x \cdot P \cdot y = \{x \cdot h \cdot y | h \in P\} & \textit{if } x \neq e \textit{ and } c \neq e \\ x \cdot y & \textit{if } x = e \textit{ and } y = e \end{cases}$$

*we call this hope $P_e$-hope. The hyperstructure $(G, \times_p)$ is an abelian $H_v$-group.*

$H_v$-structures are used in Representation Theory of $H_v$-groups which can be achieved either by generalized permutations or by $H_v$-matrices [25,] [26]. Representations by generalized permutations can be faced by translations. $H_v$-matrix is called a matrix if has entries from an $H_v$-ring. The hyperproduct of $H_v$-matrices is defined in a usual manner. The problem of the $H_v$-matrix representations is the following:

**Definition 2.6.** Let $(H, \cdot)$ be $H_v$-group, find an $H_v$-ring $R$, a set

$$M_R = \{(a_{ij}) \mid a_{ij} \in R\}$$

and a map

$$T : H \to M_R : h \mapsto T(h) \text{ such that } T(h_1h_2) \cap T(h_1)T(h_2) \neq \varnothing, \forall h_1, h_2 \in H.$$

Then, the map $T$ is called *$H_v$-matrix representation.*

If the $T(h_1h_2) \subset T(h_1)(h_2), \forall h_1, h_2 \in H$ is valid, then $T$ is called *inclusion representation.*

If $T(h_1h_2) = T(h_1)(h_2) = \{T(h)|h \in h_1h_2\}$, $\forall h_1, h_2 \in H$, then $T$ is called *good representation* and then an induced representation $T^*$ for the hypergroup algebra is obtained.

If $T$ is one to one and good then it is a *faithful representation.*

In the representations of $H_v$-groups there are two difficulties: To find an $H_v$-ring or an $H_v$-field and an appropriate set of $H_v$-matrices. Hopes on any type of ordinary matrices can be defined [17], they are called *helix hopes.*

Using several classes of $H_v$-structures one can face several representations. Some of those classes are as follows:

**Definition 2.7.** Let $\mathbf{M} = \mathbf{M}_{m\times n}$ be a module of $m \times n$ matrices over a ring $\mathbf{R}$ and $\mathbf{P} = \{P_i : i \in I\} \subseteq \mathbf{M}$. We define, a kind of, a P-hope $\underline{P}$ on $\mathbf{M}$ as follows

$$\underline{P} : \mathbf{M} \times \mathbf{M} \to \mathbf{P}(\mathbf{M}) : (A, B) \to A\underline{P}B = \{AP_i^t B : i \in I\} \subseteq \mathbf{M}$$

where $P^t$ denotes the transpose of the matrix P.

The hope $\underline{P}$, which is a bilinear map, is a generalization of Rees' operation where, instead of one sandwich matrix, a set of sandwich matrices is used. The hope $\underline{P}$ is strong associative and the inclusion distributivity with respect to addition of matrices

$$A\underline{P}(B + C) \subseteq A\underline{P}B + A\underline{P}C \text{ for all } A, B, C \text{ in } \mathbf{M}$$

is valid. Therefore, $(\mathbf{M}, +, \underline{P})$ defines a multiplicative hyperring on non-square matrices. Multiplicative hyperring means that only the multiplication is a hope.

**Definition 2.8.** Let $\mathbf{M} = \mathbf{M}_{m\times n}$ be a module of $m\times n$ matrices over R and let us take sets $\mathbf{S} = \{s_k : k \in K\} \subseteq R, \mathbf{Q} = \{Q_j : j \in J\} \subseteq \mathbf{M}, \mathbf{P} = \{P_i : i \in I\} \subseteq \mathbf{M}$. Define three hopes as follows

$$\begin{aligned}
\underline{S} : R \times \mathbf{M} \to \mathbf{P}(\mathbf{M}) : (r, A) \to r\underline{S}A &= \{(rs_k)A : k \in K\} \subseteq \mathbf{M} \\
\underline{Q}_+ : \mathbf{M} \times \mathbf{M} \to \mathbf{P}(\mathbf{M}) : (A, B) \to A\underline{Q}_+B &= \{A + Q_j + B : j \in J\} \subseteq \mathbf{M} \\
\underline{P} : \mathbf{M} \times \mathbf{M} \to \mathbf{P}(\mathbf{M}) : (A, B) \to A\underline{P}B &= \{AP_i^t B : i \in I\} \subseteq \mathbf{M}
\end{aligned}$$

Then $(\mathbf{M}, \underline{S}, \underline{Q}_+, \underline{P})$ is a hyperalgebra over $\mathbf{R}$ called *general matrix P-hyperalgebra.*

In a similar way, a generalization of this hyperalgebra can be defined if one considers an $H_v$-ring or an $H_v$-field instead of a ring and using $H_v$-matrices.

## 3. Lie-hyperalgebras

Since the algebras are defined on vector spaces, we now present the analogous to the above Theorem 2.3, on $H_v$-vector spaces. The proof is similar.

**Theorem 3.1.** *Let $(V, +)$ be an $H_v$-vector space over the $H_v$-field $F$. Denote by $U$ the set of all expressions consisting of finite hopes either on $F$ and $V$ or the external hope applied on finite sets of elements of $F$ and $V$. We define the relation $\epsilon$ in $V$ as follows:*

$$x\epsilon y \text{ iff } \{x, y\} \subset u \text{ where } u \in U.$$

*Then the relation $\epsilon^*$ is the transitive closure of the relation $\epsilon$.*

**Proof.** Let $\underline{\epsilon}$ be the transitive closure of $\epsilon$, and denote by $\underline{\epsilon}(x)$ the class of the element x. First we prove that the quotient set $\mathbf{M}/\underline{\epsilon}$ is a module over $\mathbf{R}/\gamma^*$.

In $M/\underline{\epsilon}$ the sum $(\oplus)$ and the product $(\otimes)$ using the $\gamma^*$ classes in $\mathbf{R}$, are defined in the usual manner:

$$\begin{aligned} \underline{\epsilon}(x) \oplus \underline{\epsilon}(y) &= \{\underline{\epsilon}(z) : z \in \underline{\epsilon}(x) + \underline{\epsilon}(y)\}, \\ \gamma^*(a) \otimes \underline{\epsilon}(x) &= \{\underline{\epsilon}(z) : z \in \gamma^*(a) \cdot \underline{\epsilon}(x)\} \forall a \in R,\ x, y \in \mathbf{M}. \end{aligned}$$

Take $x' \in \underline{\epsilon}(x)$, $y' \in \underline{\epsilon}(y)$. Then we have $x'\underline{\epsilon}x$ iff $\exists\, x_1, ..., x_{m+1}$ with $x_1 = x'$, $x_{m+1} = x$ and $u_1, ..., u_m \in U$ such that $\{x_i, x_{i+1}\} \subset u_i$, $i = 1, ...m$, and $y'\underline{\epsilon}y$ iff $\exists y_1, ..., y_{m+1}$ with $y_1 = y'$, $y_{n+1} = y$ and $v_1, ..., v_n \in U$ such that $\{y_j, y_{j+1}\} \subset v_j$, $i = 1, ...n$. From the above, we obtain

$$\{x_i, x_{i+1}\} + y_1 \subset u_i + v_1,\ i = 1, ...m - 1,$$

$$x_{m+1} + \{y_j, y_{j+1}\} \subset u_m + v_j,\ j = 1, ...n.$$

The sums

$$u_i + v_1 = t_i,\ i = 1, ...m - 1 \text{ and } u_m + v_j = t_{im+j-1},\ j = 1, ...n$$

are also elements of $\mathbf{U}$, therefore $t_k \in U$ for all $k \in \{1, ..., m + n - 1\}$.

Now, pick up elements $z_1, ..., z_{m+n}$ such that

$$z_i \in x_i + y_1,\ i = 1, ..., n \text{ and } z_{m+j} \in x_{m+1} + y_{j+1},\ j = 1, ..., n.$$

Therefore, using the above relations, we obtain $\{z_k, z_{k+1}\} \subset t_k,\ k = 1, ..., m+n-1$. Thus, every element $z_1 \in x_1 + y_1 = x' + y'$ is $\underline{\epsilon}$ equivalent to every element $z_{m+n} \in x_{m+1} + y_{n+1} = a + b$. Thus, $\underline{\epsilon}(x) \oplus \underline{\epsilon}(y)$ is a singleton, so we can write

$$\underline{\epsilon}(x) \oplus \underline{\epsilon}(y) = \underline{\epsilon}(z) \text{ for all } z \in \underline{\epsilon}(x) + \underline{\epsilon}(y)$$

In a similar way, using the properties of $\gamma*$ in $\mathbf{R}$, one can prove that

$$\gamma^*(a) \otimes \underline{\epsilon}(x) = \underline{\epsilon}(z) \text{ for all } z \in \gamma * (a) \cdot \underline{\epsilon}(x).$$

The WASS and the weak distributivity on $\mathbf{R}$ and $\mathbf{M}$ guarantee that the associativity and the distributivity are valid for the quotient $\mathbf{M}/\underline{\epsilon}$ over $\mathbf{R}/\gamma^*$. Therefore $\mathbf{M}/\underline{\epsilon}$ is a module over $\mathbf{R}/\gamma^*$.

Now, let $\sigma$ be an equivalence relation in $\mathbf{M}$ such that $\mathbf{M}/\sigma$ is a module over $\mathbf{R}/\gamma^*$. Denote $\sigma(x)$ the class of x. Then $\sigma(x) \oplus \sigma(y)$ and $\gamma^*(a) \otimes \sigma(x)$ are singletons for all $a \in \mathbf{R}$ and $x, y \in \mathbf{M}$, i.e.,

$$\begin{aligned} \sigma(x) \oplus \sigma(y) &= \sigma(z) \quad \text{for all } z \in \sigma(x) + \sigma(y), \\ \gamma * (a) \otimes \sigma(x) &= \sigma(z) \quad \text{for all } z \in \gamma * (a) \cdot \sigma(x). \end{aligned}$$

Thus, we can write, for every $a \in R, x, y \in M$ and $\mathbf{A} \subset \gamma^*(a)$, $\mathbf{X} \subset \sigma(x)$, $\mathbf{Y} \subset \sigma(x)$,

$$\sigma(x) \oplus \sigma(y) = \sigma(x+y) = \sigma(\mathbf{X} + \mathbf{Y}),$$

$$\gamma * (a) \otimes \sigma(x) = \sigma(ax) = \sigma(\mathbf{A} \cdot \mathbf{B})$$

By induction, we extend these relations on finite sums and products. Thus, for every $u \in U$, we have the relation $\sigma(x) = \sigma(u)$ for all $x \in u$. Consequently,

$$x \in \epsilon(x) \text{ implies } x \in \sigma(x) \text{ for every } x \in \mathbf{M}.$$

But $\sigma$ is transitively closed, so we obtain:

$$x' \in \underline{\epsilon}(x) \text{ implies } x' \in \sigma(x).$$

That means that $\underline{\epsilon}$ is the smallest equivalence relation on $\mathbf{M}$ such that $\mathbf{M}/\underline{\epsilon}$ is a module over $\mathbf{R}/\gamma^*$, i.e., $\underline{\epsilon} = \epsilon^*$. ■

The general definition of an $H_v$-Lie algebra was given in [30] as follows:

**Definition 3.2.** Let $(L, +)$ be an $H_v$-vector space over the $H_v$-field $(F, +, \cdot)$, $\phi : F \to F/\gamma^*$ the canonical map and $\omega_F = \{x \in F : \phi(x) = 0\}$, where 0 is the zero of the fundamental field $F/\gamma$. Similarly, let $\omega_L$ be the core of the canonical map $\phi' : L \to L/\epsilon^*$ and denote by the same symbol 0 the zero of $L/\epsilon^*$. Consider the *bracket (commutator) hope*:

$$[,] : L \times L \to P(L) : (x, y) \to [x, y].$$

Then $\mathbf{L}$ is an *$H_v$-Lie algebra* over F if the following axioms are satisfied:

(L1) The bracket hope is bilinear, i.e.,

$$[\lambda_1 x_1 + \lambda_2 x_2] \cap (\lambda_1[x_1, y] + \lambda_2[x_2, y]) \neq \varnothing$$

$$[x, \lambda_1 y_1 + \lambda_2 y_2] \cap (\lambda_1[x, y_1] + \lambda_2[x, y_2]) \neq \varnothing,$$

$$\forall x, x_1, x_2, y, y_1, y_2 \in L, \quad \lambda_1, \lambda_2 \in F$$

(L2) $[x, x] \cap \omega_L \neq \varnothing, \quad \forall x \in L$

(L3) $([x, [y, z]] + [y, [z, x]] + [z, [x, y]]) \cap \omega_L \neq \varnothing, \quad \forall x, y \in L$

This is a general definition thus one can use special cases in order to face problems in applied sciences.

Now, we can see theta hopes in $H_v$-vector spaces and $H_v$-Lie algebras:

**Theorem 3.3.** *Let* $(\mathbf{A}, +, \cdot)$ *be an algebra over the field F. Take any map* $f : A \to A$, *then the* $\partial$*-hope on the Lie bracket* $[x, y] = xy - yx$, *is defined as follows*

$$x\partial y = \{f(x)y - f(y)x, f(x)y - yf(x), xf(y) - f(y)x, xf(y) - yf(x)\}.$$

*Then* $(\mathbf{A}, +, \partial)$ *is an* $H_v$*-algebra over F, with respect to the* $\partial$*-hopes on Lie bracket, where the weak anti-commutativity and the inclusion linearity is valid.*

Remark that if we take the identity map $f(x) = x, \forall x \in A$, then $x\partial y = \{xy - yx\}$, thus we have not a hope and remains the same operation.

## 4. An application

During last decades hyperstructures have a variety of applications in other branches of mathematics and in many other sciences. These applications range from biomathematics -conchology, inheritance- and hadronic physics or on leptons to mention but a few. The hyperstructures theory is closely related to fuzzy theory; consequently, hyperstructures can now be widely applicable in industry and production, too. In several books and review papers [8], [10], [11], [13], [14], [15], [21], [26], [31] one can find numerous applications.

The Lie-Santilli theory on *isotopies* was born in 1970's to solve Hadronic Mechanics problems. Santilli proposed a "lifting" of the n-dimensional trivial unit matrix of a normal theory into a nowhere singular, symmetric, real-valued, positive-defined, n-dimensional new matrix. The original theory is reconstructed such as to admit the new matrix as left and right unit. The *isofields* needed in this theory correspond into the hyperstructures were introduced by Santilli and Vougiouklis in 1996 [20] and they are called *e-hyperfields*. The $H_v$-fields can give e-hyperfields which can be used in the isotopy theory in applications as in physics or biology. We present in the following the main definitions and results restricted in the $H_v$-structures.

**Definition 4.1.** A hyperstructure $(H, \cdot)$ which contain a unique scalar unit e, is called e-hyperstructure. In an e-hyperstructure, we assume that for every element x, there exists an inverse $x^{-1}$, i.e. $e \in x \cdot x^{-1} \cap x^{-1} \cdot x$. Remark that the inverses are not necessarily unique.

**Definition 4.2.** A hyperstructure $(F, +, \cdot)$, where $(+)$ is an operation and $(\cdot)$ is a hope, is called e-hyperfield if the following axioms are valid:

1. $(F, +)$ is an abelian group with the additive unit 0,

2. $(\cdot)$ is WASS,

3. $(\cdot)$ is weak distributive with respect to $(+)$,

4. 0 is absorbing element: $0 \cdot x = x \cdot 0 = 0, \forall x \in F$,
5. exist a multiplicative scalar unit 1, i.e. $1 \cdot x = x \cdot 1 = x, \forall x \in F$,
6. for every $x \in F$ there exists a unique inverse $x^{-1}$, such that $1 \in x{\cdot}x^{-1} {\cap} x^{-1}{\cdot}x$.

The elements of an e-hyperfield are called *e-hypernumbers*. In the case that the relation $1 = x \cdot x^{-1} = x^{-1} \cdot x$ is valid, we say that we have a *strong e-hyperfield.*

Now, we present a general construction which is based on the partial ordering of the $H_v$-structures and on the Little Theorem.

**Definition 4.3.** *The Main e-Construction.* Given a group $(G, \cdot)$, where e is the unit, then we define in G, a large number of hopes $(\otimes)$ as follows:

$$x \otimes y = \{xy, g_1, g_2, ...\}, \forall x, y \in G - \{e\}, \text{ and } g_1, g_2, ... \in G - \{e\}$$

$g_1, g_2$,... are not necessarily the same for each pair (x,y). Then $(G, \otimes)$ becomes an $H_v$-group, actually is an $H_b$-group which contains the $(G, \cdot)$. The $H_v$-group $(G, \otimes)$ is an e-hypergroup. Moreover, if for each x,y such that $xy = e$, so we have $x \otimes y = xy$, then $(G, \otimes)$ becomes a strong e-hypergroup

The proof is immediate since we enlarge the results of the group by putting elements from G and applying the Little Theorem. Moreover one can see that the unit e is a unique scalar and for each x in G, there exists a unique inverse $x^{-1}$, such that $1 \in x \cdot x^{-1} \cap x^{-1} \cdot x$ and if this condition is valid then we have $1 = x \cdot x^{-1} = x^{-1} \cdot x$. So the hyperstructure $(G, \otimes)$ is a strong e-hypergroup.

The above main e-construction gives an extremely large number of e-hopes. These e-hopes can be used in the several more complicate hyperstructures to obtain appropriate e-hyperstructures.

One can see that we can have more strict hopes. The reason we gave the above example is to see that there is a large variety of e-hyperstructures we can construct from given classical structures. One can see that some classes of e-hyperstructures and their properties and results connected them with the classical theory. The representation theory and the Lie algebras as well as in hypermatrix theory large classes of e-hyperstructures appear and can offer to Lie-Santilli algebraic theory models to represent their theory

## 5. The Santilli's Lie-admissibility in hyperstructures

Now, we present the Santilli's admissibility in hyperstructures in a general form:

**Definition 5.1.** Let $(L, +)$ be an $H_v$-vector space $H_v$-field $(F, +, \cdot)$, $\phi : F \to F/\gamma^*$ the canonical map and $\omega_F = \{x \in F : \phi(x) = 0\}$, where 0 is the zero of the fundamental field $F/\gamma$. Similarly, let $\omega_L$ be the core of the canonical map $\phi' : L \to L/\epsilon^*$ and denote by the same symbol 0 the zero of $L/\epsilon^*$. Consider the $H_v$*-Lie algebra* over $\mathbf{F}$ *taking the bracket(commulator) hope*:

$$[,] : \mathbf{L} \times \mathbf{L} \to \mathbf{P}(\mathbf{L}) : (x, y) \to [x, y]$$

Take any two subsets $\mathbf{R}, \mathbf{S} \subseteq \mathbf{L}$ then the **general Santilli's Lie-admissible hyperalgebra** is obtained by reformulating the Lie bracket into the hope:

$$[,]_{RS} : \mathbf{L} \times \mathbf{L} \to \mathbf{P}(\mathbf{L}) : [x, y]_{RS} = x\mathbf{R}y - y\mathbf{S}x.$$

Notice that $[x, y]_{RS}$ is not a single element but a set

$$[x, y]_{RS} = x\mathbf{R}y - y\mathbf{S}x = \{xry - ysx / r \in \mathbf{R} \text{ and } s \in \mathbf{S}\}.$$

Remark that this definition is, in some way, a generalization of the $P$-hopes.

Special cases, but not degenerate, are the "small" and "strict" ones:

(a) **R**=e. Then $[x, y]_R S = xy - y\mathbf{S}x = \{xy - ysx/s \in \mathbf{S}\}$

(b) **S**=e. Then $[x, y]_R S = x\mathbf{R}y - yx = \{xry - yx/r \in \mathbf{R}\}$

(c) $\mathbf{R} = \{r_1, r_2\}$ and $\mathbf{S} = \{s_1, s_2\}$. Then

$$[x, y]_{RS} = x\mathbf{R}y - y\mathbf{S}x = \{xr_1y - ys_1x, xr_1y - ys_2x, xr_2y - ys_1x, xr_2y - ys_2x\}$$

Since the above is the most general definition one can take special cases in order to obtain applications. Therefore if one take e-hyperstructures used in Lie-Santilli theory then the admissibility is transferred in an obvious reason. Finally using the fundamental structures the classical algebraic structures and the hyperstructures are connected.

Now, we can transfer the Santilli's admissibility problem, presented in the Introduction, into the hyperstructure theory. The Santilli's admissibility in the in the hyperstructure theory can be achieved in the following ways: (a) The use of an $H_v$-field instead of an ordinary field. (b) The replacement, or enlargement, of the single valued external or internal operations on vectors by multivalued ones. (c) The replacement of the selected elements R and S by sets of elements.

Therefore, (i) In equation (1.1), the hyperstructure form can be faced by using the elements $\lambda$ from an $H_v$-field or the external operation $\lambda A$ could be a hope or, of course, both the above generalizations can be used.

(ii) In the realization presented by equation (1.4), the hyperstructure form can be achieved by using an $H_v$-field or by replace $R$ and $S$ (consequently, $T$ and $W$) by sets of elements. In the later case, since they are P-hopes, one can use also the Construction 2.5 as well.

(iii) The case of equation (1.7) is already a generalization into the multivalued case. In fact, this generalization is an $Hb$-hope since it is an extension of an single valued operation. However, if we replace $S_1, S_2, ..., S_n$ and $R_1, R_2, ..., R_n$ by sets of elements, then we obtain enlarged hopes.

## 6. Concluding remarks

As it is well known, the correlation of two or more atoms of a DNA can yield entire organs during the growth of a biological structure with a very large number of atoms. But biological correlations can be mathematically represented with the multiplication that, for the DNA is expected to be multi-valued. Therefore, Santilli [8], [13] suggested hypermathematics with ordered basic rules (1.7) in which the product of two quantities can yield a large number of ordered results. However, rules (1.7) are classical and, as such, they cannot provide the most general possible mathematics as expected for the DNA in view of its complexity. This insufficiency has been resolved in this paper via Vougiouklis $H_v$ formulation of Santilli's classical Lie-admissible hyperstructures (1.7). Specific initial applications to the DNA are in progress for reporting in a subsequent paper.

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